Field Theory and Critical Phenomena

Lecture notes by
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FIELD THEORY AND CRITICAL PHENOMENA

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KEYWORDS

- STATISTICAL SYSTEMS MAY EXHIBIT CRITICAL POINTS ASSOCIATED TO PHASE TRANSITIONS.
- PHASE TRANSITIONS CAN BE: MAGNETIC, LIQUID-GAS, "GEOMETRICAL"
- COLLECTIVELY: CRITICAL PHENOMENA.
- APPARENT ENDLESS DIVERSITY, ACTUALLY THEY FALL INTO A GENERAL THEORETICAL FRAMEWORK.
- KEY EMERGENT PROPERTY IS UNIVERSALITY: SYSTEMS DIFFERING AT THE MICROSCOPIC LEVEL SHOW SOME COMMON CRITICAL PROPERTIES, MAINLY DICTATED BY THE DIMENTIONALITY OF THE SYSTEM AND THE SYMMETRIES.
- FIELD THEORY EXPLAINS UNIVERSALITY AND DESCRIBES QUANTITATIVELY UNIVERSALITY CLASSES.

PLAN

I. GENERAL FRAMEWORK AND PERTURBATIVE RENORMALIZATION GROUP

II. CRITICAL FLUCTUATIONS AS COLLECTIVE EXCITATIONS (PARTICLE MODES)

→ EXACT RESULTS IN LOW DIMENSIONS

→ INSIGHT ON QUANTUM CRITICAL POINTS

→ INTERFACIAL PHENOMENA

REFS:

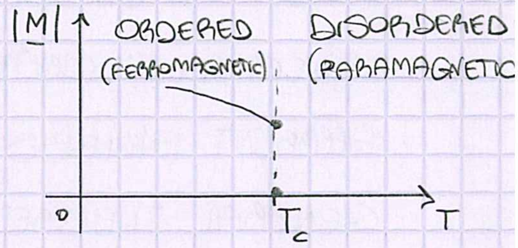
I. J. CARDY, "Scaling and renormalization in statistical physics", CAMBRIDGE, 1996

II. G. DELFINO, Fields, particles and universality, ARXIV: 1502.05538

EXAMPLES

1) FERROMAGNETS

BASIC EXAMPLE OF PHASE TRANSITIONS. THE CHANGE OF PHASE APPEARS THROUGH THE MEASURE OF THE LOCAL MAGNETIZATION (ORDER PARAMETER), $\underline{M}$.

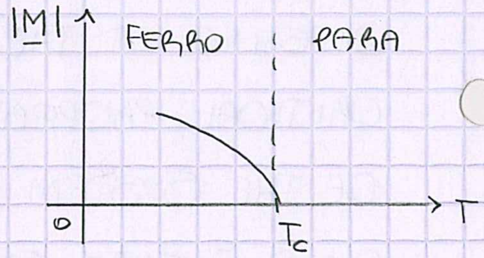


TWO CASES:

i. I ORDER TRANSITIONS

THE ORDER PARAMETER IS DISCONTINUOUS AT T_c.

DIFFERENT PHASES COEXIST AT T_c.



ii. II ORDER (OR CONTINUOUS) TRANSITIONS

THE ORDER PARAMETER IS CONTINUOUS AT T_c.

LET'S CALL

$\underline{S}(x)$ = MAGNETIC MOMENT AT POINT x.

FOR AN HOMOGENEOUS SYSTEM, WE CAN CONSIDER

$\underline{M} = \langle \underline{S}(x) \rangle$.

ANOTHER QUANTITY OF INTEREST IS THE CORRELATION FUNCTION

$\langle \underline{S}(x_1)\, \underline{S}(x_2) \rangle \simeq \langle \underline{S}(x_1) \rangle \langle \underline{S}(x_2) \rangle$ IF $|x_1 - x_2| \gg \xi$

WHERE

ξ = CORRELATION LENGTH.

EXPERIMENTALLY:

- $\xi < \infty$ AT $T \neq T_c$
- $\xi < \infty$ AT $T = T_c$ FOR I ORDER TRANSITIONS
- $\xi = \infty$ AT $T = T_c$ FOR II ORDER TRANSITIONS.

THIS LAST FACT IS THE ORIGIN OF UNIVERSALITY: SOME QUANTITIES DO NOT DEPEND ON MICROSCOPIC DETAILS, BUT ONLY ON GLOBAL FEATURES (DIMENSIONALITY, SYMMETRY).

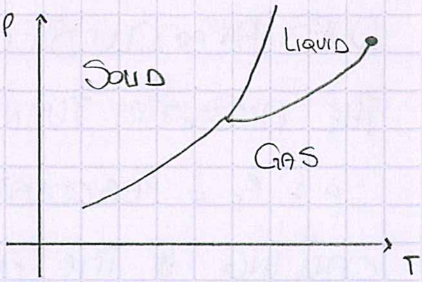
UNIVERSALITY EMERGES NEAR II ORDER TRANSITION POINTS.
TYPICAL UNIVERSAL QUANTITIES ARE THE CRITICAL EXPONENTS:

$$|\underline{M}| \propto (T_c - T)^{\beta}, \qquad T \to T_c^- \qquad (\beta > 0)$$

$$\xi \propto |T - T_c|^{-\nu}, \qquad T \to T_c^{\pm} \qquad (\nu > 0)$$

T_c, FOR INSTANCE, IS NON-UNIVERSAL.

2) SIMPLE FLUIDS



PHASE DIAGRAM IN P-T PLANE.
LET'S FOCUS ON THE LIQUID/GAS LINE, WHICH HAS AN ENDPOINT AT (T_c, P_c).
HERE THE DENSITY ρ CHANGES DISCONTINUOUSLY ACROSS THE TRANSITION LINE, HENCE MAKING IT A I ORDER ONE.
THUS, L AND G COEXIST ALONG THE LINE, WITH DENSITIES ρ_L, ρ_G.
EXPERIMENTALLY,

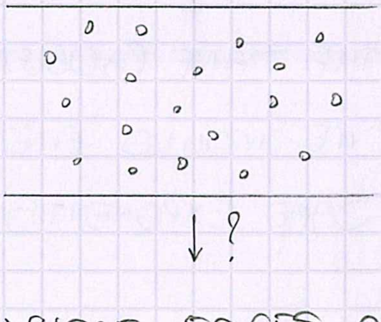
$(\rho_L - \rho_G) \to 0$ AS (T_c, P_c) IS APPROACHED.

$(\rho_L - \rho_G) \propto (T_c - T)^{\beta}$ AS $T \to T_c^-$ ALONG THE LINE

THE CORRELATION LENGTH DIVERGES AT (T_c, P_c) WITH AN EXPONENT ν

THESE EXPONENTS TURN OUT TO COINCIDE WITH THOSE OF UNIAXIAL FERROMAGNETS ($\mathbb{Z}_2$ SYMMETRY, ISING).

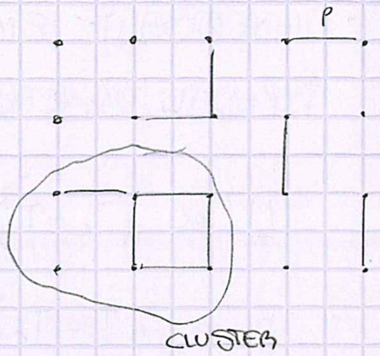
3) PERCOLATION



SUPPOSE A LIQUID IS POURED ON TOP OF A POROUS MEDIUM. WILL THE LIQUID BE ABLE TO REACH THE BOTTOM?
WE CAN MODEL THE PROBLEM ON A LATTICE WHERE EDGES ARE OPEN WITH PROBABILITY p (BOND PERCOLATION).

THE PROBABILITY OF A GIVEN CONFIGURATION IS

$$p^{\# \text{OPEN EDGES}} \, (1-p)^{\# \text{CLOSED EDGES}}$$



THE QUESTION BECOMES: IS THE PROBABILITY OF FINDING AN OPEN PATH FROM TOP TO BOTTOM > 0?

IN FACT, IT IS EASIER TO CONSIDER AN INFINITE SYSTEM AND ASK: DOES AN ∞ CLUSTER OF OPEN EDGES EXIST?

THE ANSWER TURNS OUT TO BE: YES IF

$p > p_c$ = PERCOLATION THRESHOLD.

CALLING P THE FRACTION OF THE LATTICE OCCUPIED BY THE INFINITE CLUSTER, IT IS FOUND THAT

$$P \propto (p-p_c)^{\beta} \quad \text{AS} \quad p \to p_c^+ .$$

THIS IS A RANDOM PROBLEM (NO TEMPERATURE, NO INTERACTION, NO SYMMETRY, NON-LOCAL ORDER PARAMETER): IT PROVIDES THE PROTOTYPE OF GEOMETRICAL TRANSITIONS.

WE CAN DEFINE THE

CONNECTIVITY $g(r) \equiv$ PROB. THAT TWO OPEN EDGES SEPARATED BY A DISTANCE r ARE IN THE SAME CLUSTER.

WE WOULD FIND

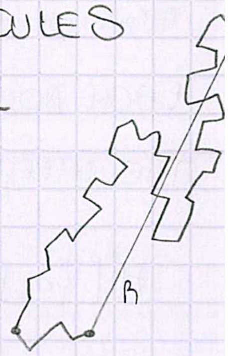
$$g(r) \sim e^{-r/\xi}, \qquad r \to \infty$$

$$\xi \propto |p-p_c|^{-\nu} \qquad \text{AS} \quad p \to p_c^{\pm}$$

THE SAME PROBLEM CAN BE RECAST INTO ONE OF SITE PERCOLATION. WE WOULD FIND A DIFFERENT p_c (NON UNIVERSAL), BUT THE SAME EXPONENTS (EVEN ON A DIFFERENT LATTICE).

4) POLYMER STATISTICS

POLYMERS ARE LONG, FLEXIBLE CHAINS MADE OF MOLECULES (MONOMERS) PLACED IN A SOLVENT. THEIR STATISTICAL PROPERTIES REVEAL CRITICAL EXPONENTS AND UNIVERSALITY.



FOR EXAMPLE, CALLING R THE END TO END DISTANCE, WHAT IS THE VALUE OF $\langle R \rangle$ WHEN THE NUMBER OF MONOMERS $N \to \infty$? IT TURNS OUT THAT

$$\langle R \rangle \propto N^{\nu} \qquad \text{AS } N \to \infty .$$

THIS BELONGS TO THE UNIVERSALITY CLASS OF SELF-AVOIDING WALKS.

EQUILIBRIUM STATISTICAL MECHANICS

10.10.19

THINK OF FERROMAGNETS MODELED BY SPIN VARIABLES LOCATED AT EACH NODE OF A REGULAR INFINITE LATTICE IN $d > 1$ DIMENSIONS (σ_i).

THE INTERACTION IS SPECIFIED BY

$\mathcal{H}[\{\sigma_i\}]$ = ENERGY FOR CONFIGURATION $\{\sigma_i\}$

$$\langle O \rangle = \frac{1}{Z} \sum_{\{\sigma_i\}} O \, e^{-\mathcal{H}[\sigma_i]/T} \qquad Z = \sum_{\{\sigma_i\}} e^{-\mathcal{H}[\sigma_i]/T} \quad \text{(PARTITION FUNCTION)}$$

THE FREE ENERGY PER SITE CAN BE DEFINED AS

$$f = -\frac{T}{N} \ln Z \qquad N \to \infty, \ \# \text{ OF SITES}$$

THE SIMPLEST MODEL EXHIBITING A PHASE TRANSITION IS THE ISING MODEL

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i \qquad \langle ij \rangle \text{ MEANS NEAREST NEIGHBOURS}$$

$$\sigma_i = \pm 1 \qquad J > 0 \text{ FOR FERROMAGNETS.}$$

IF $H = 0$, THEN $\mathcal{H}$ IS INVARIANT FOR SPIN REVERSAL:

$$\mathbb{Z}_2 : \sigma_i \to -\sigma_i \ \forall i \qquad M = \langle \sigma_i \rangle.$$

SPONTANEOUS SYMMETRY BREAKING (SSB)

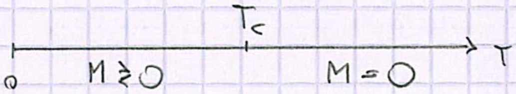
CONSIDER ISING WITH $H = 0$.

FOR $T \to \infty$, $\frac{J}{T} \to 0$ (DECOUPLED LIMIT) AND THE SYSTEM IS DISORDERED.

FOR $T \to 0$, $\frac{J}{T} \to \infty$ AND THERE ARE TWO GROUND STATES:

$$\begin{cases} \sigma_i = 1, \ \forall i \\ \sigma_i = -1, \ \forall i. \end{cases}$$

LOWERING T FROM $T = \infty$ THE SYSTEM HAS TO CHOOSE IN WHICH GROUND STATE IT WILL GO AT $T = 0$. THIS HAPPENS AT T_c, WHICH IS THE POINT OF SSB.



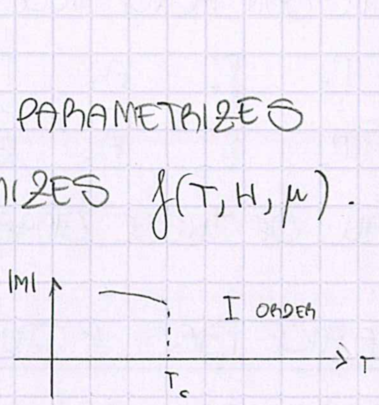
$\mathcal{H}$ IS $\mathbb{Z}_2$-INVARIANT $\forall T$, BUT BELOW T_c THERE ARE 2 EQUILIBRIUM PHASES AND THE SYSTEM CHOOSES ONE.

LANDAU PICTURE

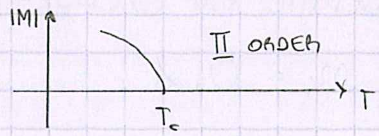
$f(T,H)$ = EQUILIBRIUM FREE ENERGY

IT IS A RESULT OF THERMODYNAMICS THAT, IF μ PARAMETRIZES DEVIATIONS FROM EQUILIBRIUM, THEN $f(T,H)$ MINIMIZES $f(T,H,\mu)$.

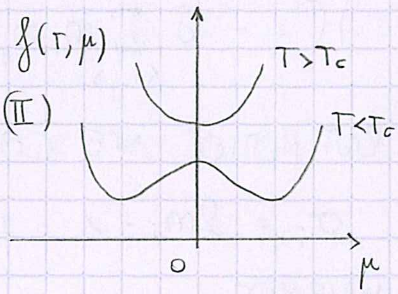
THINK OF A $\mathbb{Z}_2$-SYMMETRIC $\mathcal{H}$.



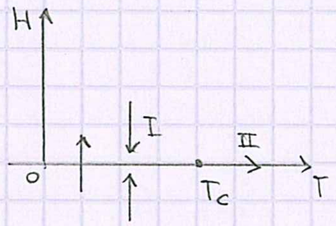
IN LANDAU THEORY M IS THE MINIMUM OF $f(T,\mu)$.

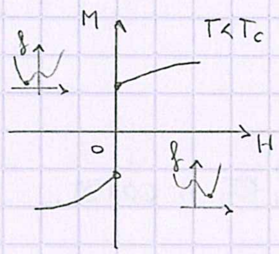


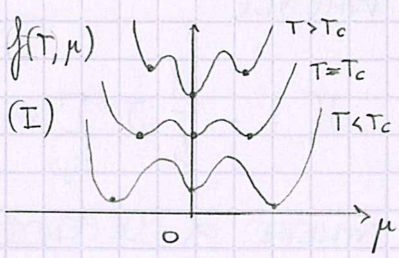
THE H = 0 ISING MODEL HAS A II ORDER TRANSITION.



A I ORDER TRANSITION (DISCONTINUITY IN M) OCCURS IN ISING WITH $H \neq 0$ (EXPLICIT SYMMETRY BREAKING).







CANONICAL OBSERVABLES

$$M = \langle \sigma_i \rangle \underset{\text{HOM.}}{=} \frac{1}{N}\sum_i \langle \sigma_i \rangle = -\frac{\partial f}{\partial H}$$ MAGNETIZATION

$$\chi = \frac{\partial M}{\partial H} = -\frac{\partial^2 f}{\partial H^2} = \frac{1}{TN}\sum_{i,j}\langle \sigma_i \sigma_j \rangle_c$$ SUSCEPTIBILITY

$$\langle \sigma_i \sigma_j \rangle_c \sim e^{-\frac{|i-j|}{\xi}}, \quad |i-j| \to \infty$$ ξ CORRELATION LENGTH

$$C = -T\frac{\partial^2 f}{\partial T^2}$$ SPECIFIC HEAT

CRITICAL EXPONENTS:

$$M = B(T_c - T)^{\beta}, \quad T \to T_c^-, \quad H = 0$$ B CRITICAL AMPLITUDE

$$\chi = \Gamma_\pm |T-T_c|^{-\gamma}, \quad T \to T_c^\pm, \quad H = 0$$

$$\xi = \xi_\pm^0 |T-T_c|^{-\nu} \quad \text{AND} \quad C = A_\pm |T-T_c|^{-\alpha}$$ IN THE SAME LIMIT.

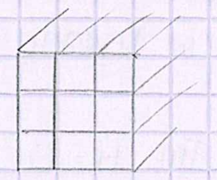
$\alpha, \beta, \gamma, \nu$ ARE UNIVERSAL, WHILE CRITICAL AMPLITUDES ARE NOT.

WE CAN NEVERTHELESS CONSTRUCT SOME UNIVERSAL COMBINATIONS, LIKE

$$\frac{\Gamma_+}{\Gamma_-}, \quad \frac{\xi_+^0}{\xi_-^0}, \quad \frac{A_+}{A_-}, \quad \frac{\Gamma_- \Gamma_+}{B^2 (\xi_+^0)^d}, \ \dots$$

ALL OF THESE CHARACTERIZE AN UNIVERSALITY CLASS.

· LATTICE GAS PICTURE OF A SIMPLE FLUID



DEFINE AN OCCUPATION NUMBER OF SITE i

$$m_i = 0, 1$$

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} m_i m_j + \mu \sum_i m_i$$

BUT NOTICE WE CAN DEFINE AN ISING VARIABLE LIKE

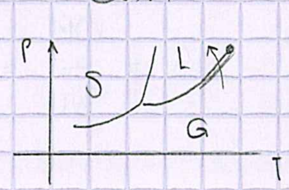
$$\sigma_i = 2m_i - 1 = \pm 1$$

WHENCE

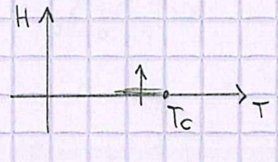
$$\mathcal{H} = -\frac{1}{4} J \sum_{\langle ij \rangle} \sigma_i \sigma_j + \left(\frac{\mu}{2} - \alpha J\right) \sum_i \sigma_i + \text{CONST.}$$

WHERE α DEPENDS ON THE COORDINATION NUMBER OF THE LATTICE (i.e. THE # OF NEAREST NEIGHBOURS, WHICH ON A CUBIC 3d LATTICE IS 6).

IT IS NOT SURPRISING, THEN, THAT A SIMPLE FLUID IS IN THE SAME UNIVERSALITY CLASS AS ISING.



NOTICE THIS UNIVERSALITY ONLY HOLDS IN THE VICINITY OF A CRITICAL POINT, WHERE ξ DIVERGES. THIS DOESN'T HAPPEN ANYWHERE ELSE IN THE P-T DIAGRAM ON THE RIGHT.



· q-STATE POTTS MODEL

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j} \qquad s_i = 1, 2, \dots, q \qquad \text{(COLORS)}$$

IT IS SYMMETRIC UNDER S_q, NAMELY A GLOBAL PERMUTATION OF COLORS.

THE MODEL ADMITS q GROUND STATES, SO IT EXHIBITS SSB OF S_q AT $T=T_c$.

DEFINING

$$\sigma_{i,\alpha} \equiv \delta_{s_i,\alpha} - \frac{1}{q} \qquad \alpha = 1,2,\dots,q \qquad \text{(I)}$$

THEN BY CONSTRUCTION (SINCE OVER T_c EACH COLOR OCCURS WITH PROBABILITY $\frac{1}{q}$),

$$\langle \sigma_{i,\alpha} \rangle_{T>T_c} = 0. \qquad \text{(II)}$$

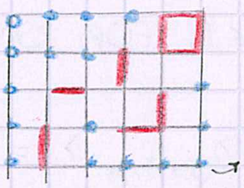
MOREOVER, SINCE

$$\sum_{\alpha=1}^{q} \sigma_{i,\alpha} = 0$$

WE DEDUCE THAT THE ORDER PARAMETER IS A $(q-1)$-COMPONENTS ONE. IN PARTICULAR, $S_2 = \mathbb{Z}_2$ (ISING).

★ LET'S STUDY ITS GRAPH EXPANSION: YOU COULD PROVE THAT

$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{-H/T} \propto \sum_{G} p^{N_b} (1-p)^{\bar{N}_b} q^{N_c}$$



ONE POSSIBLE CONFIGURATION ON A POSSIBLE LATTICE

NOTE: FORTUIN-KASTELEYN 1969.

WHERE

$G \equiv$ GRAPH MADE OF BONDS

$p \equiv 1 - e^{-J/T} \in (0,1)$ FOR $J>0$

$N_b \equiv$ # OF BONDS IN G

$\bar{N}_b \equiv$ # OF EDGES WITHOUT A BOND

$N_c \equiv$ # OF CLUSTERS IN G (CONNECTED COMPONENTS + ISOLATED SITES)

BEWARE THESE ARE NOT THE OBVIOUS SPIN CLUSTERS (SAME COLOR).

THIS MAPPING MAKES SENSE OF THE POTTS MODEL FOR CONTINUOUS q, SO IT PROVIDES AN ANALYTIC CONTINUATION OF THE MODEL FOR NON INTEGER q. IN PARTICULAR ONE COULD TAKE THE LIMIT

$q \to 1$: THE WEIGHT BECOMES $p^{N_b}(1-p)^{\bar{N}_b}$, AS IN PERCOLATION.

IT WORKS, BECAUSE $p \in (0,1)$. WE SAW IN PERCOLATION THAT

$\exists\, p_c$: FOR $p > p_c$, $\mathbb{P}$(A SITE BELONGS TO AN ∞ CLUSTER) > 0.

HENCE, WE ARE MAPPING A GEOMETRICAL TRANSITION ONTO THE $q \to 1$ POTTS FERROMAGNETIC TRANSITION.

NOTICE IT DOESN'T WORK WITH $q = 1$ (IT'S A SINGLE STATE MODEL).

* CONSIDER, IN PERCOLATION, THE MEAN # OF CLUSTERS PER SITE

$$\frac{\langle N_c \rangle}{N} = \frac{1}{N} \lim_{q \to 1} \frac{\partial}{\partial q} \log Z$$

$$= \frac{1}{N} \lim_{q \to 1} \frac{\ln Z - \overbrace{\ln Z|_{q=1}}^{=0}}{q-1} = -\lim_{q \to 1} \frac{f}{q-1} \sim |p - p_c|^{2-\alpha|_{q=1}}$$

WHICH IS FINITE, EVEN THOUGH

$$\lim_{q \to 1} f = 0.$$

NOTE: IT IS BARELY AN INCREMENTAL RATIO, IT IS NOT

$$\log x = \lim_{m \to 0} \frac{x^m - 1}{m}$$

REMEMBER WE DEFINED THE CONNECTIVITY AS

$g(x_1, x_2) \equiv$ PROB. THAT x_1 AND x_2 ARE IN THE SAME CLUSTER.

TAKING $p \leq p_c$ FOR SIMPLICITY, CONSIDER

$$\underbrace{\langle \delta_{s(x_1),\alpha} \, \delta_{s(x_2),\alpha} \rangle}_{\text{PROB. } x_1, x_2 \text{ HAVE COLOR } \alpha} = \frac{g(x_1, x_2)}{q} + \frac{1 - g(x_1, x_2)}{q^2}$$

→ DIFFERENT CLUSTERS

↳ EACH OF THE CLUSTERS WITH ITS COLOR

HENCE*

$$g(x_1, x_2) = \lim_{q \to 1} \frac{\langle \sigma_\alpha(x_1) \sigma_\alpha(x_2) \rangle}{q-1} \sim e^{-|x_1 - x_2|/\xi}, \quad |x_1 - x_2| \to \infty$$

WHICH IS ANOTHER INTERESTING CONNECTION, WHENCE

$$\xi \underset{T \to T_c}{\propto} |T - T_c|^{-\nu} \rightsquigarrow \xi \propto |p - p_c|^{-\nu|_{q=1}}.$$

THIS SETS THE CORRESPONDENCE BETWEEN

CLUSTER OBSERVABLES - SPIN OBSERVABLES

PERCOLATION EXPONENTS - FERROMAGNET EXPONENTS.

*NOTE: WE USED THE DEFINITION (I) AND PROPERTY (II), AND ACTUALLY $g(x_1, x_2) = \lim_{q \to 1} q^2 \frac{\langle \sigma_1 \sigma_2 \rangle}{q-1}$.

· O(m) VECTOR MODEL

15.10.19

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \underline{S}_i \cdot \underline{S}_j \qquad \underline{S}_i \text{ m-COMPONENT UNIT VECTOR}$$

THIS HAMILTONIAN IS INDEED INVARIANT UNDER THE O(m) GROUP OF ROTATIONS (A CONTINUOUS SYMMETRY). FOR

$m = 1 \rightarrow$ ISING MODEL

$m = 2 \rightarrow$ XY MODEL

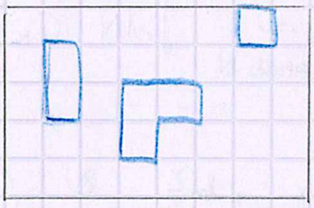
$m = 3 \rightarrow$ HEISENBERG MODEL.

AS FOR THE POTTS MODEL, THERE IS A GRAPH EXPANSION. MATHEMATICALLY, THE EXPANSION HOLDS FOR

$$\tilde{\mathcal{H}} = -\sum_{\langle i,j \rangle} \ln\left(1 + x\, \underline{S}_i \cdot \underline{S}_j\right) \qquad x \sim \frac{J}{T}$$

WHICH COINCIDES WITH $\mathcal{H}$ FOR SMALL x, AND SHARES ITS SAME SYMMETRY: FOR UNIVERSALITY PURPOSES, IT IS EQUALLY OK BECAUSE IT IS IN THE SAME UNIVERSALITY CLASS. THEN

$$\mathcal{Z} = \sum_{\{\underline{S}_i\}} e^{-\frac{\tilde{\mathcal{H}}}{T}} \rightsquigarrow \sum_{\text{LOOPS}} x^{\#\text{BONDS}}\, m^{\#\text{LOOPS}}.$$

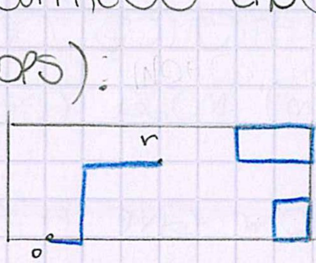


THIS MAKES SENSE OF m REAL (i.e. NOT NECESSARILY INTEGER), LIKE $m \to 0$.

INDEED, THIS LIMIT CAN BE SHOWN TO SUPPRESS CROSSINGS: WE ONLY RETAIN SELF-AVOIDING PATHS (LOOPS).

DEFINING

$$G(r) \equiv \langle S_1(\underline{r}) S_1(\underline{0}) \rangle$$



WE NEED TO AVERAGE OVER CONFIGURATIONS LIKE THE ONE ABOVE WHERE AN OPEN PATH CONNECTING $\underline{r}$ AND $\underline{0}$ APPEARS AS WELL, WEIGHTED WITH THE POSSIBLE CONFIGURATIONS OF THE OTHER CLOSED LOOPS.

ACTUALLY, DUE TO THE FACTOR $m^{\#\text{LOOPS}}$, THE LIMIT $m \to 0$ SUPPRESSES LOOPS IN GENERAL. THEN

NOTE: ONLY THE TERM $0^0 = 1$ SURVIVES, i.e. $\lim_{m\to 0} m^0 = 1$

$$\lim_{m\to 0} G(r) = \sum_N{}' C_N(r)\, X^N$$

$C_N(r)$ = # OF N-STEP S.A.W.'S BETWEEN $\underline{0}$ AND $\underline{r}$.

THIS IS INTERESTING FOR POLYMER STATISTICS. FOR INSTANCE,

$$C_N = \sum_r{}' C_N(r) = \text{TOTAL \# OF N-STEP S.A.W.'S}$$

HENCE

$$\sum_N{}' C_N X^N = \lim_{m\to 0} \underbrace{\sum_r G(r)}_{\text{SUSCEPTIBILITY, } \chi \sim |T-T_c|^{-\gamma}} \underset{X\to X_c}{\sim} \left(1 - \frac{X}{X_c}\right)^{-\gamma_0} \qquad \gamma_0 \equiv \gamma|_{m=0}.$$

WE NOW VERIFY THAT THIS SINGULAR BEHAVIOR CORRESPONDS TO

$$\underline{C_N \propto X_c^{-N} N^{\gamma_0 - 1} \quad \text{AS } N\to\infty}.$$

INDEED,

$$\sum_N C_N X^N \underset{\text{LARGE N}}{\sim} \int dN\, C_N X^N \underset{C_N \propto X_c^{-N} N^{\gamma_0-1}}{\propto} \int \frac{dN}{N} N^{\gamma_0} \left(\frac{X}{X_c}\right)^N$$

$$= \int \frac{dN}{N} N^{\gamma_0} e^{N \ln\frac{X}{X_c}} \underset{Y \equiv N\ln\frac{X}{X_c}}{\overset{(*)}{\propto}} \left(\ln\frac{X}{X_c}\right)^{-\gamma_0} = \left[\ln\left(1 + \frac{X-X_c}{X_c}\right)\right]^{-\gamma_0} \underset{X\to X_c}{\simeq} \left(\frac{X-X_c}{X_c}\right)^{-\gamma_0}.$$

POLYMER STATISTICS CAN THUS BE MAPPED ON THE $m\to 0$ LIMIT OF AN $O(m)$ MODEL FOR A FERROMAGNET.

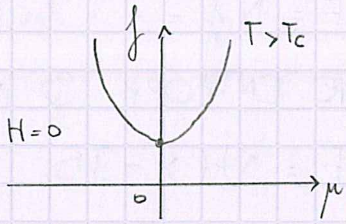
WE WILL NOW GIVE A FIRST ESTIMATE OF ITS CRITICAL EXPONENTS.

(*) NOTE: CALLING $A \equiv \ln\frac{X}{X_c} < 0$, $Y = NA$,

$$\int dN\, N^{\gamma_0 - 1} e^{NA} = \int \frac{dY}{A}\left(\frac{Y}{A}\right)^{\gamma_0-1} e^Y \propto A^{-\gamma_0}.$$

MEAN FIELD CRITICAL EXPONENTS

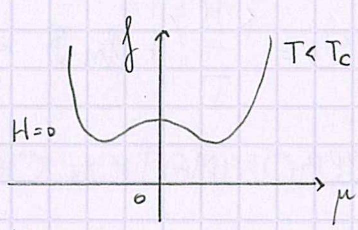
ONE WRITES THE SIMPLEST FREE ENERGY ENCODING SSB WITHIN LANDAU'S PICTURE.



FOR ISING ($\mathbb{Z}_2$ SYMMETRY),

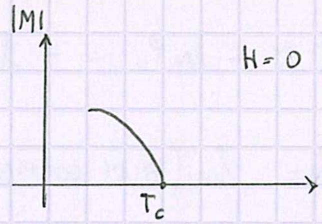
$$f(\mu) = a + b\tau\mu^2 + c\mu^4 + H\mu d$$

$$b, c > 0, \qquad \tau \equiv T - T_c$$



$M \equiv$ VALUE OF μ WHICH MINIMIZES $f(\mu)$

$$= \begin{cases} 0 & T > T_c \\ \pm\left[-\dfrac{b\tau}{2c}\right]^{1/2} & T < T_c \end{cases}$$



HENCE ($H=0$)

$$M \sim |T_c - T|^\beta, \qquad T \to T_c^- \qquad \Rightarrow \qquad \beta_{MF} = \frac{1}{2}.$$

IF $H \to 0$, $\tau > 0$ WE WILL HAVE $M \simeq \tilde{e} H$. THEN

NOTE: USE $f'(\mu) = 2b\tau\mu + 4c\mu^3 + Hd$

$$0 \equiv \left.\frac{\partial f}{\partial \mu}\right|_{\mu = \tilde{e}H} = 2b\tau\tilde{e}H + Hd + O(H^3) \qquad \Rightarrow \qquad \tilde{e} = -\frac{d}{2b\tau}$$

$$\chi = \frac{\partial M}{\partial H} = \tilde{e} \propto \frac{1}{\tau} \qquad \Rightarrow \qquad \gamma_{MF} = 1.$$

FOR $H = 0$, THE FREE ENERGY COMPUTED IN THE MINIMUM IS

$$f(M) = \begin{cases} a, & \tau > 0 \\ a + O(\tau^2), & \tau \to 0^- \quad (M \sim \sqrt{-\tau}). \end{cases}$$

THE SPECIFIC HEAT IS GIVEN BY

$$C = -\frac{\partial^2 f}{\partial \tau^2} = \begin{cases} 0, & \tau > 0 \\ \text{CONST.}, & \tau \to 0^- \end{cases} \qquad \Rightarrow \qquad \alpha_{MF} = 0.$$

ANTICIPATING A BIT, IF τ CORRESPONDS TO THE MASS TERM $m^2\varphi^2$ IN FIELD THEORY, THEN

$$\tau \sim m^2 \qquad m \sim 1/\xi.$$

NONE OF THESE DEPENDS ON THE DIMENSION, SO THEY CANNOT BE CORRECT!

THEN IN MF

$$\xi \sim \tau^{-1/2}, \qquad \tau \to 0^\pm \qquad \Rightarrow \qquad \nu_{MF} = 1/2.$$

ENTROPY

FREE ENERGY:

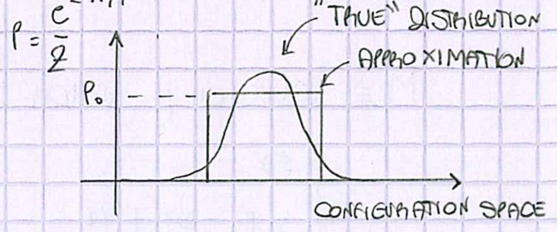
$$F = N f = -T \ln Z.$$

THE ENTROPY S IS DEFINED BY THE RELATION

$$F = \langle \mathcal{H} \rangle - TS$$

$$S = \frac{\langle \mathcal{H} \rangle}{T} + \ln Z = -\left\langle \ln \frac{e^{-\mathcal{H}/T}}{Z} \right\rangle = -\langle \ln(\text{PROB. OF A CONFIGURATION}) \rangle.$$

APPROXIMATION OF EQUIPROBABILITY:



$$S \approx -\ln p_0 = -\ln \frac{1}{\#\text{ OF CONFIGS. WITH } p \neq 0}$$

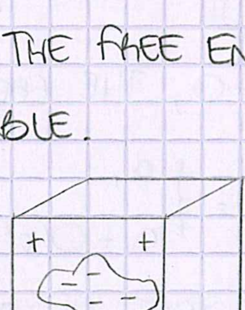
$$= \ln(\#\text{ OF CONFIGS. WITH } p \neq 0)$$

OR, MORE REALISTICALLY, S IS A LOGARITHMIC MEASURE OF THE # OF CONFIGS WITH SIGNIFICANT PROBABILITY.

HEURISTIC ARGUMENTS ON STABILITY OF ORDERED PHASE IN ISING

START WITH ALL SPINS UP AND CHECK THE FREE ENERGY COST ΔF FOR THE FORMATION OF A DOMAIN OF DOWN SPINS OF LINEAR SIZE l.

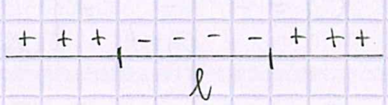
IF THE FORMATION OF A LARGE DOMAIN LOWERS THE FREE ENERGY ($\Delta F < 0$), THEN THE ORDERED PHASE IS UNSTABLE.



$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

$$\Delta F = 2J \cdot \text{PERIMETER} - T \cdot \text{ENTROPY}.$$

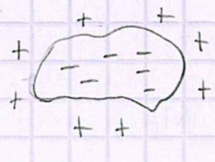
IN $d = 1$,

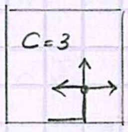


$$\text{PERIMETER} = 2, \quad \text{ENTROPY} \sim \underset{\text{PLACES WHERE I CAN PUT THEM}}{\log(l)}$$

$$\Delta F \sim 4J - T \ln l < 0 \quad \text{FOR } T \neq 0 \text{ AND LARGE ENOUGH } l.$$

IN $d = 2$,





$$\text{PERIMETER} = l, \quad \text{ENTROPY} \sim \ln(c^l) = l \cdot \ln \underset{\#\text{ OF CHOICES}}{c}$$

$$\Delta F \sim 2Jl - Tl \ln c > 0 \quad \text{FOR } T \text{ LOW ENOUGH}.$$

HENCE WE SEE LOW-T ORDER IS STABLE IN $d \geq 2$.

THE ARGUMENT GENERALIZES TO THE OTHER MODELS WITH DISCRETE SYMMETRIES, HENCE

SSB OF DISCRETE SYMMETRIES OCCURS FOR $d > 1$.

HOWEVER, THE ARGUMENT BREAKS DOWN FOR CONTINUOUS SYMMETRY BECAUSE OF SLOW SPIN REVERSAL. $\uparrow \nearrow \nearrow \nearrow \nearrow \rightarrow \searrow \downarrow$

IT CAN BE SHOWN (MERMIN-WAGNER-HOHENBERG THEOREM) THAT CONTINUOUS SYMMETRY ONLY BREAKS SPONTANEOUSLY FOR $d > 2$.

TO SUM UP, CALLING

d_L = LOWER CRITICAL DIMENSION

= LARGEST DIMENSION FOR WHICH THE PHASE TRANSITION DOES NOT OCCUR

WE FOUND THAT

$d_L = 1$ FOR DISCRETE SYMMETRIES

$d_L = 2$ FOR CONTINUOUS SYMMETRIES.

IN $d = 2$, THE XY MODEL ($O(2)$) HAS A TRANSITION (BKT), BUT IT DOES NOT INVOLVE SSB AND THERE IS NO ORDERED PHASE. THERE IS THUS NO VIOLATION OF THE THEOREM.

FIELD THEORY

17.10.19

UNIVERSALITY CLOSE TO II ORDER TRANSITION POINTS IS RELATED TO THE DIVERGENCE OF THE CORRELATION LENGTH. IT SHOULD BE BEST DESCRIBED WITHIN A CONTINUOUS APPROACH: FIELD THEORY.

LATTICE	CONTINUUM
SITE i	$x = (x_1, \dots, x_d) \in \mathbb{R}^d$
σ_i	$\sigma(x)$ SPIN FIELD
$\sum_{j \text{ n.n. } i} \sigma_i \sigma_j$	$\varepsilon(x)$ ENERGY DENSITY FIELD
H/T	A EUCLIDEAN ACTION
Z	$Z = \sum_{\text{FIELD CONFIGS}} e^{-A} \sim \int D\phi\, e^{-A[\phi]}$.

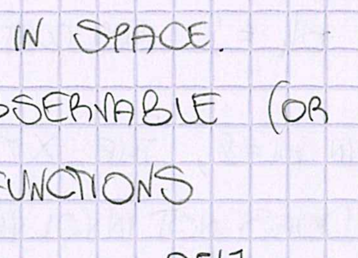
FIELDS ARE NONTRIVIAL OBJECTS WHOSE NATURE WILL BECOME PROGRESSIVELY CLEAR.

THE SUM OVER CONFIGURATIONS MEANS THAT WE DEAL WITH STATISTICAL FT (AND NOT CLASSICAL). WE ARE THUS NEVER INTERESTED IN THE VALUE OF A FIELD AT POINT x IN SPACE. FIELDS ARE NOT THEMSELVES OBSERVABLES. OBSERVABLE (OR MEASURABLE) QUANTITIES ARE THE CORRELATION FUNCTIONS

$$\langle \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\rangle = \frac{1}{Z}\int D\phi\, \phi_1(x_1)\dots\phi_n(x_n)\, e^{-A[\phi]}.$$

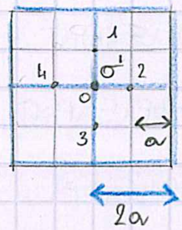
THERE ARE INFINITELY MANY FIELDS (THINK OF THE DERIVATIVES OF ϕ, FOR INSTANCE). THERE IS AN ∞-DIMENSIONAL BASIS $\{\psi_i\}$ SUCH THAT PRODUCTS OF FIELDS CAN BE EXPANDED ON THIS BASIS. THEN WE CAN THINK OF WRITING GENERICALLY

$$A = \sum_i g_i \int d^d x\, \psi_i(x) \qquad \underline{g} = (g_1, g_2, \dots).$$



· RENORMALIZATION GROUP IDEA

★ LATTICE: CONSIDER $H[\{\sigma_i\}]$, $\sigma_i = \pm 1$.



$a \to 2a$

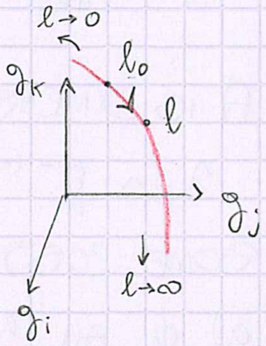
$$\sigma' = \mathrm{sign}\left(\sum_{j=0}^{4} \sigma_j\right) = \pm 1 \qquad H[\sigma] \to H'[\sigma'].$$

IF THE HAMILTONIAN IS EXPRESSED IN THE MOST GENERAL FORM WITH ALL POSSIBLE COUPLINGS $\underline{J} = (J_1, J_2, \dots)$, THIS OPERATION AMOUNTS TO CHANGING

$$\underline{J} \to \underline{J}'.$$

THE PASSAGE TO SPIN VARIABLES REFERRING TO A LARGER SCALE IS EXPRESSED THROUGH A CHANGE IN THE COUPLING VECTOR: THIS IS A RG TRANSFORMATION.

NOTICE IT'S IRREVERSIBLE, BECAUSE WE HAVE LOST INFORMATION.



★ FIELD THEORY: CONTINUOUS CHANGE OF SCALE

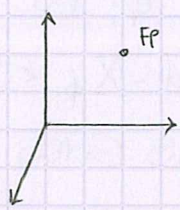
→ RG TRAJECTORY

$$\underline{g}(\ell_0) \to \underline{g}(\ell).$$

ITS DIRECTION IS TOWARDS LARGER DISTANCES.

★ RG FIXED POINTS (FPs): THESE ARE THEORIES WITHOUT ANY DIMENSIONAL PARAMETER, AND ARE THUS SCALE INVARIANT, i.e.

$$\frac{dg_j}{d\ell} = 0 \quad \forall j.$$



IN A HOMOGENEOUS SYSTEM WHICH IS MOREOVER INVARIANT UNDER ROTATIONS (SCALAR), THE CORRELATION FUNCTION AT A F.P. TAKES THE SCALE INVARIANT FORM

$$\langle \phi(x)\phi(y) \rangle = \frac{\text{CONST.}}{|x-y|^{2X_\phi}}, \qquad \xi = \infty$$

X_ϕ = SCALING DIMENSION OF ϕ AT THE GIVEN F.P. ($\phi \sim \text{LENGTH}^{-X_\phi}$).

IN A SENSE WHICH WILL BE CLARIFIED, ϕ IS SAID TO BE

- RELEVANT, IF $X_\phi < d$
- IRRELEVANT, IF $X_\phi > d$
- MARGINAL, IF $X_\phi = d$.

NOTE: "WE CALL THE OPERATORS WHOSE COEFFICIENTS GROW DURING THE RECURSION PROCEDURE RELEVANT OPERATORS" (PESKIN P. 402).

CORRELATIONS DECAY WITH DISTANCE, HENCE

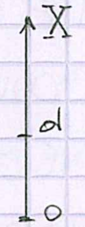
$X_\phi > 0$

NOTE: LONG DISTANCE => IR, LOW ENERGIES.

(APART FROM THE TRIVIAL IDENTITY FIELD I, FOR WHICH $X_I = 0$).

THE SPECTRUM OF SCALING DIMENSIONS IS USUALLY DISCRETE, SO WE EXPECT A FINITE NUMBER OF RELEVANT FIELDS.

THE # OF IRRELEVANT FIELDS IS ALWAYS INFINITE.



- BEYOND FPs

$$A = A_{FP} + \sum_i \lambda_i \int d^d x \, \phi_i(x)$$

A IS DIMENSIONLESS (e^{-A}), SO

$$\lambda_i \sim \xi^{-(d - X_{\phi_i})}$$

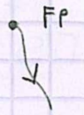
ξ = CHARACTERISTIC LENGTH.

NOTE: THIS IS THE CORRELATION LENGTH ONLY WHEN IT IS FINITE, OTHERWISE IT'S SOME OTHER LENGTH (LIKE $\tilde{\xi}$).

SOME POSSIBLE SCENARIOS:

a) ϕ_i ALL RELEVANT

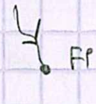
$d - X_{\phi_i} > 0 \quad \forall i \quad \Rightarrow \quad \lambda_i \to 0$ WHEN $\xi \gg l$;



THIS MEANS THE F.P. IS AN ULTRAVIOLET F.P. ($l \to 0$, i.e. OBSERVING THE SYSTEM AT SMALLER SCALES, IS THE REVERSE RG FLOW).

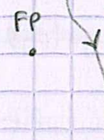
b) ϕ_i ALL IRRELEVANT

$d - X_{\phi_i} < 0 \quad \forall i \quad \Rightarrow \quad \lambda_i \to 0$ WHEN $\xi \ll l$.



THIS GIVES AN INFRARED F.P..

c) SOME ϕ_i ARE RELEVANT, SOME ARE IRRELEVANT



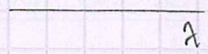
λ_i CANNOT ALL VANISH SIMULTANEOUSLY.

d) $A = A_{FP} + \lambda \int d^d x \, \phi(x)$ WITH ϕ MARGINAL

λ IS SUPERFICIALLY DIMENSIONLESS (SUPERFICIAL SCALE INVARIANCE).

LOGARITHMIC CORRECTIONS CAN MAKE THE FIELD MARGINALLY RELEVANT/IRRELEVANT.

HOWEVER, IF THERE ARE NO logs, THEN ϕ STAYS "TRULY MARGINAL" AND SCALE INVARIANCE IS REALLY PRESERVED.



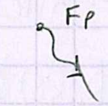
THIS PRODUCES A LINE OF FPs PARAMETRIZED BY λ.
THIS IS, FOR INSTANCE, WHAT'S BEYOND THE BKT TRANSITION AND THE LUTTINGER LIQUID.

· <u>TWO - POINT FUNCTION</u>

TAKE

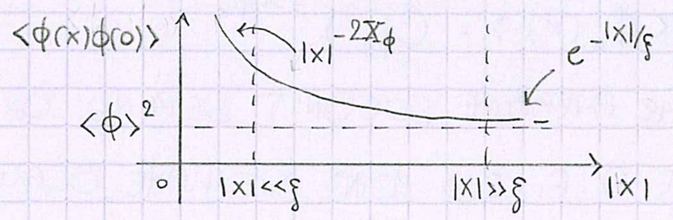
$$A = A_{FP} + \sum_i \lambda_i \int d^d x\, \phi_i(x) \qquad \phi_i \text{ ALL RELEVANT.}$$

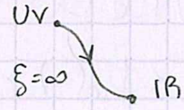
NOTE: AT A CRITICAL POINT, THIS IS CLEARLY NOT THE CASE!



1. GENERIC FLOW: THE THEORY DEVELOPS A FINITE CORRELATION LENGTH.

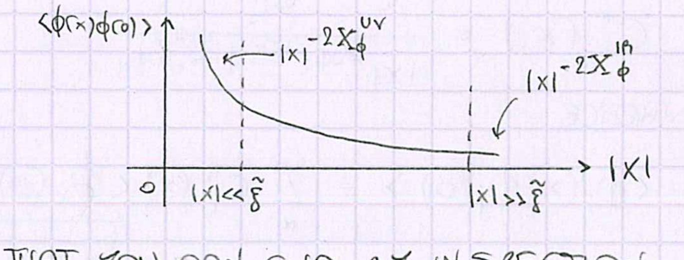
$|x|$ PLAYS THE ROLE OF l, THE DISTANCE AT WHICH WE PROBE THE SYSTEM.





2. CROSSOVER FLOW

THEN $\xi = \infty$ AT EACH OF THE 2 FPs, SO IT STAYS SO IN BETWEEN (NON-MONOTONIC BEHAVIOR OF ξ DOES NOT OCCUR WITH RG TRANSFORMATIONS).



THERE IS A CROSSOVER LENGTH $\tilde{\xi}$ THAT YOU CAN FIND BY INSPECTION.

· <u>FOCUS</u>: RELEVANT/IRRELEVANT IN QFT (PESKIN, P. 402)

"AN OPERATOR WITH MASS DIMENSION X_i HAS A COEFFICIENT $[\lambda_i]$ WITH DIMENSION $(\text{MASS})^{d-X_i}$. THE NATURAL ORDER OF MAGNITUDE FOR THIS MASS IS THE CUTOFF Λ [i.e. $\frac{1}{a}$]. THUS, IF $d - X_i > 0$, THE PERTURBATION IS INCREASINGLY IMPORTANT AT LOW MOMENTA [AND WE CALL THIS OPERATOR RELEVANT, BECAUSE ITS COEFFICIENT GROWS DURING THE RECURSION PROCEDURE, THAT IS GOING TOWARDS BIGGER $l = \frac{1}{p}$].

$\left(\frac{1}{p}\right)^{d-X_i}$

ON THE OTHER HAND, IF $d - X_i < 0$, THE RELATIVE SIZE OF THIS TERM DECREASES AS $\left(\frac{p}{\Lambda}\right)^{X_i - d}$ AS THE MOMENTUM $p \to 0$; THUS THE TERM IS TRULY IRRELEVANT".

$\hookrightarrow \left[\left(\frac{l}{a}\right)^{d-X_i}\right]$. BUT WE DON'T USE THE CUTOFF: a DOESN'T EXIST IN OUR CONTINUUM FORMULATION.

UNIVERSALITY 21.10.19

· OPERATOR PRODUCT EXPANSION (OPE)

$$\phi_i(x)\phi_j(y) = \sum_k \underbrace{C^k_{ij}(x-y)}_{\text{STRUCTURE FUNCTIONS}}\phi_k(y)$$

$\{\phi_k\}$ BASIS.

NOTE: THIS IS ONLY TRUE IN A WEAK SENSE, I.E. INSIDE CORRELATION FUNCTIONS.

IN A CORRELATION FUNCTION,

$$\langle \dots \phi_i(x)\phi_j(y) \dots \rangle = \sum_k C^k_{ij}(x-y)\langle \dots \phi_k(y) \dots \rangle.$$

IN PARTICULAR

$$\langle \phi_i(x)\phi_j(y)\rangle = \sum_k C^k_{ij}(x-y)\langle\phi_k(y)\rangle$$

NOTE: IT SHOULD BE $|x-y| \ll \xi$ (MUSSARDO). THIS ON THE LEFT IS THE OPERATOR ALGEBRA IF ϕ_i, ϕ_j BELONG TO THE BASIS $\{\phi_k\}$.

BUT USUALLY OUR SYSTEMS ARE HOMOGENEOUS, SO

$$\langle \phi_k(x)\rangle = C_{\phi_k}\,\xi^{-X_{\phi_k}}$$

(FOR DIMENSIONAL REASONS)

(THE AVERAGE DOESN'T DEPEND ON x). C_{ϕ_k} IS THE NORMALIZATION. AT A FIXED POINT WE HAVE SCALE INVARIANCE, HENCE $\langle\phi_k\rangle = 0$ UNLESS $X_{\phi_k}=0$ (WHICH IS ONLY TRUE FOR THE IDENTITY). WE CAN WRITE

$$\langle\phi_k(x)\rangle = \delta_{\phi_k, I}$$

AT A F.P..

DIMENSIONALLY,

$$C^k_{ij}(x) = \frac{C^k_{ij}}{|x|^{X_{\phi_i}+X_{\phi_j}-X_{\phi_k}}}$$

NOTE: STRUCTURE CONSTANTS OF THE O. ALGEBRA

$C^k_{ij} \equiv$ OPE COEFFICIENTS

WHENCE

$$\langle\phi_i(x)\phi_i(0)\rangle \underset{\text{F.P.}}{=} \sum_k C^k_{ii}(x)\underbrace{\langle\phi_k(0)\rangle}_{\delta_{\phi_k, I}} = C^I_{ii}(x) = \frac{C^I_{ii}}{|x|^{2X_{\phi_i}}}$$

IN AGREEMENT WITH WHAT WE STATED LAST TIME*.

NOTE: $\langle\phi(x)\phi(y)\rangle = \frac{\text{CONST.}}{|x-y|^{2X_\phi}}$ IF $\xi=\infty$

· SCALING LIMIT AND UNIVERSALITY

CONSIDER A LATTICE MODEL DEPENDING ON A SINGLE PARAMETER $\frac{J}{T}$, INVARIANT UNDER A SYMMETRY GROUP G, AND UNDERGOING A II ORDER TRANSITION AT T_c. THEN WE CAN STEP TO THE CONTINUUM LIMIT

$$\mathcal{H} \xrightarrow[\xi \gg a]{T\to T_c} A = A_{FP} + \sum_i \lambda_i \int d^dx\, \phi_i(x)$$

*NOTE: SOLVING THE THEORY MEANS FINDING ALL THE X_i AND THE STRUCTURE CONSTANTS C^k_{ij}.

BOTH A_{FP} AND ϕ_i SHOULD BE G-INVARIANT.

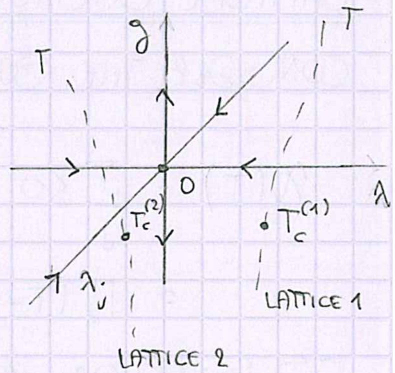
THE SUM RUNS OVER ∞-LY MANY G-INVARIANT ϕ_i'S, A FINITE NUMBER OF WHICH ARE RELEVANT.

FOR REASONS THAT WILL BECOME IMMEDIATELY CLEAR, TAKE ONLY ONE RELEVANT FIELD $\mathcal{E}(x)$ IN A,

$$A = A_{FP} + g\int d^dx\, \mathcal{E}(x) + \sum_{j=1}^{\infty} \lambda_j \int d^dx\, \psi_j(x) \qquad \begin{cases} X_{\mathcal{E}} < d \\ X_{\psi_j} > d \end{cases}.$$

BEING T THE ONLY PARAMETER IN $\mathcal{H}$, WE EXPECT

$$\begin{cases} g = g(T) \\ \lambda_j = \lambda_j(T). \end{cases}$$



EACH DASHED LINE IS AN IMAGE OF T IN THE COUPLING CONSTANTS SPACE: THERE IS ONE IMAGE FOR EACH MICROSCOPIC REALIZATION (e.g. CUBIC, TRIANGULAR LATTICE). BY DEFINITION,

$g = 0$: SUBSPACE OF RG TRAJECTORIES ENDING AT LARGE DISTANCES ON A FP, WHENCE* $\xi = \infty$.

THIS HYPERPLANE CONTAINS T_c. THEN

$$g(T_c) = 0$$
$$g(T) = b(T - T_c) + O\left((T-T_c)^2\right).$$

*NOTE: IF IT WEREN'T ∞ IN THE FIRST PLACE, THEN THE RG FLOW WOULD MAKE IT ZERO. BUT THIS IS IMPOSSIBLE, BECAUSE THE TRAJECTORY ENDS ON A FP, WHERE $\xi = \infty$.

UNIVERSALITY IS EXPECTED WHEN THE SYSTEM IS OBSERVED AT LARGE DISTANCES ($\ell \gg a$):

$$\lambda_j \to 0 \text{ FOR } \ell \text{ LARGE.}$$

NOTA: CAPITO? PER OGNI T. POI A T_c DEVE ESSERE NULLO ANCHE PERCHÉ A SIA INVARIANTE.

THIS MEANS ALL IMAGES OF $\mathcal{H}$ ARE PROJECTED ONTO A UNIQUE THEORY

$$A_{SCALING} = A_{FP} + g\int d^dx\, \mathcal{E}(x).$$

IN PARTICULAR, THE CRITICAL POINT IS PROJECTED ONTO THE F.P. ($g(T_c) = 0$)

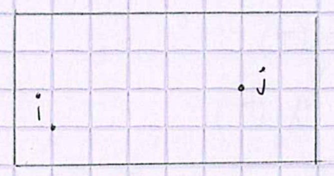
THIS IS THE RG MECHANISM FOR UNIVERSALITY: ALL NON-UNIVERSAL PROPERTIES CORRESPOND TO IRRELEVANT FIELDS.

SINCE ONLY RELEVANT FIELDS SURVIVE IN THE SCALING ACTION, $A_{SCALING}$, THEN THEIR NUMBER IS EQUAL TO THE NUMBER OF PARAMETERS IN $\mathcal{H}$.

THE ENERGY DENSITY FIELD $\varepsilon(x)$ CAN BE DEFINED AS THE MOST RELEVANT G-INVARIANT FIELD (i.e. LOWEST X, EXCLUDING THE IDENTITY).

· CRITICAL EXPONENTS AND AMPLITUDES

CONSIDER THE SUSCEPTIBILITY



$$\chi(T) = \sum_j \langle \sigma_i \sigma_j \rangle_{\mathcal{H}}$$

$$\simeq \Gamma_{\pm} |T - T_c|^{-\gamma} \quad \text{AS} \quad T \to T_c^{\pm}.$$

ITS SINGULAR PART, χ_{SING}, IS DETERMINED BY LARGE DISTANCES $|i-j|$ AND $T \to T_c$: BUT THESE ARE EXACTLY THE CONDITION UNDER WHICH $A \rightsquigarrow A_{SCALING}$.

IN THE CONTINUUM LIMIT,

$$\sigma_j \to \sum_k c_k \, \sigma_k(x)$$

NOTE: i.e. A COMBINATION OF THE "FIELDS" $\sigma_k(x)$.

WHERE $\sigma_k(x)$ ARE FIELDS WITH THE SYMMETRY OF THE ORDER PARAMETER. WE ORDER THEM BY THEIR SCALING DIMENSION,

$$X_{\sigma_1} < X_{\sigma_2} < \dots$$

SIMILARLY,

$$\sum_j \to J \int d^d x$$

WHERE THE CONSTANT J PRESERVES IN GENERAL SOME LATTICE DEPENDENCE. THE SINGULAR PART OF χ HENCE LOOKS LIKE (CLOSE TO THE FP, $\xi < \infty$)

$$\chi_{SING}(T) = J \sum_{k,\ell} c_k c_\ell \underbrace{\int d^d x \, \langle \sigma_k(x) \sigma_\ell(0) \rangle_{A_{SCALING}}}_{\Gamma_{k\ell}\, \xi^{d - X_{\sigma_k} - X_{\sigma_\ell}}}$$

(BY DIMENSIONAL ANALYSIS). AT T_c,

$$\chi_{SING} \xrightarrow[T \to T_c^{\pm}]{\xi \to \infty} J c_1^2 \, \Gamma_{11}^{\pm} \, \xi^{d - 2X_{\sigma_1}}. \qquad \text{(I)}$$

IN OTHER WORDS, THE BEHAVIOR IS DOMINATED BY THE MOST RELEVANT FIELD σ_1, TO WHICH FROM NOW ON WE WILL REFER AS

$$\sigma_1 \equiv \sigma \equiv \text{SPIN FIELD}.$$

* THE RELATION

NOTE: $\left[g \int d^d x\, \varepsilon(x)\right] = 0$.

$$g \sim \xi^{-(d-X_\varepsilon)}$$

CAN BE INVERTED TO GIVE

$$\xi \sim |g|^{-\frac{1}{d-X_\varepsilon}} \sim |T-T_c|^{-\frac{1}{d-X_\varepsilon}} \equiv |T-T_c|^{-\nu}$$

WHENCE

$$\underline{\nu = \frac{1}{d-X_\varepsilon}}.$$

IT FOLLOWS FROM (I)

$$\chi_{SING}(T) \underset{T\to T_c^\pm}{=} \int C_1^2\, \Gamma_{11}^{\pm}\, |T-T_c|^{-\frac{d-2X_\sigma}{d-X_\varepsilon}} \equiv |T-T_c|^{-\gamma}$$

WHICH GIVES

$$\underline{\gamma = \frac{d-2X_\sigma}{d-X_\varepsilon}}.$$

MOREOVER, WE FIND THE UNIVERSAL RATIO BETWEEN CRITICAL AMPLITUDES

$$\frac{\Gamma_+}{\Gamma_-} = \frac{\Gamma_{11}^+}{\Gamma_{11}^-} = \frac{\int d^d x\, \langle \sigma(x)\sigma(0)\rangle_{g>0}}{\int d^d x\, \langle \sigma(x)\sigma(0)\rangle_{g<0}}.$$

THE ONLY NON-UNIVERSAL QUANTITY HERE IS THE NORMALIZATION OF THE SPIN FIELD, WHICH HOWEVER CANCELS OUT IN THE RATIO. NOTICE THIS IS NOT DETERMINED BY THE FP, BECAUSE WE ARE INTEGRATING ALONG THE TRAJECTORIES.

FINALLY,

$$\langle \sigma \rangle \sim \xi^{-X_\sigma} \sim |T-T_c|^{\nu X_\sigma} \equiv |T-T_c|^{\beta}$$

$$\underline{\beta = \frac{X_\sigma}{d-X_\varepsilon}}$$

AND

$$C = -\frac{\partial^2 f}{\partial T^2} \sim \int d^d x \, \langle \varepsilon(x)\varepsilon(0)\rangle \sim \xi^{d-2X_\varepsilon} \sim |T-T_c|^{-\underbrace{\nu(d-2X_\varepsilon)}_{\equiv\alpha}}$$

$$\underline{\alpha = \frac{d-2X_\varepsilon}{d-X_\varepsilon}}.$$

MORAL: X_σ AND X_ε DETERMINE ALL THE CRITICAL EXPONENTS.
THIS EXPLAINS THE EARLY DAYS SCALING RELATIONS

$$\begin{cases} \alpha + 2\beta + \gamma = 2 \\ 2 - \alpha = \nu d. \end{cases}$$

· <u>SPATIAL SYMMETRIES AND THE STRESS TENSOR</u>

THIS IS THE EQUIVALENT, IN EUCLIDEAN SPACE, OF THE ENERGY-MOMENTUM TENSOR IN MINKOWSKI SPACETIME.
IT GIVES US THE RESPONSE OF THE ACTION UNDER AN INFINITESIMAL COORDINATE TRANSFORMATION. THE STRESS TENSOR $T_{\mu\nu}(x)$ IS INDEED DEFINED BY (SUM OVER μ, ν)

$$\delta A = \frac{1}{S_d} \int d^d x \, T_{\mu\nu}(x)\, \partial_\mu \delta x_\nu \qquad \text{UNDER } x_\mu \to x_\mu + \delta x_\mu$$

$$\mu = 1, \dots, d$$

$$S_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} = \text{SURFACE OF THE UNIT SPHERE}.$$

NOTE: $T_{\mu\nu}$ COINCIDES, UP TO A FACTOR $\frac{1}{S_d}$ INSERTED FOR CONVENIENCE, WITH THE USUAL DEFINITION FROM NOETHER'S THEOREM

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi - \mathcal{L}\,\delta^\mu{}_\nu .$$

ITS SCALING DIMENSION IS

$$X_{T_{\mu\nu}} = d.$$

1. TRANSLATIONS

$$\delta x_\nu = a_\nu \qquad \Rightarrow \qquad \delta A = 0$$

BECAUSE $\partial_\mu a_\nu = 0$. TRANSLATIONAL INVARIANCE IS ALREADY ENCODED.

2. ROTATIONS (IN $\mu\nu$-PLANE)

$$\delta x_\nu \propto \varepsilon_{\mu\nu} x_\mu \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$$

$$\partial_\mu \delta x_\nu \propto \varepsilon_{\mu\nu} \qquad \Rightarrow \qquad \delta A = 0 \text{ IF } T_{\mu\nu} = T_{\nu\mu}.$$

3. DILATIONS

$$\delta x_\nu = \alpha x_\nu \quad \Rightarrow \quad \delta A = \frac{\alpha}{S_d} \int d^d x \, \underbrace{T_{\mu\nu} \delta_{\mu\nu}}_{= T_{\mu\mu} \equiv \Theta}$$

$$x_\nu \rightarrow (1+\alpha) x_\nu$$

SO SCALE INVARIANCE REQUIRES A TRACELESS STRESS TENSOR,

$$\Theta(x) = 0 .$$

AT A FP, $T_{\mu\nu}$ IS TRACELESS.

· FOCUS: ROTATIONS

$$x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda^\mu{}_\nu \simeq \delta^\mu{}_\nu + \epsilon^\mu{}_\nu .$$

IMPOSING THE INVARIANCE OF THE METRIC,

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu g_{\alpha\beta} = g_{\mu\nu} \quad \Rightarrow \quad g_{\alpha\nu} \epsilon^\alpha{}_\mu + g_{\mu\beta} \epsilon^\beta{}_\nu = 0$$

WHICH SHOWS THAT $\epsilon^\alpha{}_\beta$ IS NOT ANTISYMMETRICAL (IT IS NOT EVEN A TENSOR), BUT

$$\epsilon_{\nu\mu} + \epsilon_{\mu\nu} = 0 \quad \Rightarrow \quad \epsilon_{\mu\nu} = - \epsilon_{\nu\mu} .$$

UNDER THIS COORDINATE TRANSFORMATION Λ, FIELDS TRANSFORM AS

$$\varphi'(x') = S(\Lambda) \varphi(x) \qquad S(\Lambda) = \mathbb{1} + \frac{1}{2} \epsilon_{\mu\nu} M^{\mu\nu}$$

WHERE $M^{\mu\nu}$ DEPENDS ON THE NATURE OF THE FIELD:

$S(\Lambda) = \mathbb{1}$ SCALAR

$S(\Lambda) = \Lambda$ VECTOR

$S(\Lambda) = \mathbb{1} + \frac{1}{8} [\gamma^\mu, \gamma^\nu] \epsilon_{\mu\nu}$ SPINOR.

GENERAL FACTS ABOUT $T_{\mu\nu}$

22.10.19

★ AT A F.P.,

$$\langle \phi(x)\phi(0)\rangle = \frac{C^{I}_{\phi\phi}}{|x|^{2X_\phi}}.$$

UNDER A DILATION

$$\delta x_\mu = \alpha x_\mu .$$

THE VARIATION OF THE TWO-POINTS FUNCTION IS

$$\delta\langle \phi(x)\phi(0)\rangle = -2X_\phi C^{I}_{\phi\phi}\alpha|x|^{-2X_\phi} = -2\alpha X_\phi \langle\phi(x)\phi(0)\rangle \qquad (\mathrm{I})$$

BUT THE SAME CALCULATION CAN BE PERFORMED AS

$$\delta\langle \phi(x)\phi(0)\rangle = \delta\int D\varphi\, \phi(x)\phi(0)\frac{e^{-A_{FP}}}{Z}$$

$$\underset{\substack{\uparrow\\ \delta A_{FP}=0}}{=} \langle\delta\phi(x)\phi(0)\rangle + \langle\phi(x)\delta\phi(0)\rangle. \qquad (\mathrm{II})$$

COMPARING (I) AND (II) WE FIND

$$\underline{\delta\phi(x) = -\alpha X_\phi \phi(x)}.$$

NOTE: A NULL DILATION HAS $\alpha=1$, NOT $\alpha=0$. IN PRACTICE

$b = 1+\alpha$ $\qquad x' = bx = x + \underbrace{\alpha x}_{\delta x}$

$d^dx' = (1+\alpha)^d d^dx$

$\simeq (1+\alpha d)\cdot d^dx.$

★ IN THE VICINITY OF A F.P,

$$A = A_{FP} + \sum_i \lambda_i \int d^dx\, \phi_i(x).$$

UNDER THE SAME DILATION, WE GET

$$\delta A = \sum_i \lambda_i \int\left[\delta(d^dx)\phi_i(x) + d^dx\,\delta\phi_i(x)\right]$$

$$= \sum_i \lambda_i \int d^dx\left[\alpha d - \alpha X_{\phi_i}\right]\phi_i(x). \qquad (\mathrm{III})$$

BUT RECALL, BY DEFINITION,

$$\delta A = \frac{1}{S_d}\int d^dx\, T_{\mu\nu}(x)\partial_\mu\delta x_\nu \overset{\substack{\text{DILATION}\\ \downarrow}}{=} \frac{\alpha}{S_d}\int d^dx\,\Theta(x) \qquad (\mathrm{IV})$$

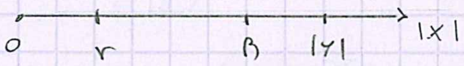
$\Theta(x) = T_{\mu\mu}(x).$

COMPARING (III) AND (IV) WE GET

$$\theta(x) = S_d \sum_i \lambda_i (d - X_{\phi_i}) \phi_i(x) .$$

★ BACK AT THE F.P., CONSIDER THE TRANSFORMATION

$$\delta x_\mu = \begin{cases} \alpha x_\mu & |x| < r \\ f(x) & r < |x| < R \\ 0 & r < R < |x| \end{cases}$$



WHERE f IS AN ARBITRARY FUNCTION INTERPOLATING BETWEEN THE VALUES AT r AND R.

TAKING y AS IN THE FIGURE,

$$\delta \langle \phi(y) \phi(0) \rangle = \delta \frac{C^I_{\phi\phi}}{|y|^{2X_\phi}} = 0$$

NOTE: A_{FP} IS ONLY INVARIANT UNDER DILATIONS, NOT UNDER A GENERIC $f(x)$.
$\langle \phi(x) \rangle \propto \frac{1}{\xi^{X_\phi}}$
SO IT'S NULL AT A F.P. UNLESS $X_\phi = 0$.

BECAUSE $|y| > R$. ON THE OTHER HAND, BEING $A = A_{FP}$,

$$\delta \langle \phi(y) \phi(0) \rangle = \int \mathcal{D}\varphi \, \phi(y) \, \delta\left(\phi(0) \frac{e^{-A}}{Z} \right)$$

$$= \int \mathcal{D}\varphi \, \phi(y) \left[\delta\phi(0) - \phi(0) \left(\delta A + \frac{\delta Z}{Z} \right) \right] \frac{e^{-A}}{Z} .$$

BUT

$$\frac{\delta Z}{Z} = - \langle \delta A \rangle \propto \int d^d x \, \langle T_{\mu\nu}(x) \rangle \, \partial_\mu \delta x_\nu = 0$$

SINCE $\langle T_{\mu\nu} \rangle = 0$ AT A F.P. (ONLY THE IDENTITY HAS A NONZERO EXPECTATION VALUE AT A F.P.). WE CONCLUDE THAT

$$\delta \phi(0) = \delta A \cdot \phi(0) . \qquad (V)$$

NOTICE $\delta A \neq 0$ ONLY IN $r \leq |x| \leq R$ (FOR $x \leq r$ IT'S A DILATION).

INTEGRATING δA BY PARTS IN (V),

$$\delta \phi(0) = \frac{1}{S_d} \left\{ - \int_{|x|=r} d\sigma_\mu \underbrace{\delta x_\nu}_{=\alpha x_\nu} T_{\mu\nu}(x) \phi(0) - \int_{r<|x|<R} d^d x \, \delta x_\nu \, \partial_\mu T_{\mu\nu}(x) \phi(0) \right\}$$

$$d\sigma_\mu \equiv \frac{x_\mu}{|x|} |x|^{d-1} d\Omega \quad \text{(INGOING)} \qquad \int d\Omega = S_d .$$

THE L.H.S. DOES NOT DEPEND ON THE ARBITRARY INTERPOLATION, WHICH IS ONLY POSSIBLE IF

$$\underline{\partial_\mu T_{\mu\nu} = 0} \qquad \text{CONSERVATION OF } T_{\mu\nu}$$

HENCE

$$\delta\phi(0) = -\frac{\alpha}{S_d}\int d\Omega\, |x|^{d-2} x_\mu x_\nu T_{\mu\nu}(x)\phi(0) \qquad (\text{VI})$$

$$= -\alpha X_\phi \phi(0) \,.$$

RECALL THAT $T_{\mu\nu}$ IS SYMMETRIC AND (SINCE WE ARE AT A F.P.) TRACELESS. WE ALSO KNOW IT IS CONSERVED AND $X_{T_{\mu\nu}} = d$. THIS ALLOWS US TO WRITE THE O.P.E.

$$T_{\mu\nu}(x)\phi(0) = C^{\phi}_{T_{\mu\nu},\phi}(x)\phi(0) + \dots$$

NOTE: WHAT HAPPENS TO THE REMAINING PART OF THE EXPANSION? ALL OF THIS WAS 1ST ORDER, SO THE OTHER TERMS SHOULD BE SUBLEADING.

WHERE THE STRUCTURE CONSTANT IS NECESSARILY

$$C^{\phi}_{T_{\mu\nu},\phi} = a_\phi \frac{x_\mu x_\nu - \frac{|x|^2}{d}\delta_{\mu\nu}}{|x|^{d+2}}$$

WHERE THE CONSTANT a_ϕ IS FIXED BY (VI):

$$a_\phi = \frac{X_\phi\, d}{d-1} \,.$$

· CONFORMAL TRANSFORMATIONS

THEY CHANGE THE METRIC MULTIPLICATIVELY IN A POINT-DEPENDENT WAY.

NOTE: SEE FOCUS IN A FEW PAGES.

THIS MEANS

$$\delta(dx_\mu dx_\mu) = f(x)\, dx_\mu dx_\mu$$

$$= 2 dx_\mu \delta dx_\mu = 2 dx_\mu (\partial_\nu \delta x_\mu) dx_\nu = (\partial_\nu \delta x_\mu + \partial_\mu \delta x_\nu) dx_\mu dx_\nu$$

WHICH IMPLIES

$$\partial_\nu \delta x_\mu + \partial_\mu \delta x_\nu = \delta_{\mu\nu} f(x) \,.$$

MULTIPLYING BOTH SIDES BY $\delta_{\mu\nu}$, WE GET

$$2\partial_\mu \delta x_\mu = f(x)\cdot d.$$

WHENCE

$$\underline{\partial_\nu \delta x_\mu + \partial_\mu \delta x_\nu = \frac{2}{d}\delta_{\mu\nu}\partial_\lambda \delta x_\lambda .} \qquad (*)$$

UNDER AN INFINITESIMAL CONFORMAL TRANSFORMATION,

$$\delta A \propto \int d^d x\, T_{\mu\nu}(x)\,\delta_{\mu\nu}\,\partial_\lambda \delta x_\lambda = \int d^d x\, \Theta(x)\,\partial_\lambda \delta x_\lambda .$$

THIS TELLS US THAT AT FP'S, WHERE $\Theta(x)=0$, THERE IS ALSO CONFORMAL INVARIANCE!

EXPANDING δx_μ IN POWERS OF X AND PLUGGING IT INTO THE DEFINING EQUATION (*) GIVES, ORDER BY ORDER:

	GENERATORS	
$O(1)$: a_μ	d	TRANSLATIONS
$O(x)$: αx_μ, $\varepsilon_{\mu\nu} x_\nu$	$1+\frac{d(d-1)}{2}$	DILATIONS, ROTATIONS
$O(x^2)$: $b_\mu \lvert x\rvert^2 - 2 b_\lambda x_\lambda x_\mu$	d	SPECIAL CONFORMAL TRANSFORMATIONS (NOTE: i.e. INVERSION + TRANSLATION)
$O(>2)$: NOTHING FOR $d>2$		

IN $d>2$, THE TOTAL NUMBER OF GENERATORS (i.e. THE DIMENSION OF THE SYMMETRY GROUP) IS

$$\frac{(d+1)(d+2)}{2}.$$

NOT SO IN $d=2$, WHERE THE GROUP BECOMES ∞-DIMENSIONAL.

* TWO CONSEQUENCES OF CONFORMAL INVARIANCE AT FP'S:

1. IT MUST BE

$$\langle \phi_i(x)\phi_j(0)\rangle = \delta_{ij}\,\frac{C^{I}_{\phi_i\phi_i}}{|x|^{2x_{\phi_i}}}.$$

INDEED,

$$\langle \phi_1(y_1)\phi_2(y_2)\rangle - \langle \phi_1(y_2)\phi_2(y_1)\rangle = 0$$

AND SO IS ITS VARIATION, WHICH READS EXPLICITLY, UNDER A

LOCAL DILATION,

NOTE: RECALL $\delta\langle\phi(x)\phi(0)\rangle = -2X_\phi\langle\phi(x)\phi(0)\rangle$.
IT WORKS NOT ONLY FOR LOCAL DILATIONS, BUT "CONFORMAL".

$$0 = \delta\left[\langle\phi_1(r_1)\phi_2(r_2)\rangle - \langle\phi_1(r_2)\phi_2(r_1)\rangle\right]$$

$$= -\left[\alpha(r_1)X_{\phi_1} + \alpha(r_2)X_{\phi_2} - \alpha(r_2)X_{\phi_1} - \alpha(r_1)X_{\phi_2}\right]\langle\phi_1(r_1)\phi_2(r_2)\rangle$$

$$= -\left[\alpha(r_1) - \alpha(r_2)\right]\left[X_{\phi_1} - X_{\phi_2}\right]\langle\phi_1(r_1)\phi_2(r_2)\rangle$$

WHICH MEANS THAT, IF THE DILATION IS REALLY POINT-DEPENDENT, i.e.

$$\alpha(r_1) \neq \alpha(r_2)$$

THEN

$$\langle\phi_1(r_1)\phi_2(r_2)\rangle = 0 \qquad \text{IF } X_{\phi_1} \neq X_{\phi_2}.$$

2. FOR A 3-POINT FUNCTION AT A f.p., TRANSLATIONAL, ROTATIONAL AND SCALE INVARIANCE GIVE

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{\phi_1\phi_2\phi_3}}{|x_1-x_2|^a|x_1-x_3|^b|x_2-x_3|^c}$$

WITH

$$a+b+c = X_1 + X_2 + X_3 \qquad X_{\phi_i} \equiv X_i .$$

TAKING THE LIMIT $x_2 \to x_1$,

$$\longrightarrow \frac{C_{\phi_1\phi_2\phi_3}}{|x_1-x_2|^a|x_1-x_3|^{b+c}} . \qquad (\mathrm{VII})$$

ON THE OTHER HAND, BY O.P.E. WE CAN WRITE

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \sum_k \frac{C^{\phi_k}_{\phi_1\phi_2}}{|x_1-x_2|^{X_1+X_2-X_k}}\langle\phi_k(x_1)\phi_3(x_3)\rangle$$

$$\underset{\substack{\uparrow\\ \text{PROPERTY 1}}}{=} \frac{C^{\phi_3}_{\phi_1\phi_2}C^{I}_{\phi_3\phi_3}}{|x_1-x_2|^{X_1+X_2-X_3}|x_1-x_3|^{2X_3}} . \qquad (\mathrm{VIII})$$

COMPARING (VII) AND (VIII) WE GET

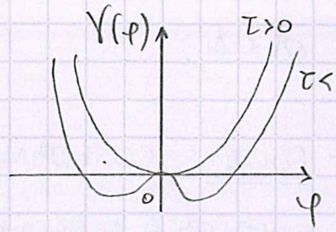
$$\begin{cases} a = X_1 + X_2 - X_3 \\ b = X_1 + X_3 - X_2 \\ c = X_2 + X_3 - X_1 \end{cases} \qquad \text{AND} \qquad C_{\phi_1\phi_2\phi_3} = C^{\phi_3}_{\phi_1\phi_2}C^{I}_{\phi_3\phi_3} .$$

- LANDAU - GINZBURG ACTION

THIS IS THE SIMPLEST ACTION ACCOUNTING FOR SPONTANEOUS SYMMETRY BREAKING.

LET'S CALL $\sigma(x) = \varphi(x)$ AND TAKE $G = \mathbb{Z}_2$,

$\mathbb{Z}_2 : \varphi \to -\varphi$.



THE L-G ACTION READS

$$A_{LG} = \underbrace{\frac{1}{2}\int d^dx\, (\nabla\varphi)^2}_{A^{FP}_{GAUSS}} + \int d^dx \underbrace{\left[\tau\varphi^2 + \lambda\varphi^4\right]}_{V(\varphi)} \qquad \begin{cases} \tau \sim T - T_c \\ \lambda > 0 \end{cases}$$

WHERE A GAUSSIAN FP IS ONE MADE UP OF NON INTERACTING SCALAR FIELDS.

NOTICE WE ARE SEARCHING FOR

$$A_{ISING} = \underbrace{A^{FP}_{ISING} + \tau \int d^dx\, \mathcal{E}(x)}_{\text{SCALING ACTION}} + \text{IRRELEVANT}.$$

DOES A_{LG} REALLY DESCRIBE QUANTITATIVELY A_{ISING}?

FIRST OF ALL,

$$A^{FP}_{ISING} \stackrel{?}{=} A^{FP}_{GAUSS}.$$

NOTE: RECALL THERE CAN ONLY BE ONE RELEVANT FIELD IN A_{ISING}, BECAUSE τ IS THE ONLY PHYSICAL PARAMETER.

BY DEFINITION, $\mathcal{E}(x)$ IS THE ONLY $\mathbb{Z}_2$-EVEN FIELD AT THE ISING FP.

AT THE GAUSSIAN FP,

NOTE: I THINK HE MEANS "THE ONLY RELEVANT FIELD".

$$X^G_\varphi = \frac{d-2}{2} \quad \underset{\text{NO INTERACTION}}{\Longrightarrow} \quad X^G_{\varphi^m} = m X^G_\varphi = m\frac{d-2}{2}.$$

IN PARTICULAR,

$$X^G_{\varphi^2} = d-2 < d \qquad\qquad X^G_{\varphi^4} = 2(d-2) > d \text{ ONLY FOR } d>4.$$

HENCE WE IDENTIFY

$$\varphi^2 \equiv \mathcal{E} \qquad \text{(ALWAYS RELEVANT AT GAUSSIAN FP)}$$

AND WE CONCLUDE THAT

$$A^{FP}_{ISING} \neq A^{FP}_{GAUSS} \qquad \text{FOR } d<4.$$

IN FACT, THERE IS MORE THAN ONE $\mathbb{Z}_2$-EVEN RELEVANT FIELD IN THE GAUSSIAN FP.

THUS WE SAY THAT A^{FP}_{ISING} IS "NONTRIVIAL" (i.e. $\neq A^{FP}_{GAUSS}$) WHEN $d<4$.

· FOCUS: CONFORMAL AND NON-CONFORMAL TRANSFORMATIONS

CONSIDER A COORDINATE TRANSFORMATION (DIFFERENTIABLE, i.e. A DIFFEOMORPHISM)

$$\Lambda^{\alpha'}{}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} \simeq \delta^{\alpha'}{}_{\beta} + \varepsilon^{\alpha'}{}_{\beta} \qquad x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^{\beta} \simeq x^{\alpha} + \delta x^{\alpha}$$

THE DISTANCE $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ IS A SCALAR (i.e. INVARIANT UNDER COORDINATE TRANSFORMATIONS). THE INVARIANT VOLUME ELEMENT IS $\sqrt{-g}\, d^d x$, BECAUSE

$$d^d x' = |J| d^d x = |\det \Lambda^{\alpha'}{}_{\beta}| d^d x \qquad\qquad d^d x' = \sqrt{\frac{-g}{-g'}}\, d^d x$$

$$g_{\mu\nu} = \Lambda^{\alpha'}{}_{\mu} \Lambda^{\beta'}{}_{\nu}\, g_{\alpha'\beta'} \quad \xrightarrow{\det} \quad g = J^2 g'.$$

A CONFORMAL MAPPING IS A DIFFEOMORPHISM WHICH HAS THE EFFECT OF LOCALLY RESCALING THE METRIC TENSOR. WE CAN DETERMINE ITS FORM BY IMPOSING

$$\begin{cases} x^{\mu'} = x^{\mu} + \epsilon^{\mu}(x) \\ g'_{\mu\nu}(x') = A(x)\, g_{\mu\nu}(x) \end{cases} \qquad \Lambda^{\alpha'}{}_{\mu} = \delta^{\alpha}{}_{\mu} + \partial_{\mu}\epsilon^{\alpha}(x) \qquad A(x) = e^{\rho(x)} \simeq 1 + \rho(x).$$

LETTING

$$g_{\mu'\nu'} = \Lambda^{\alpha}{}_{\mu'} \Lambda^{\beta}{}_{\nu'} g_{\alpha\beta} \simeq g_{\mu\nu} - \left(\delta^{\alpha}{}_{\mu} \partial_{\nu}\epsilon^{\beta} + \delta^{\beta}{}_{\nu}\partial_{\mu}\epsilon^{\alpha}\right) g_{\mu\nu} = g_{\mu\nu} - \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right)$$

AVREI DOVUTO FARE IL CONTRARIO, MA ESCE GIUSTO

$$\overset{\text{III}}{(1+\rho(x))\, g_{\mu\nu}} \qquad \Rightarrow \qquad \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = -\rho(x)\, g_{\mu\nu}$$

SATURATING OVER $g^{\mu\nu}$ GIVES

$$\rho(x) = -\frac{2}{d} g^{\mu\nu} \partial_{\mu}\epsilon_{\nu} = -\frac{2}{d}\partial\cdot\epsilon \qquad \Rightarrow \qquad \underline{\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = g_{\mu\nu}\frac{2}{d}\partial\cdot\epsilon}.$$

ANOTHER WAY TO GET TO THIS RESULT IS TO KEEP THE BACKGROUND METRIC FIXED (FLAT, $\delta_{\mu\nu}$) AND ASSUME ds^2 TO VARY. THIS WOULD GIVE

$$ds^2 = \delta_{\mu\nu} dx^{\mu} dx^{\nu} = dx^{\mu} dx_{\mu} \overset{\text{FLAT}}{=} dx_{\mu} dx_{\mu}$$

$$\delta(dx_{\mu}dx_{\mu}) = 2dx_{\mu}\,\delta(dx_{\mu}) = 2(\partial_{\nu}\delta x_{\mu})\, dx_{\mu} dx_{\nu}$$

$$\overset{\text{III}}{f(x)\, dx_{\mu} dx_{\mu}} \qquad \Rightarrow \qquad 2(\partial_{\nu}\delta x_{\mu}) = \delta_{\mu\nu} f(x).$$

FINALLY, A WEYL TRANSFORMATION IS A LOCAL DILATION OF THE METRIC $g_{\mu\nu}(x) \to \Omega(x)\, g_{\mu\nu}(x)$.

IT CAN CHANGE THE PHYSICAL CURVATURE PROPERTIES OF THE SPACE, SO IT IS <u>NOT</u> A CHANGE OF COORDS.

LANDAU GINZBURG - ENCORE

29.10.19

$$\mathcal{A}_{LG} = \underbrace{\int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2\right.}_{\mathcal{A}^{FP}_{GAUSS}} + \underbrace{\left.\tau\varphi^2 + \lambda\varphi^4\right\}}_{V(\varphi)} \qquad \begin{cases} \tau \sim T - T_c \\ \lambda > 0 \end{cases}$$

WHILE

$$\mathcal{A}_{ISING} = \mathcal{A}^{FP}_{ISING} + \tau \int d^d x \ \varepsilon(x) + \text{(IRRELEVANT)}.$$

WE OBSERVED THAT

$$\mathcal{A}^{FP}_{GAUSS} \neq \mathcal{A}^{FP}_{ISING} \qquad \text{FOR } d < 4.$$

FOR $d > 4$, THE GAUSSIAN FP HAS THE RIGHT FIELD CONTENT AND SO IT COINCIDES WITH ISING FP. THERE ARE, HOWEVER, SOME SUBTLETIES

GAUSSIAN EXPONENTS:

$$\varepsilon \sim \varphi^2, \quad \sigma \sim \varphi$$

	GAUSS	SIMULATIONS $d \geq 4$	
α	$2 - d/2$	0	MF
β	$1/4(d-2)$	$1/2$	
γ	1	1	
ν	$1/2$	$1/2$	

$$\nu_G = \frac{1}{d - X_\varepsilon} = \frac{1}{d - X_{\varphi^2}} = \frac{1}{d-(d-2)} = \frac{1}{2}$$

$$\alpha = (d - 2X_\varepsilon)\nu, \quad \beta = X_\sigma \nu, \quad \gamma = (d - 2X_\sigma)\nu. \qquad (I)$$

HOWEVER, IN ACTUAL SIMULATIONS IN $d \geq 4$ WE GET EXPONENTS WHICH DO NOT DEPEND ON d AND COINCIDE WITH THE MEAN FIELD ONES. THE REASON FOR THE MISMATCH IN α, β IS THE FOLLOWING:

$$\beta_G = \frac{d-2}{4} > \beta_{MF} = \frac{1}{2} \qquad \text{FOR } d > 4.$$

SINCE $M \sim (T - T_c)^\beta$, THE "SCALING" EXPONENT $\beta_G = X_\sigma \nu$ GIVES, FOR $d > 4$, AN EFFECT WHICH IS SUBLEADING WRT THAT OF THE DISPLACEMENT OF THE MINIMUM OF THE FREE ENERGY DUE TO SSB AND TAKEN INTO ACCOUNT BY MF. SIMILARLY,

$$\alpha_G = 2 - \frac{d}{2} < \alpha_{MF} = 0 \qquad \text{FOR } d > 4$$

WITH $C \sim |T - T_c|^{-\alpha}$.

* SUMMARIZING, FOR $d \geq 4$ THE FP IS GAUSSIAN; THE EXPONENTS ARE THE MEAN FIELD ONES AND COINCIDE WITH THE GAUSSIAN EXPS EVALUATED AT $d=4$, WHICH IS CALLED THE UPPER CRITICAL DIMENSION (d_c) OF THE ISING MODEL.

MORE GENERALLY, d_c IS DEFINED AS THE DIMENSION BELOW WHICH THE FP IS NON-TRIVIAL (i.e. NON-GAUSSIAN).

FOR A LATTICE MODEL INVARIANT UNDER A GROUP G AND CONTAINING m PARAMETERS,

d_c = dim FOR WHICH THE $(m+1)$-TH MOST RELEVANT G-INVARIANT FIELD AT THE GAUSSIAN FP BECOMES MARGINAL.

· EXAMPLE: ISING

NOTE: IF $X_\phi < d$, THEN THE FIELD IS RELEVANT.

$m=1$, 2^{nd} MOST RELEVANT $\mathbb{Z}_2$-EVEN FIELD IS φ^4 AND

$$X^G_{\varphi^4} = 4\,\frac{d-2}{2} \underset{\substack{\uparrow \\ \text{MARGINALITY CONDITION}}}{\equiv} d \qquad \Rightarrow \qquad d_c = 4.$$

IN GENERAL, AT $d<d_c$ YOU GET $(m+1)$ G-INVARIANT RELEVANT FIELDS (WHEN YOU WANT ONLY m) IN GAUSSIAN FP: THEN THE FP IS NON-TRIVIAL.

FOR $d \geq d_c$, THE FP IS GAUSSIAN AND THE EXPONENTS ARE MF:

MF EXPS $\equiv$ GAUSSIAN EXPS EVALUATED AT $d=d_c$.

· ISING MODEL WITH VACANCIES (OR BLUME-CAPEL MODEL)

$$\mathcal{H} = -J \sum_{\langle i,j\rangle} \sigma_i \sigma_j t_i t_j + \Delta \cdot \sum_i t_i \qquad \sigma_i = \pm 1, \quad t_i = 0,1 \quad (\{\sigma_i\}, \{t_i\}).$$

DEFINING A NEW SET OF VARIABLES

$$S_i \equiv \sigma_i t_i \qquad S_i^2 = t_i$$

$$\mathcal{H} = -J \sum_{\langle i,j\rangle} S_i S_j + \Delta \sum_i S_i^2 \qquad S_i = -1, 0, 1$$

WHICH IS THE BLUME-CAPEL FORMULATION.

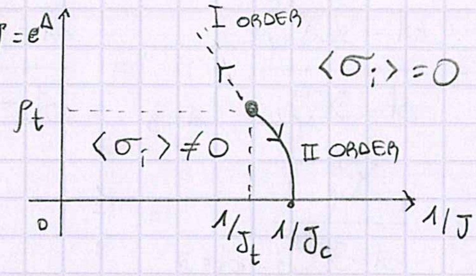
IN QUANTE d?!

ITS SYMMETRY IS STILL

$\mathbb{Z}_2: \sigma_i \to -\sigma_i \ \forall i$ $\quad (s_i \to -s_i)$.

THE LIMIT $\Delta \to -\infty$ IMPLIES $t_i = 1 \ \forall i$, WHICH IS THE PURE ISING MODEL

THE TRANSITION STAYS II ORDER UP TO ρ_t, THEN IT BECOMES I ORDER. THE CORRELATION LENGTH IS ∞ ALONG THE WHOLE SOLID LINE*.



THIS ENDS WITH A SCALE-INVARIANT TRICRITICAL POINT AT ρ_t: THIS IS A FP WITH AN ADDITIONAL $\mathbb{Z}_2$-EVEN FIELD (THE VACANCY DENSITY - COMPARE WITH THE ENERGY DENSITY).

THIS IS THE SIMPLEST EXAMPLE OF A CROSSOVER FLOW, FROM TRICRITICAL TO CRITICAL ISING. THE LINES IN THE DIAGRAM ARE ACTUALLY RG TRAJECTORIES.

LESSON: THE SYMMETRY ALONE DOES NOT NECESSARILY DETERMINE THE UNIVERSALITY CLASS, BECAUSE THERE CAN BE MULTICRITICAL POINTS

THE NEW FP IS DESCRIBED BY

$$A = A^{FP}_{\text{TRICRITICAL}} + \tau \int d^d x \, \varepsilon(x) + g \int d^d x \, \psi(x) + (\text{IRRELEVANT})$$

$$\begin{cases} \tau \sim J_t - J \\ g \sim \rho - \rho_t . \end{cases}$$

*NOTE: GOT IT? THAT'S A RG TRAJECTORY WHICH ENDS ON A FP, HENCE BY CONSTRUCTION $\xi = \infty$. BUT THIS IS PRECISELY WHAT HAPPENS AT A II ORDER TRANSITION.

* WE CAN ALSO CONSTRUCT A LG ACTION FOR THIS PROBLEM:

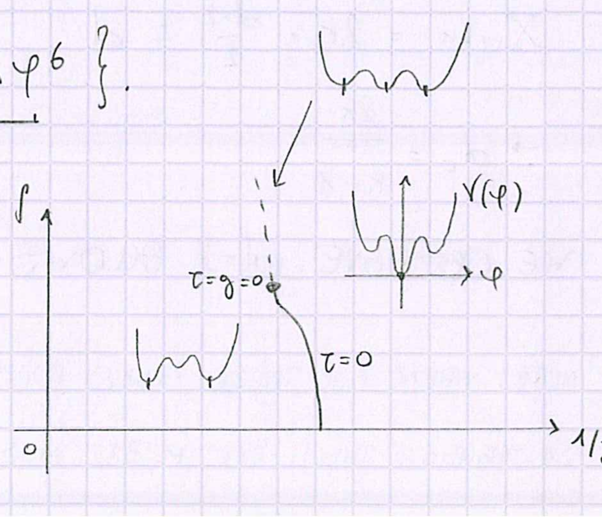
$$A_{LG} = \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \underbrace{\tau \overbrace{\varphi^2}^{\varepsilon} + g \overbrace{\varphi^4}^{\psi} + \lambda \varphi^6}_{V(\varphi)} \right\}.$$

WHAT IS d_c?

$G = \mathbb{Z}_2, \ m = 2$

SO IT IS DETERMINED BY

$$X_{\varphi^6} = 6\,\frac{d-2}{2} \equiv d \ \to \ d_c = 3 .$$



EXPONENTS ARE THUS THE MF ONES AT TRICRITICAL POINT IN $d=3$.

THE TRICRITICAL FP IS NON-TRIVIAL IN $d=2$.

THE MF EXPS ARE THE GAUSSIAN EXPS EVALUATED AT $d=3$: USING (I),

$$\begin{cases} \alpha_{MF} = \left.\dfrac{d-2X^G_{\varphi^2}}{d-X^G_{\varphi^2}}\right|_{d=3} = \left.\dfrac{d-2(d-2)}{d-(d-2)}\right|_{d=3} = \dfrac{1}{2} \\ \beta_{MF} = \left.\dfrac{X^G_{\varphi}}{d-X^G_{\varphi^2}}\right|_{d=3} = \dfrac{1}{4} \\ \gamma_{MF} = 1, \quad \nu_{MF} = \dfrac{1}{2} \end{cases}$$

(THEY DON'T DEPEND ON d)

FOR $J \to J_t$ AT $\rho = \rho_t$. LET'S CHECK THIS AGAINST LANDAU:

$$f(\mu) = a + \tau\mu^2 + g\mu^4 + \lambda\varphi^6 \qquad \lambda > 0.$$

TRICRITICALITY: $\tau = g = 0$. DEVIATIONS IN J ONLY: $g \equiv 0$. THEN

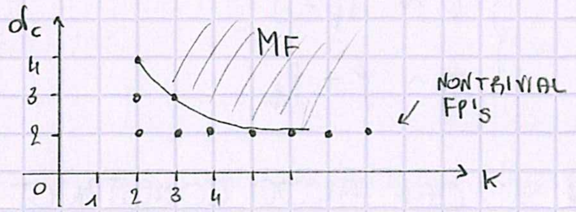
$$0 \equiv \partial_\mu f(\mu)\Big|_{g=0} = 2\tau\mu + 6\lambda\mu \quad \Rightarrow \quad M = \begin{cases} 0 & \tau > 0 \\ \sim(-\tau)^{\frac{1}{4}} & \tau < 0 \end{cases} \to \beta_{MF} = \frac{1}{4}$$

$$f(M)\Big|_{\substack{\tau<0 \\ g=0}} = a + \text{const.}(-\tau)^{3/2} \quad \Rightarrow \quad C \sim \partial_\tau^2 f \sim \tau^{-\frac{1}{2}} \to \alpha_{MF} = \frac{1}{2}.$$

· $\underline{\mathbb{Z}_2}$ - MULTICRITICALITY

K-CRITICALITY REQUIRES THE TUNING OF $(K-1)$ PARAMETERS. THEN

$$A_{LG} = \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \sum_{j=1}^{K} g_j \varphi^{2j} \right\}.$$



d_c IS AGAIN DETERMINED BY

$$X_{\varphi^{2K}} = 2K \cdot \frac{d-2}{2} \equiv d$$

$$\to d_c = \frac{2K}{K-1}.$$

WE OBSERVE $d=2$ ALLOWS FOR INFINITELY MANY NON-TRIVIAL FP'S.

NOTE: WHERE ARE THESE FIXED POINTS? THERE IS AN INFINITE, DISCRETE SET OF CFT'S DESCRIBING THEM. SEE P. 382 MUSSARDO.

LANDAU-GINZBURG: OTHER SYMMETRIES

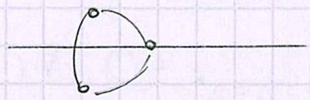
30.10.19

$$A_{LG}^{O(m)} = \int d^dx \left\{ \frac{1}{2}(\nabla \underline{\varphi})^2 + \tau\, \underline{\varphi}\cdot\underline{\varphi} + \lambda (\underline{\varphi}\cdot\underline{\varphi})^2 \right\}$$

$\underline{\varphi} = (\varphi_1, \dots, \varphi_m)$.

AGAIN, $d_c = 4$. LET'S TAKE INSTEAD

$S_3 \sim \mathbb{Z}_3 \times \mathbb{Z}_2$



THIS HAS A 2-COMPONENTS ORDER PARAMETER, WHICH CAN BE TAKEN A

$$\begin{cases} \varphi = \varphi_1 + i\varphi_2 \\ \varphi^* = \varphi_1 - i\varphi_2 \end{cases}$$

NOTE: IN GENERAL, $\mathcal{H}$ CAN CONTAIN
$\mathcal{H} = \partial\varphi\partial\varphi^+ + w(\varphi\varphi^+) + z(\varphi^3) + z(\varphi$

$$A_{LG}^{S_3} = \int d^dx \left\{ \nabla\varphi\nabla\varphi^* + \tau\varphi\varphi^* + g(\varphi^3 + \varphi^{*3}) + \lambda(\varphi\varphi^*)^2 \right\}.$$

THE SYMMETRY IS REALIZED BY

$$\begin{cases} \mathbb{Z}_3 : \varphi \to e^{i2\pi/3}\varphi,\ \varphi^* \to e^{-i2\pi/3}\varphi^* \\ \mathbb{Z}_2 : \varphi \to \varphi^*. \end{cases}$$

NOTE: I GUESS BOTH φ^3 OR φ^{*3} COULD DO DOWN HERE.

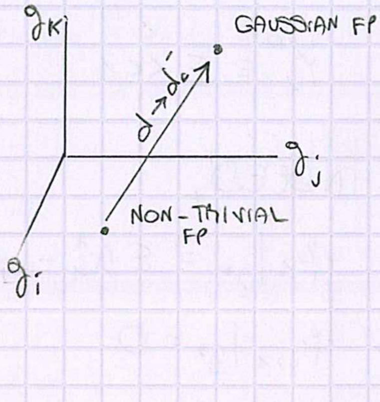
ITS UPPER CRITICAL DIMENSION CAN BE FOUND BY IMPOSING (g TERM)

$$X_{\varphi^3}^G = 3\,\frac{d-2}{2} \equiv d \quad \Rightarrow \quad d_c = 6.$$

* ABOVE d_c THE THEORY IS GAUSSIAN. WHAT HAPPENS BELOW d_c? THIS IS NORMALLY THE MOST INTERESTING CASE, WHICH REQUIRES THAT WE FACE NON-TRIVIAL FP'S.

NORMALLY THERE IS NO SMALL PARAMETER FOR PERTURBATIVE CALCULATIONS.

WILSON-FISHER: CONSIDER d AS A CONTINUOUS PARAMETER. THEN, FOR $d \to d_c^-$, THE NON-TRIVIAL FP APPROACHES THE GAUSSIAN FP.



· PERTURBATIVE RG

$$A = \underset{\substack{\uparrow \\ \text{FP ACTION}}}{A_0} + \sum_i \lambda_i \int d^dx\, \phi_i(x)$$

$X_i^0 \equiv X_{\phi_i}^0$

$Y_i \equiv d - X_i$ (RG EIGENVALUE

LET'S DEFINE THE β-FUNCTIONS

(**) NOTE: CHECK. ϕ_i RELEVANT $\to y_i > 0 \to \lambda_i - \lambda_i^* \sim l^{y_i} \xrightarrow[l\to\infty]{} \infty$
i.e. GROWS UNDER RG FLOW, LEAVING THE FP.

$$\beta_i \equiv l \frac{\partial \lambda_i}{\partial l}$$

l = SCALE AT WHICH WE OBSERVE THE SYSTEM

SO THAT IF

$$\beta_i \big|_{\underline{\lambda} = \underline{\lambda}^*} = 0 \quad \forall i \qquad \Rightarrow \qquad \underline{\lambda}^* \text{ IS A FIXED POINT.}$$

NOTE: $[\lambda_i] = \xi^{d - x_i} = \xi^{y_i}$. WHAT COUNTS IS HOWEVER THE RATIO ξ / l (THIS IS THE MEANING OF "$\rightsquigarrow$").

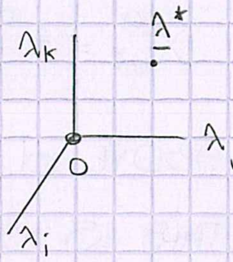
WHEN WE APPROACH A FP(**),

$$\underline{\lambda} \to \underline{\lambda}^* \qquad \Rightarrow \qquad \lambda_i - \lambda_i^* \sim \xi^{-y_i^*} \rightsquigarrow \text{"} l^{y_i^*} \text{"}$$

(NOTICE IN GENERAL $y_i \neq y_i^*$, AND y_i^* IS DIFFERENT AT EACH FP. IT IS ONLY AT THE GAUSSIAN FP THAT THE SCALING DIMENSION CAN BE FOUND BY DIMENSIONAL ANALYSIS, BECAUSE THERE IS NO INTERACTION). HENCE,

$$\beta_i \simeq y_i^* (\lambda_i - \lambda_i^*) \qquad \Rightarrow \qquad \underline{\left.\frac{\partial \beta_i}{\partial \lambda_i}\right|_{\underline{\lambda}^*} = y_i^*.}$$

IN PARTICULAR, EXPANDING(*) AROUND $\underline{\lambda} = 0$ FP,



$$\beta_i = y_i^0 \lambda_i - \sum_{j,k}' \alpha^i_{jk} \lambda_j \lambda_k + O(\lambda^3)$$

WHICH ARE KNOWN AS <u>RG EQUATIONS</u>.

* NOW SUPPOSE ϕ_1 IS ONLY SLIGHTLY RELEVANT. THEN

$$y_1^0 \equiv \varepsilon \ll 1 \qquad \Rightarrow \qquad \lambda_1^* = \frac{\varepsilon}{\alpha^1_{11}}, \quad \lambda^*_{j>1} = 0 \text{ IS A FP.}$$

INDEED,

$$\begin{cases} \beta_1 \big|_{\underline{\lambda}^*} = \varepsilon \lambda_1^* - \alpha^1_{11} \lambda_1^{*2} = 0 \\ \beta_{i>1} \big|_{\underline{\lambda}^*} = 0. \end{cases}$$

(*) NOTE:

$$\frac{\partial}{\partial \lambda_j}\left(l \frac{\partial \lambda_i}{\partial l} \right) = y_i^0 \delta_{ij}$$

$$\beta_i \simeq \overbrace{\beta_i(\underline{\lambda} = 0)}^{=0} + \underline{\nabla}_{\underline{\lambda}} \beta_i \cdot \underline{\lambda} + \sum_{j,k}' \underbrace{\frac{\partial^2 \beta_i}{\partial \lambda_j \partial \lambda_k}}_{\equiv -\alpha_{ij} \text{ BY CONVENTI}} \lambda_j \lambda_k .$$

NOTE: AT LEAST $\beta_{i>1} \big|_{\underline{\lambda}^*} = O(\varepsilon^2)$.

LET'S THEN EVALUATE

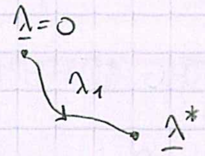
$$y_i^* = \left.\frac{\partial \beta_i}{\partial \lambda_i}\right|_{\underline{\lambda}^*} = y_i^0 - 2 \sum_k \alpha^i_{ik} \lambda_k \Big|_{\underline{\lambda}^*} = y_i^0 - 2 \frac{\alpha^i_{i1}}{\alpha^1_{11}} \varepsilon$$

AND WE CAN CALCULATE

$$\delta^* X_i \equiv X_i^* - X_i^\circ = -\left(y_i^* - y_i^\circ \right) = \frac{2\alpha^i_{i1}}{\alpha^1_{11}}\,\varepsilon \qquad (\mathrm{I}$$

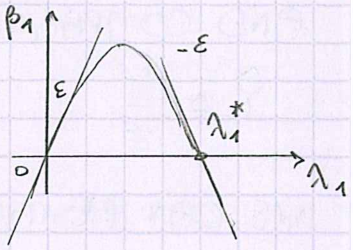
WHENCE

$$\underline{\delta^* X_1 = 2\varepsilon}$$



(λ_1 IS THE ONLY COMPONENT THAT IS FLOWING). WE HAVE

$$\beta_1 = \varepsilon \lambda_1 - \alpha^1_{11} \lambda_1^2$$



WHICH GIVES REASON OF THE SLOPE DIFFERENCE $\delta^* X_1 = 2\varepsilon$. AT THE NEW FIXED POINT $\underline{\lambda}^*$, ϕ_1 BECOMES SLIGHTLY IRRELEVANT.

* LET'S NOW CONSIDER

$$\delta \langle \phi(x)\phi(0)\rangle \equiv \langle \phi(x)\phi(0)\rangle_{\lambda_1} - \langle \phi(x)\phi(0)\rangle_{\lambda_1 = 0}$$

$$\underset{\lambda_1 \to 0}{\simeq} -\lambda_1 \int d^d y \, \langle \phi(x)\phi(0)\phi_1(y)\rangle_0$$

$$= -\lambda_1 \int d^d y \, \frac{C^\circ_{\phi\phi\phi_1}}{|x|^{2X^\circ_\phi - X^\circ_1} |x-y|^{X^\circ_1} |y|^{X^\circ_1}}$$

$$= -\lambda_1 C^\circ_{\phi\phi\phi_1} |x|^{-2X^\circ_\phi} \underbrace{\int d^d y \left(\frac{|x|}{|x-y||y|} \right)^{X^\circ_1}}_{\equiv \mathrm{I}} .$$

NOTE: WE DERIVED THE ACTION BY λ_1, BECAUSE

$$\delta \langle \phi(x)\phi(0)\rangle \simeq \lambda_1 \frac{\partial}{\partial \lambda_1} \langle \phi(x)\phi($$

NOTE: ON 22.10.19 (CONFORMAL INVARIANCE) WE DERIVED

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{\phi_1\phi_2\phi_3}}{|x_1-x_2|^{1+2-3}|x_1-x_3|^{1+3-2}|x_2-x_3|^{2+}}$$

WITH "1" $\equiv X_{\phi_1}$, AND

$$C_{\phi_1\phi_2\phi_3} = C^{\phi_3}_{\phi_1\phi_2} C^{I}_{\phi_3\phi_3} .$$

WE NOTICE(*)

$$\mathrm{I} = \tilde{\gamma} \, |x|^{d - X_1^\circ}$$

NOTE: $= \gamma |x|^\varepsilon$.

AND IT IS FINITE FOR ε SMALL AND POSITIVE. INDEED, AS $y \to \infty$

$$\int_{|y|>R} d^d y \, \frac{1}{|y|^{2(d-\varepsilon)}} < \infty \qquad \text{IF } 2(d-\varepsilon) > d$$

NOTE: RECALL $d - X_1^\circ = y_1^\circ \equiv \varepsilon > 0$
$\to X_1^\circ = d - \varepsilon$.

WHILE FOR $y \to 0$

$$\int_{|y|<R} d^d y \, \frac{1}{|y|^{d-\varepsilon}} < \infty \qquad \text{IF } \varepsilon > 0$$

(AND SIMILARLY FOR $y \to x$).

(*) NOTE: I THINK IT'S JUST PLA[IN] DIMENTIONAL ANALYSIS, AND TH[E] ONLY LENGTH IS $|x|$.

WE OBSERVE A log UV DIVERGENCE FOR $\varepsilon = 0$:

$$I \sim \ln \frac{|x|}{r} \qquad r \equiv \text{UV CUTOFF}.$$

ON THE OTHER HAND,

$$I = \tilde{\gamma} |x|^{\varepsilon} = \tilde{\gamma}\, e^{\varepsilon \ln|x|} \underset{\varepsilon \to 0}{\simeq} \tilde{\gamma}(1 + \varepsilon \ln|x|)$$

AND COMPARING THE TWO WE CONCLUDE IT MUST BE $\tilde{\gamma}\varepsilon = \text{CONST.}$, i.e.

$$\tilde{\gamma} \simeq \frac{\gamma}{\varepsilon} \qquad \gamma \text{ FINITE AS } \varepsilon \to 0.$$

NOTE: BY OPE, $\langle \phi(x)\phi(0)\rangle_0 = \dfrac{C^I_{\phi\phi}}{|x|^{2X_\phi}}$ AT A F.P.

WE CAN REWRITE, FOR $\varepsilon \ll 1$,

$$\delta \langle \phi(x)\phi(0)\rangle = -\langle \phi(x)\phi(0)\rangle_0 \, \lambda_1 \frac{C^{\circ}_{\phi\phi\phi_1}}{(C^I_{\phi\phi})^{\circ}} \frac{\gamma}{\varepsilon}(1 + \varepsilon \ln|x|).$$

WE ALSO HAVE, BY O.P.E.,

$$\langle \phi(x)\phi(0)\rangle_* = (C^I_{\phi\phi})^* |x|^{-2X^*_\phi} = \left((C^I_{\phi\phi})^{\circ} + \delta^* C^I_{\phi\phi}\right)|x|^{-2(X^{\circ}_\phi + \delta^* X_\phi)}$$

NOTE: $|x|^{-2\delta^* X_\phi} = e^{-2\delta^* X_\phi \ln|x|} \simeq 1 - 2\delta^* X_\phi \ln|x|$

$$\simeq \langle \phi(x)\phi(0)\rangle_0 \left(1 + \frac{\delta^* C^I_{\phi\phi}}{(C^I_{\phi\phi})^{\circ}}\right)(1 - 2\delta^* X_\phi \cdot \ln|x|).$$

EQUATING THE TERMS IN $\ln|x|$, WE FIND (RECALL $\lambda^*_1 = O(\varepsilon)$)

$$\delta^* X_\phi = \frac{1}{2}\lambda^*_1 \gamma \frac{C^{\circ}_{\phi\phi\phi_1}}{(C^I_{\phi\phi})^{\circ}} = \frac{1}{2}\lambda^*_1 \gamma \left(C^{\phi}_{\phi\phi_1}\right)^{\circ} \qquad \text{(II)}$$

BECAUSE, AS WE DERIVED A FEW LECTURES AGO,

$$\underset{\substack{\uparrow \\ \text{NUMBER}}}{C_{\phi_1\phi_2\phi_3}} = \underset{\substack{\uparrow \\ \text{OPE COEFFICIENTS}}}{C^{\phi_3}_{\phi_1\phi_2}} C^I_{\phi_3\phi_3}.$$

NOTE: BY CONSTRUCTION, $C_{\phi_1\phi_2\phi_3}$ DOES NOT DEPEND ON THE ORDER OF THE ϕ_i'S AND $C^{\phi_3}_{\phi_1\phi_2} = C^{\phi_3}_{\phi_2\phi_1}$

NOW RECALL $\delta^* X_1 = 2\varepsilon$. PLUGGING IT INTO (II),

$$\frac{1}{2}\lambda^*_1 \gamma = \frac{2\varepsilon}{(C^{\phi_1}_{\phi_1\phi_1})^{\circ}} \quad \overset{\text{(II)}}{\Rightarrow} \quad \underline{\delta^* X_\phi = 2 C_\phi \varepsilon + O(\varepsilon^2)}$$

WHERE C_ϕ IS A UNIVERSAL NUMBER

$$C_\phi \equiv \left(\frac{C^{\phi}_{\phi\phi_1}}{C^{\phi_1}_{\phi_1\phi_1}}\right)^{\circ}$$

(THE OPE COEFFICIENTS ARE NOT, THEY DEPEND ON THE FIELD NORMALIZATION).

NOW WE ALSO RECOGNIZE

$$\alpha^i_{jk} = \left(C^{\phi_i}_{\phi_j \phi_k} \right)^\circ$$

NOTE: BY COMPARISON WITH (I).

SO THAT WE CAN REWRITE THE RG EQUATIONS AS

$$\beta_i = y_i^\circ \lambda_i - \sum_{j,k} (C^i_{jk})^\circ \lambda_j \lambda_k + O(\lambda^3) . \qquad (\mathrm{III})$$

NOTE: FOR FUTURE USE, RECALL $y_i^\circ \equiv d - X^\circ_{\varphi_i}$.

· THEORIES IN $d_c - \varepsilon$ DIMENSIONS

THERE IS A SLIGHTLY RELEVANT FIELD, SO WE CAN APPLY (III).

FOR THE ISING MODEL,

$$A_{LG} = \int d^d x \left\{ \frac{1}{2} (\nabla \varphi)^2 + \tau \varphi^2 + \lambda \varphi^4 \right\} .$$

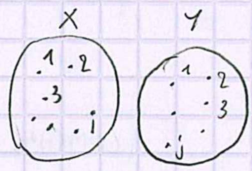
TO MATCH WITH THE ABOVE CALCULATION,

$$\begin{cases} A = A_{LG}, \quad A_0 = A^{FP}_{GAUSS} & \phi_1 = \varphi^4 \\ \varepsilon = d - X^G_{\varphi^4} = d - 2(d-2) = 4 - d = d_c - d . \end{cases}$$

WE JUST NEED TO EVALUATE THE OPE COEFFICIENTS AT FP^{GAUSS}:

$$\varphi^i \cdot \varphi^j \sim \sum_k C^{\varphi^k}_{\varphi^i \varphi^j} \varphi^k \qquad \text{(FORMALLY)} .$$

PICTORIALLY,

$$\varphi^i(x) \varphi^j(y) \rightsquigarrow$$  .

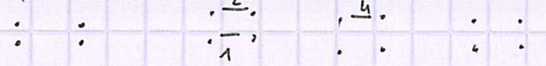
NOTE: THIS EVENTUALLY BOILS DOWN TO WICK'S THEOREM.

IN THE GAUSSIAN FP,

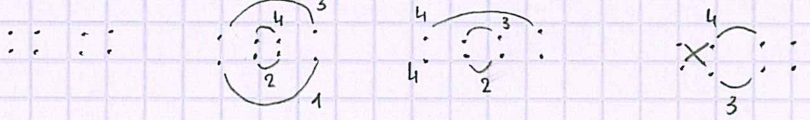
$C^{\varphi^k}_{\varphi^i \varphi^j}$ = # OF WAYS OF CONNECTING POINTS IN X TO POINTS IN Y SO THAT K POINTS REMAIN UNCONNECTED.

· EXAMPLES

$$\varphi^2 \cdot \varphi^2 \sim 2 I + 4 \varphi^2 + \varphi^4$$



$$\varphi^4 \cdot \varphi^4 \sim 4!\, I + 4 \cdot 4!\, \varphi^2 + 6 \cdot 4 \cdot 3\, \varphi^4 + \ldots = 24 I + 96 \varphi^2 + 72 \varphi^4 + \ldots$$



HENCE

$$X^*_{\varphi^2} = X^0_{\varphi^2} + 2\, C_{\varphi^2}\, \varepsilon + O(\varepsilon^2) \qquad\qquad C_{\varphi^2} = \frac{C^{\varphi^2}_{\varphi^2\varphi^4}}{C^{\varphi^4}_{\varphi^4\varphi^4}}.$$

BUT WE CAN ALSO RESHUFFLE INDICES AS

$$C_{\varphi^2} = \frac{C^{\varphi^4}_{\varphi^2\varphi^2}}{C^{I}_{\varphi^2\varphi^2}} \cdot \frac{C^{I}_{\varphi^4\varphi^4}}{C^{\varphi^4}_{\varphi^4\varphi^4}} = \frac{1\cdot 24}{2\cdot 72} = \frac{1}{6}$$

NOTE:
$$C_{\varphi^2\varphi^4\varphi^2} = C^{\varphi^2}_{\varphi^2\varphi^4}\, C^{I}_{\varphi^2\varphi^2}$$
$$\parallel$$
$$C_{\varphi^2\varphi^2\varphi^4} = C^{\varphi^4}_{\varphi^2\varphi^2}\, C^{I}_{\varphi^4\varphi^4}.$$

THUS

$$X^*_{\varphi^2} = d-2 + 2\cdot\frac{1}{6}\,\varepsilon + O(\varepsilon^2)$$

$$\nu_*^{-1} = d - X^*_{\varepsilon} = d - X^*_{\varphi^2} \simeq d - \left(d-2+2\frac{\varepsilon}{6}\right) = 2\left(1-\frac{\varepsilon}{6}\right)$$

$$\Rightarrow \quad \underline{\nu_* \simeq \frac{1}{2}\left(1+\frac{\varepsilon}{6}\right).}$$

WHAT HAPPENS IF WE EXTRAPOLATE THIS LINEARLY, UP TO $d=3$?

SETTING $\varepsilon = 1$,

$$\nu_*\big|_{\varepsilon=1} = \frac{1}{2}\left(1+\frac{1}{6}\right) = \frac{7}{12} \simeq 0.58\,. \qquad \left(\text{COMPARE WITH } \nu^{\text{MEAS}} \simeq 0.63,\ \nu^{\text{MF}} = \frac{1}{2}\right)$$

WE CAN ALSO EVALUATE, FOR THE SPIN FIELD φ,

$$\delta^* X_\varphi \propto C^{\varphi}_{\varphi\varphi^4} = 0 \qquad \Rightarrow \qquad \beta_* = X^*_\varphi\, \nu_* \simeq \frac{d-2}{2}\cdot\frac{1}{2}\left(1+\frac{\varepsilon}{6}\right)$$

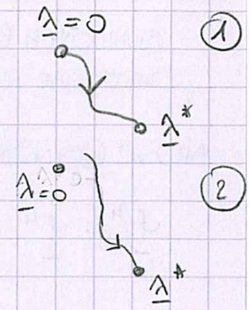
WHENCE

$$\beta_*\big|_{d=3} \simeq \frac{1}{2}\,\frac{7}{12} \simeq 0.29 \qquad \left(\text{COMPARE WITH } \beta^{\text{MEAS}} \simeq 0.32,\ \beta^{\text{MF}} = \frac{1}{2}\right).$$

SUMMING UP: PERTURBATIVE RG 05.11.19

$$A = A_0 + \sum_i \lambda_i \int d^d x \; \phi_i(x)$$

WITH ϕ_1 SLIGHTLY RELEVANT ($X_{\phi_1} = d - \varepsilon$, $\varepsilon \ll 1$). WE AREN'T SAYING ANYTHING ABOUT THE OTHER ϕ_i's; ACTUALLY, IF WE DRAW IT LIKE ON THE RIGHT, THEN IT'S LIKE WE ARE ASSUMING THEM TO BE ALL RELEVANT (OTHERWISE IT WOULD LOOK LIKE ②).



THERE IS A FP AT $\underline{\lambda} = \underline{\lambda}^* = O(\varepsilon)$, AND WE FOUND

$$X^*_\phi = X^0_\phi + 2 C_\phi \varepsilon + O(\varepsilon^2) \qquad C_\phi \equiv \left(\frac{C^{\phi}_{\phi\phi_1}}{C^{\phi_1}_{\phi_1\phi_1}} \right)^0$$

ONE APPLICATION ARE THE THEORIES IN $d = d_c - \varepsilon$ (WILSON-FISHER CASE). FOR ISING ($d_c = 4$),

PERTURBATIVE RG:	A	A_0	ϕ_1
ISING IN $(4-\varepsilon)$:	A_{LG}	A^{FP}_{GAUSS}	φ^4.

· O(m) MODEL

$$H = -J \sum_{\langle i,j \rangle} \underline{S}_i \cdot \underline{S}_j \qquad \underline{S}_i \equiv m\text{-COMPONENT UNIT VECTOR.}$$

IN THE CONTINUUM,

$$A_{LG} = \int d^d x \left\{ \frac{1}{2} (\nabla \underline{\varphi})^2 + \tau \underline{\varphi}^2 + \lambda \underbrace{\underline{\varphi}^4}_{\phi_i} \right\}$$

WHERE

$$\underline{\varphi}^2 \equiv \underline{\varphi} \cdot \underline{\varphi} \qquad \underline{\varphi}^4 \equiv (\underline{\varphi} \cdot \underline{\varphi})^2$$

AND $d_c = 4$. IN $d = 4 - \varepsilon$,

$$X^*_{\underline{\varphi}^2} = X^G_{\underline{\varphi}^2} + 2 C_{\underline{\varphi}^2} \varepsilon + O(\varepsilon^2)$$

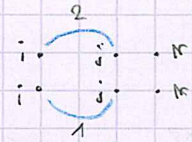
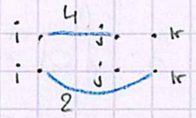
WHERE

$$C_{\underline{\varphi}^2} = \frac{C^{\underline{\varphi}^2}_{\underline{\varphi}^2 \underline{\varphi}^4}}{C^{\underline{\varphi}^4}_{\underline{\varphi}^4 \underline{\varphi}^4}}.$$

WE NEED TO COUNT THE POSSIBLE COMBINATIONS (*):

$\underline{\varphi}^2 \cdot \underline{\varphi}^4$

$\varphi_i \varphi_i \cdot \varphi_j \varphi_j \varphi_k \varphi_k$

LOOK FOR $\underline{\varphi}^2$ ⇝  + 

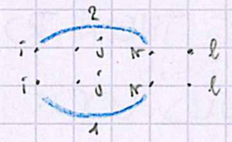
SUM OVER REPEATED INDICES (THESE ARE ALL ACTUALLY SUMS) →

$$\underbrace{2\delta_{ij}\delta_{ij}}_{=m}\varphi_k\varphi_k \cdot \underset{(j\leftrightarrow k)}{2} + 8\delta_{ij}\delta_{ik}\varphi_j\varphi_k = \underbrace{4(m+2)}_{C^{\underline{\varphi}^2}_{\underline{\varphi}^2\underline{\varphi}^4}}\underline{\varphi}^2 .$$

(SAME RIGHT INDICES) (DIFFERENT R.I.)

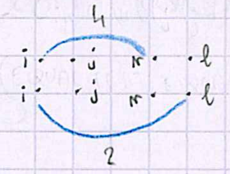
NOW CONSIDER

$\underline{\varphi}^4 \cdot \underline{\varphi}^4$

LOOK FOR $\underline{\varphi}^4$ ⇝ 

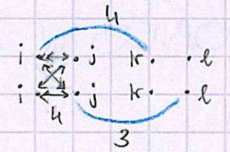
$$2\delta_{ik}\delta_{ik}\varphi_j\varphi_j\varphi_l\varphi_l \cdot \underset{(i\leftrightarrow j)}{2} \cdot \underset{(k\leftrightarrow l)}{2}$$

(EQUAL LEFT, EQUAL RIGHT)

\+ 

$$8\delta_{ik}\delta_{il}\varphi_j\varphi_j\varphi_l\varphi_k \cdot \underset{(i\leftrightarrow j)}{2}$$

(EQUAL LEFT, DIFFERENT RIGHT)

\+ 

$$4 \cdot 12\,\delta_{ik}\delta_{jk}\varphi_i\varphi_j\varphi_k\varphi_l$$

(DIFFERENT LEFT, ANY RIGHT)

$$= 8(m+8)\underline{\varphi}^4 \equiv C^{\underline{\varphi}^4}_{\underline{\varphi}^4\underline{\varphi}^4}\underline{\varphi}^4 .$$

FINALLY

$$C_{\underline{\varphi}^2} = \frac{4(m+2)}{8(m+8)} = \frac{m+2}{2(m+8)} .$$

AS A CHECK, WE GET $\frac{1}{6}$ FOR $m=1$, AS WE HAD ALREADY FOUND FOR ISING. THEN

$$X^*_{\underline{\varphi}^2} = d-2+\frac{m+2}{(m+8)}\varepsilon + O(\varepsilon^2)$$

WHENCE

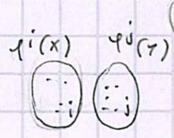
$$\nu_*^{-1} = d - X^*_{\underline{\varphi}^2} = 2 - \frac{m+2}{(m+8)}\varepsilon + O(\varepsilon^2).$$

MOREOVER,

$$\delta^* X_{\underline{\varphi}} \propto C^{\underline{\varphi}}_{\underline{\varphi}\underline{\varphi}^4} = 0$$

SO THAT

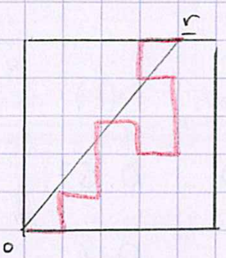
$$\beta_* = X^*_{\underline{\varphi}}\nu_* = \frac{d-2}{2}\left(2 - \frac{m+2}{m+8}\varepsilon\right)^{-1} + O(\varepsilon^2).$$

(*) NOTE: RECALL THAT, IN THE GAUSSIAN FP, WICK'S THEOREM TELLS US THAT 

$C^{\varphi^K}_{\varphi^i\varphi^j}$ = # OF WAYS OF CONNECTING POINTS IN x TO POINTS IN y SO THAT K REMAIN UNCONNECTED.

· MEAN SQUARE END-TO-END DISTANCE FOR N-STEP SAW's

$$\langle r^2 \rangle = \frac{1}{C_N} \sum_{\underline{r}} r^2 C_N(r)$$



$C_N(r) \equiv$ # OF N-STEP SAW's BETWEEN O AND $\underline{r}$

$C_N \equiv$ # OF N-STEP SAW's ORIGINATING FROM O.

WE ARE INTERESTED IN LARGE N ASYMPTOTICS. WE KNOW THAT

$$\sum_N C_N(r) X^N = \lim_{m\to 0} G(r)$$

NOTE: HAVE A LOOK BACK A
15.10.19 BEFORE YOU CARRY O

WHERE $G(r)$ IS THAT OF THE $O(m)$ MODEL,

$$G(r) = \langle S_1(\underline{r}) S_1(0) \rangle \qquad X \sim \frac{J}{T}.$$

HENCE

$$\sum_N \sum_{\underline{r}} r^2 C_N(r) X^N = \sum_{\underline{r}} r^2 G(r) \simeq \int d^d r\, r^2 G(r) \sim \xi^{2+d-2X_S}$$

(DIMENSIONAL ANALYSIS)

$$\sim (X - X_c)^{-\nu(2+d-2X_S)} \underset{\gamma = \nu(d-2X_\sigma)}{=} (X - X_c)^{-2\nu-\gamma}$$

($m \to 0$ UNDERSTOOD)

WE CAN EASILY CHECK THAT THIS AMOUNTS TO

$$\sum_{\underline{r}} r^2 C_N(r) \sim N^{a-1} X_c^{-N} \qquad \begin{cases} N \to \infty \\ a = 2\nu + \gamma. \end{cases} \qquad \text{(I)}$$

IN FACT

$$\sum_N \sum_{\underline{r}} r^2 C_N(r) X^N \overset{\text{(I)}}{\sim} \int dN\, N^{a-1} \left(\frac{X}{X_c}\right)^N = \int \frac{dN}{N} N^a e^{N \ln\left(\frac{X}{X_c}\right)}$$

$$\underset{Y = N \ln \frac{X}{X_c}}{\propto} \left(\ln \frac{X}{X_c}\right)^{-a} = \left[\ln\left(1 + \frac{X - X_c}{X_c}\right)\right]^{-a} \simeq \left(\frac{X - X_c}{X_c}\right)^{-a}.$$

γ, ν ARE $O(m)$ EXPS
EVALUATED AT $m = 0$.

WE ALREADY KNOW THAT

$$C_N \sim N^{\gamma-1} X_c^{-N} \qquad N \to \infty$$

HENCE

$$\langle r^2 \rangle \sim \frac{N^{2\nu+\gamma-1} X_c^{-N}}{N^{\gamma-1} X_c^{-N}} = N^{2\nu} \qquad N \to \infty.$$

EXPONENTS FOR THE O(m) MODEL IN $d=3$

LET'S COMPARE

	ν		γ		
m	$O(\varepsilon)$	MEASURED	$O(\varepsilon)$	MEASURED	NAME
0	0.57	0.587	1.14	1.15	SAW's
1	0.6	0.63	1.2	1.23	ISING
2	0.625	0.67	1.25	1.31	XY
3	0.65	0.7	1.3	1.4	HEISENBERG

IN FACT

$$\gamma_* = \left(d - 2X_s^*\right)\nu_* = \left(d - 2\frac{d-2}{2}\right)\nu_* + O(\varepsilon^2) = 2\nu_* + O(\varepsilon^2).$$

QUENCHED DISORDER

CONSIDER A MAGNET (e.g. ISING) WITH IMPURITIES (NON-MAGNETIC SITES, VACANCIES).

IMPURITIES ARE THE DISORDER. TWO CASES:

1. ANNEALED DISORDER: IMPURITIES ARE IN THERMAL EQUILIBRIUM WITH THE MAGNETIC DEGREES OF FREEDOM.

EXAMPLE:

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} t_i t_j \sigma_i \sigma_j + \Delta \sum_j t_j \qquad \sigma_i = \pm 1,\ t_i = \overset{\substack{\text{IMPURITY}\\ \downarrow}}{0}, 1$$

$$Z = \sum_{\{\sigma_i\},\{t_i\}} e^{-\mathcal{H}/T}$$

ANNEALED DISORDER IS, IN A NUTSHELL, SOMETHING WE ALREADY KNOW.

2. QUENCHED DISORDER: THE IMPURITIES REQUIRE A MUCH LONGER TIME THAN MAGNETIC DEGREES OF FREEDOM IN ORDER TO REACH THERMAL EQUILIBRIUM.

THEN IT MAKES SENSE TO CONSIDER EQUILIBRIUM ONLY FOR MAGNETIC DEGREES OF FREEDOM, CONSIDERING INSTEAD THE POSITIONS OF

THE IMPURITIES AS FIXED.

EXAMPLE:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} t_i t_j \sigma_i \sigma_j \qquad \sigma_i = \pm 1, \; t_i = 0,1$$

$$Z[\{t_i\}] = \sum_{\{\sigma_i\}} e^{-\mathcal{H}/T}.$$

IMPURITIES WILL BE RANDOMLY DISTRIBUTED. WE THEN AVERAGE THERMODYNAMIC OBSERVABLES (MAGNETIZATION, SPECIFIC HEAT, SUSCEPTIBILITY) OVER DISORDER ACCORDING TO A PROBABILITY DISTRIBUTION $P(\{t_i\})$.

THESE OBSERVABLES ARE DERIVATIVES OF

$$f = -\frac{1}{V} \ln Z.$$

IT IS SUFFICIENT TO AVERAGE f:

$$\bar{f} \equiv \sum_{\{t_i\}} P(\{t_i\}) \, f(\{t_i\}).$$

· RECAP: QUENCHED DISORDER 07.11.19

$$\begin{cases} \mathcal{H} = -J \sum_{\langle i,j \rangle} t_i t_j \sigma_i \sigma_j \\ Z[\{t_i\}] = \sum_{\{\sigma_i\}} e^{-\mathcal{H}/T} \end{cases} \qquad \sigma_i = \pm 1, \; t_i = 0,1$$

$$\Rightarrow \quad \overline{f} = \sum_{\{t_i\}} P(\{t_i\}) f(\{t_i\}).$$

· REPLICA METHOD

WE KNOW THAT $f = -\frac{T}{V} \ln Z$, BUT

$$\ln Z = \lim_{m \to 0} \frac{Z^m - 1}{m}.$$

CONSIDERING $\overline{f}$ AMOUNTS TO DEALING WITH $\overline{Z^m}$, $m \to 0$.

CALLING a THE REPLICA INDEX, $\sigma_{a,i}$ THE SPIN AT SITE i IN REPLICA a,

$$[Z(\{t_i\})]^m = \prod_{a=1}^{m} \left[\sum_{\{\sigma_{a,i}\}} e^{-\mathcal{H}[\{\sigma_{a,i}\},\{t_i\}]} \right]$$

$$= \sum_{\{\sigma_{1,i}\}\dots\{\sigma_{m,i}\}} e^{-\sum_{a=1}^{m} \mathcal{H}[\{\sigma_{a,i}\},\{t_i\}]}$$

SO THAT

$$\overline{Z^m} = \sum_{\{t_i\}} P(\{t_i\}) \sum_{\{\sigma_{1,i}\}\dots\{\sigma_{m,i}\}} e^{-\sum_{a=1}^{m} \mathcal{H}[\{\sigma_{a,i}\},\{t_i\}]}$$

WHICH CONTAINS m REPLICAS COUPLED BY DISORDER AVERAGE (IN FACT, $\{t_i\}$ IS COMMON TO ALL REPLICAS). DISORDER BECOMES THE SOURCE OF AN EFFECTIVE INTERACTION AMONG m REPLICAS, WITH $m \to 0$.

WE TREAT THE t_i'S AS INDEPENDENT RANDOM VARIABLES, TAKING FOR EXAMPLE

$$P(\{t_i\}) = \prod_i \left[p\, \delta_{t_i,1} + (1-p)\, \delta_{t_i,0} \right], \qquad (1-p) = \text{CONCENTRATION OF IMPURITIES}$$

THIS IS A RANDOMLY DILUTED MAGNET.

WE CAN ALSO CONSIDER BOND DISORDER:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j \qquad\qquad J_{ij} \text{ RANDOM VARIABLES.}$$

· <u>TYPICAL EXAMPLES</u>

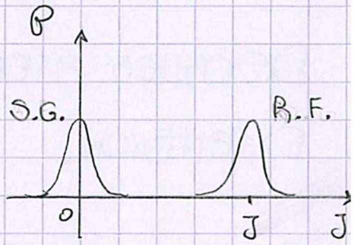
1) BIMODAL

$$P(\{J_{ij}\}) = \prod_{\langle i,j \rangle} \left[p\,\delta_{J_{ij},J_1} + (1-p)\,\delta_{J_{ij},J_2} \right]$$

$$\begin{cases} J_1, J_2 > 0 : \text{ RANDOM FERROMAGNET} \\ J_1 > 0,\ J_2 < 0 : \text{ SPIN GLASS.} \end{cases}$$

2) GAUSSIAN

$$P(\{J_{ij}\}) = \prod_{\langle i,j \rangle} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(J_{ij}-J)^2}.$$



★ EXPERIMENTAL AND NUMERICAL STUDIES SHOW THE EXISTENCE OF DISORDERED (OR RANDOM) CRITICAL POINTS WITH EXPONENTS DIFFERING FROM THOSE OF THE PURE CASES; THESE EXPONENTS SATISFY THE USUAL SCALING RELATIONS. THE EXPONENTS DEPEND ON DISORDER, BUT APPEAR TO FALL INTO UNIVERSALITY CLASSES OF <u>RANDOM CRITICAL BEHAVIOR</u>.
IN PARTICULAR, A WEAKLY DISORDERED FERROMAGNET SHOWS THE SAME EXPONENTS FOR DIFFERENT DISORDER REALIZATIONS:

- WEAKLY RANDOMLY DILUTED FERROMAGNET (VACANCY CONCENTRATION $1-p \ll$
- FERROMAGNET WITH WEAK BOND RANDOMNESS ($J_1 > 0$, ANY J_2, AND $1-p \ll 1$, OR INSTEAD GAUSSIAN WITH $J > 0$, $\sigma \ll J$).

FOR WEAKLY DISORDERED FERROMAGNETS, IT MAKES SENSE TO LOOK FOR A FIELD THEORY DESCRIPTION THAT STARTS FROM PURE SYSTEMS.

WE WRITE AN ACTION AS

$$A = A_0 + \int d^d x \, m(x) \, \mathcal{E}(x)$$

WHERE

A_0 = FP OF THE PURE FERROMAGNET

$\mathcal{E}(x)$ = ENERGY DENSITY FIELD OF THE PURE FERROMAGNET

$m(x)$ = RANDOM (SITE OR BOND) VARIABLE

(NOTICE $m(x)$ IS NOT A FIELD; IT'S ACTUALLY A COUPLING). THEN

$$\overline{Z^m} = \sum_{\{m(x)\}} P(\{m(x)\}) \int \mathcal{D}\phi \, e^{-\sum_{a=1}^{m} \left[A_a^0 + \int d^d x \, m(x) \, \mathcal{E}_a(x) \right]}$$

NOTE: $\mathcal{D}\phi \equiv$ ALL THE FIELDS $\phi_i^{(a)}$.

DISORDER AVERAGE CAN BE PERFORMED WITHIN A CUMULANT EXPANSION.

RANDOM VARIABLES AND CUMULANT EXPANSION

LET y BE A RANDOM VARIABLE WITH PROBABILITY DISTRIBUTION $P(y)$ s.t. $\int dy \, P(y) = 1$

$$\overline{f} \equiv \int dy \, P(y) \, f(y).$$

CONSIDER

$$G(k) \equiv \overline{e^{ky}} = \int dy \, P(y) \, e^{ky} = \sum_{m=0}^{\infty} \frac{k^m}{m!} \overline{y^m}.$$

THIS CAN BE REWRITTEN AS

$$G(k) = e^{\sum_{j=1}^{\infty} \frac{k^j}{j!} C_j} \qquad C_j \equiv \text{CUMULANTS}$$

$$C_1 = \overline{y}$$

$$C_2 = \overline{y^2} - (\overline{y})^2$$

$$C_3 = \overline{y^3} - 3\,\overline{y^2}\,\overline{y} + 2(\overline{y})^3$$

AND SO ON.

FOR A GAUSSIAN DISTRIBUTION,

$$P(Y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(Y-\bar{Y})^2}$$

AND

$$G(K) = e^{\bar{Y}K + \frac{1}{2}\sigma^2 K^2} \qquad \begin{cases} C_2 = \sigma^2 \\ C_{j>2} = 0 \end{cases}.$$

* IN OUR CONTEXT,

$$Y \rightsquigarrow m(x)$$

$$\overline{e^{KY}} \rightsquigarrow \sum_{\{m(x)\}} P(\{m(x)\}) \int \mathcal{D}\phi\, e^{\cdots m(x) \cdots}$$

$$K \rightsquigarrow -\sum_{a=1}^{m} \varepsilon_a(x)$$

NOTE: ACTUALLY $-\int d^dx \sum_{a=1}^{m} \varepsilon_a(x)$.

HENCE

$$\overline{Z^m} = \int \mathcal{D}\phi\, e^{-\sum_a [A_a^0 + \bar{m}\int d^dx\, \varepsilon_a(x)] + \frac{1}{2!}\int d^dx_1\, d^dx_2 \left(\overline{m(x_1)m(x_2)} - \bar{m}^2\right)\sum_{a,b} \varepsilon_a(x_1)\varepsilon_b(x_2)}$$

BUT NOTICE, SINCE $m(x_1)$ AND $m(x_2)$ ARE INDEPENDENT VARIABLES,

$$\overline{m(x_1)m(x_2)} - \bar{m}^2 = C_2\, \delta(x_1 - x_2)$$

THUS

$$\overline{Z^m} = \int \mathcal{D}\phi\, e^{-\sum_a [A_a^0 + \bar{m}\int d^dx\, \varepsilon_a(x)] + \frac{C_2}{2}\sum_{a,b}\int d^dx\, \varepsilon_a(x)\varepsilon_b(x) + \cdots}.$$

NOTICE THE FIRST CUMULANT IN

$$\bar{m}\int d^dx\, \varepsilon_a(x)$$

DOES NOT COUPLE THE REPLICAS AND CANNOT PRODUCE NEW CRITICAL BEHAVIOR. THE SAME IS TRUE FOR THE DIAGONAL PART $\varepsilon_a \cdot \varepsilon_a \sim I + \varepsilon_a \ldots$
INSTEAD $\varepsilon_a \cdot \varepsilon_b$, WITH $a \neq b$, HAS SCALING DIMENSION $X^0_{\varepsilon_a\varepsilon_b}$ AT THE PURE FP, WITH

$$A^0 = \sum_{a=1}^{m} A_a^0 \quad \Rightarrow \quad \underline{X^0_{\varepsilon_a\varepsilon_b} = 2X^0_\varepsilon}.$$

THIS MEANS THE EFFECT OF WEAK DISORDER IS IRRELEVANT IN THE RG SENSE IF

$$\underline{2X^0_\varepsilon > d} \qquad \text{HARRIS CRITERION.}$$

PURE

· QUENCHED DISORDER - RELOADED 12.11.19

LAST TIME WE FOUND

$$\overline{Z^m} = \int \mathcal{D}\phi \, e^{-\sum_{a=1}^{m}\left[A_a^0 + \bar{m}\int d^dx\, \varepsilon_a(x)\right] + \frac{C_2}{2}\sum_{a,b}\int d^dx\, \varepsilon_a(x)\varepsilon_b(x) + \dots}$$

AND WE CONCLUDED THAT WEAK DISORDER IS IRRELEVANT IN RG SENSE (AND THEN IT DOES NOT PRODUCE NEW CRITICAL EXPONENTS) IF

$$X^0_{\varepsilon_a,\varepsilon_b \atop a\neq b} = 2X^0_\varepsilon > d \qquad \text{(HARRIS CRITERION).}$$

SINCE

$$\nu = \frac{1}{d - X_\varepsilon} \quad \Rightarrow \quad \nu_0 d > 2$$

$$\alpha = (d - 2X_\varepsilon)\nu \quad \Rightarrow \quad \alpha_0 < 0$$

WHICH ARE ALL EQUIVALENT FORMS OF THE CRITERION.

* QUITE GENERALLY, WE CAN EXPRESS THE EFFECTIVE ACTION IN THE PRESENCE OF QUENCHED DISORDER AS

$$A = \sum_{a=1}^{m} A_a^0 + g\sum_{a=1}^{m}\int d^dx\, E_a(x) - \frac{C_2}{2}\int d^dx \sum_{a\neq b} E_a(x)E_b(x).$$

WE CHANGED THE NOTATION TO

$\varepsilon(x) \to E(x)$

TO AVOID CONFUSION WITH $\varepsilon \ll 1$.

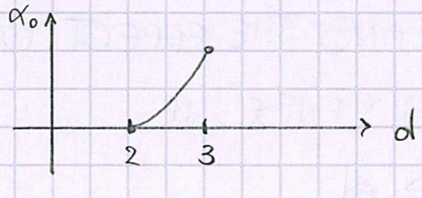
· PERTURBATIVE APPROACH TO A RANDOM FP

FOR ISING,

$d = 2 \quad \Rightarrow \quad \alpha_0 = 0$ → WEAK RANDOMNESS IS MARGINAL

$d = 3 \quad \Rightarrow \quad \alpha_0 > 0$ → W.R. IS RELEVANT: THERE IS A RANDOM FP.

THE SIMPLE EXPECTATION IS A MONOTONIC INTERPOLATION FROM $d=2$ TO $d=3$.

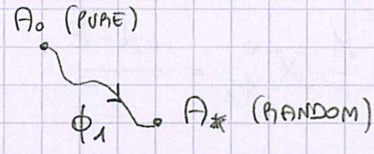


THUS, THERE IS A PERTURBATIVE RANDOM FP FOR $d \to 2^+$, i.e.

$$2X_E^\circ = d - \varepsilon, \quad \varepsilon << 1 \qquad \text{FOR } d \to 2^+.$$

THE FIELD WHICH IS SLIGHTLY RELEVANT IN THE ACTION (I) IS

$$\phi_1 = \sum_{a \neq b} E_a(x) E_b(x)$$



SO THAT IN FACT

$$2X_E^\circ = d - \varepsilon \equiv X_{\phi_1}^\circ$$

AND WE CAN APPLY THE FORMULA WE DERIVED

$$X_\phi^* = X_\phi^\circ + 2C_\phi \varepsilon + O(\varepsilon^2) \qquad C_\phi = \left(\frac{C_{\phi\phi_1}^{\phi}}{C_{\phi_1\phi_1}^{\phi_1}} \right).$$

WE PERTURB AROUND THE TRUE (NON-TRIVIAL) ISING FP IN $d \to 2^+$.

· OPE'S AT PURE FP

$$E_a \cdot E_b \sim \delta_{ab} \left(C_{E_a E_a}^{I} \right)^\circ I + \delta_{ab} \left(C_{E_a E_a}^{E_a} \right)^\circ E_a + \ldots$$

NOTE: WE ARE EXPANDING IN A GENERIC BASI[S] WITH THE SAME SYMMETRY OF $E_a \cdot E_b$. IF $a \neq b$, I AND E_a ARE ABSENT, BECAUSE THEY CANNOT CARRY 2 INDICES.

WE KNOW THAT

$$\left(C_{E_a E_a}^{E_a} \right)^\circ = O(\varepsilon)$$

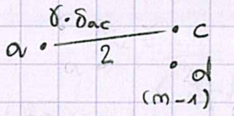
BECAUSE IT VANISHES IN $d = 2$ BY DUALITY (WE'LL COME BACK TO IT)
WE CAN SET IT TO ZERO IN $O(\varepsilon)$ CALCULATIONS. WE CALL FOR BREVITY

$$\gamma \equiv \left(C_{E_a E_a}^{I} \right)^\circ.$$

NOTE: DOWN HERE I LOOK FOR A TERM $\sum_\alpha' E_\alpha$. I'M INTERESTED ONLY IN THE COEFFICIENT IN FRONT OF TH[E] IDENTITY WHICH COMES FROM THE EXPANSION OF THE OTHER 2 SUMS. I'M NOT USING WICK (IT'S NOT GAUSSI[AN])

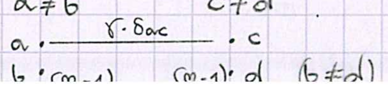
NOW CONSIDER

$$\underbrace{\sum_a' E_a}_{\equiv E} \cdot \phi_1 = \sum_a' E_a \cdot \underbrace{\sum_{c \neq d}' E_c E_d}_{m(m-1) \text{ TERMS}} \xrightarrow[\text{FOR } E]{\text{LOOK}} 2(m-1)\gamma \underbrace{\sum_a' E_a}_{E} \equiv \left(C_{E\phi_1}^{E} \right)^\circ \cdot E$$



AND NOW LET'S LOOK FOR THE DENOMINATOR OF C_E:

$$\phi_1 \cdot \phi_1 = \overbrace{\sum_{a \neq b}' E_a E_b}^{m(m-1)} \cdot \overbrace{\sum_{c \neq d}' E_c E_d}^{m(m-1)} \xrightarrow[\text{FOR } \phi_1]{\text{LOOK}} 4(m-2)\gamma \overbrace{\sum_{a \neq b}' E_a E_b}^{m(m-1)} \equiv \left(C_{\phi_1\phi_1}^{\phi_1} \right)^\circ \phi_1.$$



WE FOUND

$$\begin{cases} \left(C^{E}_{E\phi_1}\right)^{\circ} = 2(m-1)\gamma \\ \left(C^{\phi_1}_{\phi_1\phi_1}\right)^{\circ} = 4(m-2)\gamma \end{cases}$$

WHENCE

$$X^{\circ}_{E} = \frac{1}{2} X^{\circ}_{\phi_1} = \frac{d-\varepsilon}{2}$$

$$\delta^{*} X_{E} = 2\left(\frac{C^{E}_{E\phi_1}}{C^{\phi_1}_{\phi_1\phi_1}}\right)^{\circ} \varepsilon + O(\varepsilon^2) = 2\left.\frac{2(m-1)\gamma}{4(m-2)\gamma}\right|_{m=0} \cdot \varepsilon + O(\varepsilon^2)$$

$$= \frac{\varepsilon}{2} + O(\varepsilon^2)$$

NOTE: "And you see this is magic".

WHERE WE SET THE NUMBER m OF REPLICAS EQUAL TO ZERO.

THIS GIVES

$$X^{*}_{E} = \frac{d}{2} + O(\varepsilon^2) \quad \Rightarrow \quad \nu_{*} = \frac{1}{d - X^{*}_{E}} = \frac{2}{d} + O(\varepsilon^2).$$

MOREOVER,

$$\delta^{*}_{X_\sigma} \propto \left(C^{\sigma_a}_{\sigma_a\phi_1}\right)^{\circ} = 0 \quad \text{AT } O(\varepsilon), \text{ i.e. } \delta^{*} X_{\sigma} = O(\varepsilon^2).$$

NOTE: YOU NEVER FIND A TERM WITH σ_a ALONE IN $\sigma_a \cdot \phi_1$.

IN FACT,

$$\sigma_a \cdot E_b \sim \begin{cases} \sigma_a + \dots & a = b \\ \sigma_a E_b + \dots & a \neq b \end{cases}$$

$$\sigma_a \cdot \phi_1 = \sigma_a \cdot \sum_{b \neq c}' E_b E_c \sim \sigma_a \sum_{b \neq a}' E_b + \sigma_a \sum_{b \neq c \neq a}' E_b E_c .$$

WE CAN ALSO COMPUTE THE RATIOS

$$\frac{\beta_{*}}{\beta_0} = \frac{X^{*}_{a} \nu_{*}}{X^{\circ}_{a} \nu_0} = \frac{\nu_{*}}{\nu_0} + O(\varepsilon^2)$$

$$= \nu_{*}(d - X^{\circ}_{E}) + O(\varepsilon^2) = \frac{2}{d}\left(d - \frac{d-\varepsilon}{2}\right) + O(\varepsilon^2) = 1 + \frac{\varepsilon}{d} + O(\varepsilon^2)$$

$$\frac{\gamma_{*}}{\gamma_0} = \frac{(d - 2X^{*}_{E})\nu_{*}}{(d - 2X^{\circ}_{E})\nu_0} = \frac{\nu_{*}}{\nu_0} + O(\varepsilon^2) = 1 + \frac{\varepsilon}{d} + O(\varepsilon^2).$$

THEN

$$\alpha_0 = \frac{d - 2X_E^\circ}{d - X_E^\circ} = \frac{2}{d}\varepsilon + O(\varepsilon^2) \qquad \alpha_0 \sim \varepsilon$$

SO THAT α_0 COULD BE USED AS THE EXPANSION PARAMETER.
THIS IS USEFUL IF WE LOOK AT THE MEASURED EXPS IN $d=3$:

	ν	β	α	γ
PURE	0.63	0.32	0.11	1.23
RANDOM	0.68	0.35	-0.05	1.33

WHERE WE SEE THAT $\alpha_0^{(d=3)} = 0.11$ IS STILL SMALL (WE ONLY KNEW IT WAS SMALL CLOSE TO $d=2$).

α_0 SMALL IN $d=3$ MAKES $O(\varepsilon)$ RESULTS RELIABLE IN $d=3$.

FOR INSTANCE, WE FIND

$$\nu_*\big|_{d=3} \simeq \frac{2}{3} \simeq 0.66$$

UP TO $O(\alpha_0^2) \simeq 10^{-2}$ CORRECTION. SIMILARLY,

$$\alpha_* = \frac{d - 2X_E^*}{d - E_E^*} = O(\varepsilon^2) = O(\alpha_0^2)$$

WHICH INDEED IS TRUE.

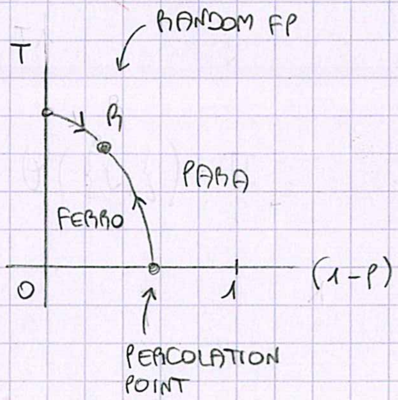
WE ALSO FIND

$$\frac{\beta_*}{\beta_0} \simeq \frac{\nu_*}{\nu_0} \simeq \frac{\gamma_*}{\gamma_0} \simeq 1 + \frac{\alpha_0}{2} \simeq 1.055$$

WHICH COMPARES WELL WITH MEASURED RATIOS $1.07 \div 1.08$.

· OBSERVED PHASE DIAGRAMS: ISING

* $d=3$, RANDOM DILUTION



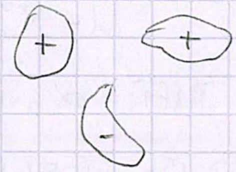
$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j t_i t_j$$

$$P(\{t_i\}) = \prod_i \left[p\,\delta_{t_i,1} + (1-p)\,\delta_{t_i,0} \right]$$

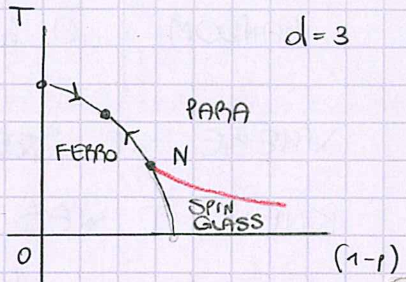
NOTE: $(1-p)$ IS THE FRACTION OF HOLES.

WHAT HAPPENS AT LOW T IN THE PRESENCE OF VACANCIES?

WE GET ISLANDS (CLUSTERS) OF PERFECTLY ALIGNED SPINS. THE TOTAL M AVERAGES OUT TO ZERO, UNLESS $(1-p)$ IS SMALL ENOUGH SO THAT WE CAN HAVE PERCOLATION.



★ EDWARDS-ANDERSON (OR $\pm J$) MODEL



$$H = -\sum_{\langle i,j \rangle} J_{ij}\,\sigma_i \sigma_j$$

$$P(\{J_{ij}\}) = \prod_{\langle i,j \rangle} \left[p\,\delta(J_{ij}-J) + (1-p)\,\delta(J_{ij}+J) \right]$$

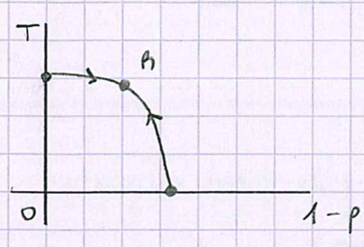
A NEW, NON-PERTURBATIVE FP APPEARS (N, NISHIMORI POINT) ABOUT WHICH WE ONLY HAVE NUMERICAL RESULTS. IT IS STILL AN OPEN QUESTION IF THERE EXISTS ANOTHER LINE (RED); THERE IS NUMERICAL EVIDENCE OF THE EXISTENCE OF A SPIN GLASS PHASE. THERE EXISTS AN ANALYTICAL TREATMENT OF A VARIANT OF THE MODEL CALLED SHERRINGTON-KIRKPATRICK,

$$H = -\sum_{i,j} J_{ij}\,\sigma_i \sigma_j .$$

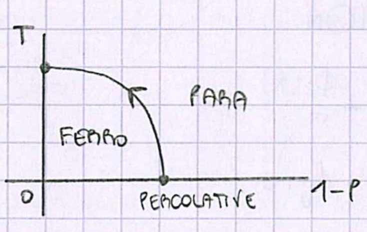
RECAP: PHASE DIAGRAMS FOR QUENCHED DISORDER 13.11.19

RANDOM SITE DILUTION

$d=3$

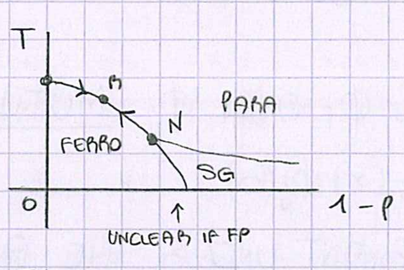


$d=2$

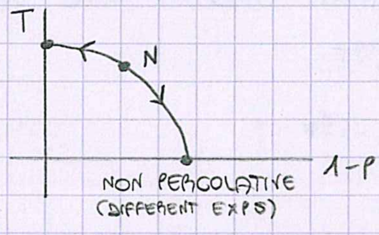


±J BOND RANDOMNESS

$d=3$



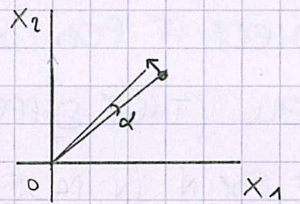
$d=2$



IN $d=2$, $\alpha_0 = 0$ AND SO A MORE REFINED ANALYSIS HAS BEEN REQUIRED TO ESTABLISH THAT DISORDER IS MARGINALLY IRRELEVANT.

· A CLOSER LOOK TO $d=2$

$$\begin{cases} z = x_1 + i x_2 \rightarrow e^{i\alpha} z \\ \bar{z} = x_1 - i x_2 \rightarrow e^{-i\alpha} \bar{z} \end{cases}$$



UNDER A ROTATION BY α.

A FIELD $\phi(x)$ IS CHARACTERIZED BY

$$\begin{cases} X_\phi = \text{SCALING DIMENSION} \\ S_\phi = \text{EUCLIDEAN SPIN} \end{cases}$$

$\phi \rightarrow e^{-i S_\phi \alpha} \phi$ UNDER ROTATION BY α

FOR INSTANCE, FOR SCALAR FIELDS

$S\phi_{SCALAR} = 0$.

NOTE: I THINK THIS COMES FROM
$\varphi'_r(x') = S_{rs}(\Lambda)\varphi_s(x)$ (CLASSICAL)
AND NOT FROM (QUANTUM)
$S^{-1}_{rs}(\Lambda)\hat{\varphi}_s(\Lambda x + a) = U(\Lambda,a)\hat{\varphi}_r(x)U^+(\Lambda,a)$.

OPE AT FP'S THEN READS

$$\phi_i(x)\phi_j(0) = \sum_k C^k_{ij} \cdot (z\bar{z})^{\frac{1}{2}(X_k - X_i - X_j)} \cdot \left(\frac{z}{\bar{z}}\right)^{\frac{1}{2}(S_k - S_i - S_j)} \cdot \phi_k(0).$$

IN FACT, $z\bar{z}$ IS THE SQUARE OF THE DISTANCE; MOREOVER, THE SPIN HAS TO BE MATCHED AS WELL. THIS CAN BE REWRITTEN AS

$$\phi_i(x)\phi_j(0) = \sum_k C^k_{ij}\, z^{(\Delta_k - \Delta_i - \Delta_j)}\, \bar{z}^{(\bar{\Delta}_k - \bar{\Delta}_i - \bar{\Delta}_j)}\, \phi_k(0)$$

WHERE WE DEFINED

Δ = RIGHT DIMENSION

$\bar{\Delta}$ = LEFT DIMENSION

$$\begin{cases} X_\phi = \Delta_\phi + \bar{\Delta}_\phi \\ S_\phi = \Delta_\phi - \bar{\Delta}_\phi . \end{cases}$$

WE CALL ϕ_i AND ϕ_j <u>MUTUALLY LOCAL</u> IF ANY

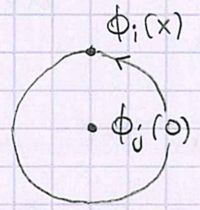
$\langle \dots \phi_i(x)\phi_j(0) \dots \rangle$

NOTE: IF $\langle\ \rangle$ IS "SINGLE VALUED".

IS INVARIANT UNDER THE ANALYTIC CONTINUATION

$z \to z e^{i2\pi}$

$\bar{z} \to \bar{z} e^{-i2\pi}$.



FROM THE OPE EXPANSION, WE KNOW
THIS HAPPENS IF

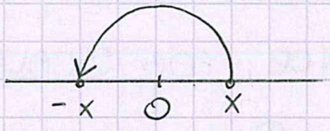
$S_k - S_i - S_j \in \mathbb{Z}$ $\forall k$ IN THE RHS OF OPE .

IN PRACTICE, CHECK IT FOR ONE FIELD ON THE R.H.S. AND IT WILL BE TRUE FOR ALL (THEY DIFFER FROM ONE ANOTHER BY AN INTEGER VALUE OF THE SPIN IN MOST PHYSICALLY INTERESTING CASES).

* CONSIDER

$$\langle \phi(x)\phi(0) \rangle \xrightarrow{\pi \text{ ROTATION}} e^{-2\pi i S_\phi} \langle \phi(-x)\phi(0) \rangle = e^{-2\pi i S_\phi} \langle \phi(0)\phi(x) \rangle .$$

WE SAY THAT ϕ IS



- BOSONIC, IF S_ϕ IS INTEGER
- FERMIONIC, IF S_ϕ IS HALF-INTEGER
- PARAFERMIONIC, OTHERWISE.

NOTE: $T_{\mu\nu}$ IS A SYMMETRIC 2×2 MATRIX, SO IT CONTAINS 3 INDEPENDENT QUANTITIES. HERE WE TAKE $T, \bar{T}, \theta$ AND WE EXPRESS $\partial_z = \partial_1 - i\partial_2$, $\partial_{\bar{z}} = \partial_1 + i\partial_2$.

· C-THEOREM

WE HAVE SEEN THE STRESS-ENERGY TENSOR SATISFIES

$$\partial_\mu T_{\mu\nu} = 0 \quad \to \quad \begin{cases} \bar{\partial} T + \frac{1}{4}\partial\theta = 0 \\ \partial \bar{T} + \frac{1}{4}\bar{\partial}\theta = 0 \end{cases}$$

WHERE

$\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$

$\theta = T_{11} + T_{22}$

$T = \frac{1}{4}(T_{11} - T_{22} + 2i\,T_{12})$

$\bar{T} = \frac{1}{4}(T_{11} - T_{22} - 2i\,T_{12})$.

COMPARE

	z	$\bar z$	∂	$\bar\partial$	θ	T	$\bar T$
Δ	-1	0	1	0	1	2	0
$\bar\Delta$	0	-1	0	1	1	0	2

NOTE: e.g.
$S_\theta = 0$ (SCALAR)
$X_\theta = d = 2$
THE LAST 2 $(T, \bar T)$ COME FROM THE EQUATION OF CONSERVATION. USE
$\Delta_i; \bar\Delta_i = (X_i \pm S_i)\cdot\frac{1}{2}$
ALSO RECALL
$\delta A \propto \int d^d x\, T_{\mu\nu}(x)\, \partial_\mu \delta x^\nu$
$X_A = 0$
$X_T = d, \; X_\theta = X_T$.

FROM THIS TABLE, WE CAN DEDUCE

$$\langle T(x) T(0)\rangle = \frac{F(z\bar z)}{z^4}$$

NOTE: IN OTHER WORDS, $\Delta_F = \bar\Delta_F = 0$.

WITH F ROTATIONALLY INVARIANT AND DIMENSIONLESS (IT WOULD BE A CONSTANT AT THE FP). SIMILARLY,

$$\langle T(x)\theta(0)\rangle = \frac{G(z\bar z)}{z^3 \bar z}$$

NOTE: THERE IS NO NEED TO MATCH X_ϕ AND S_ϕ SEPARATELY, JUST MATC[H] $\Delta\phi$.

$$\langle \theta(x)\theta(0)\rangle = \frac{H(z\bar z)}{z^2 \bar z^2}$$

(G, H WITH THE SAME PROPERTIES AS F).

NOW CONSIDER THE QUANTITY

$$0 = \langle \underbrace{\left(\bar\partial T(x) + \frac{1}{4}\partial\theta(x)\right)}_{=0} T(0)\rangle = \bar\partial \frac{F}{z^4} + \frac{1}{4}\partial\frac{G}{z^3\bar z} = \frac{1}{z^4 \bar z}\left\{\dot F + \frac{1}{4}(\dot G - 3G)\right\}$$

WHERE WE DEFINED

$$\dot f \equiv z\bar z \frac{\partial f}{\partial(z\bar z)}.$$

NOTE: I THINK IT'S ONLY A WAY OF MAKING THE DERIVATIVE ADIMENSIONAL

SIMILARLY,

$$0 = \langle\left(\bar\partial T(x) + \frac{1}{4}\partial\theta(x)\right)\theta(0)\rangle = \bar\partial\frac{G}{z^3\bar z} + \frac{1}{4}\partial\frac{H}{z^2\bar z^2} = \frac{1}{z^3\bar z^2}\left\{\dot G - G + \frac{1}{4}(\dot H - 2H\right.$$

THIS GIVES US A SYSTEM OF DIFFERENTIAL EQUATIONS. ELIMINATING G AND DEFINING

$$C \equiv 2F - G - \frac{3}{8}H$$

WE GET

$$\underline{\dot{C} = -\frac{3}{4} H}.$$

NOTE: I THINK REFLECTION POSITIVITY JUST MEANS $\langle \phi(x)\phi(0)\rangle > 0$, THE OTHER IS A CONSEQUENCE.

NOTE: THIS IS OFTEN CALLED UNITARITY IN LITERATURE.

* WE CALL <u>REFLECTION POSITIVITY</u> THE PROPERTY s.t.

$$\langle \phi(x)\phi(0)\rangle > 0 \qquad \Rightarrow \qquad H \geq 0$$

($H = 0$ AT THE FP, WHERE THE TRACE VANISHES).

THEN WE HAVE TWO CASES:

1. $\theta \neq 0$ (NON SCALE-INVARIANT THEORIES)

$$\dot{C} < 0.$$

THEN C MONOTONICALLY DECREASES ALONG RG FLOW.

INDEED, $\dot{}$ DENOTES A DERIVATIVE WRT THE DISTANCE $z\bar{z}$.

2. $\theta = 0$ (FP)

$$\dot{C} = 0.$$

THIS MEANS THAT C IS STATIONARY AT FP'S AND TAKES THE VALUE

$$C = 2F = 2C^{I}_{TT} \equiv c \qquad \underline{\text{CENTRAL CHARGE}}$$

($G = H = 0$ AT THE FP, BECAUSE $\theta = 0$).

IN FACT

$$T(x)T(0) = \frac{c}{2z^4} + \dots$$

NOTE: THIS (THE OPE) IS VALID NEARBY A F.P.. ON THE FP, $\langle \phi_k(0)\rangle = \delta_{\phi_k, I}$ AND $\langle T(x)T(0)\rangle = \frac{c}{z^4}$.

REFLECTION POSITIVITY IMPLIES

$$c > 0.$$

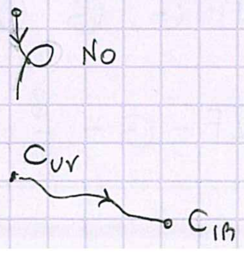
* SUMMARIZING, <u>C-THEOREM</u> (Zamolodchikov, 1986):

IN $d = 2$ THERE IS C THAT DECREASES MONOTONICALLY ALONG RG FLOWS AND IS STATIONARY AT FP'S.

THIS IMPLIES:

i) IRREVERSIBILITY OF RG FLOWS

ii) HIERARCHICAL ORDERING OF FP'S.



CROSSOVER: $C_{UV} > C_{IR}$

FOR INSTANCE,

$$C_{GAUSS} > C_{ISING}$$

$$C_{\substack{TRICRITICAL \\ ISING}} > C_{ISING} .$$

* AT A FP,

$$\Theta = 0 \qquad \Rightarrow \qquad \bar{\partial} T = \partial \bar{T} = 0 .$$

THEN

$$T = T(z) , \quad \bar{T} = \bar{T}(\bar{z}) .$$

NOTE: FOR THE SAKE OF THE ARGUMENT, IT'S ENOUGH TO REQUIRE THAT $T \cdot \phi$ PRODUCES ϕ IN ITS OPE (NOT THAT IT DOESN'T START WITH I).

NOTICE

$$T_{\mu\nu} \cdot \phi \rightsquigarrow \phi \qquad T \cdot \phi \sim \phi$$

HENCE WE CAN CHECK THAT

NOTE: WE ARE USING THE DEF. OF MUTUAL LOCAL

$$S_T + \cancel{S_\phi} - \cancel{S_\phi} \in \mathbb{Z} \qquad (S_T = 2)$$

i.e. ALL FIELDS ARE LOCAL WITH RESPECT TO T (AND $\bar{T}$)

THEN

$$T(z) \phi(0) = \sum_{m=-\infty}^{+\infty} z^{-2-m} L_m \phi(0)$$

WHERE $(-2-m)$ IS CHOSEN FOR CONVENIENCE, AND $L_m \phi$ IS A NOTATION FOR "THE FIELDS WE PRODUCE". BY DIMENSIONAL ANALYSIS, $L_m \phi$ HAS DIMENSIONS

$$(\Delta, \bar{\Delta}) = (\Delta_\phi - m, \bar{\Delta}_\phi) .$$

NOTE: IN FACT, Δ DECREASES IF WE APPLY L_m

A FIELD φ SUCH THAT $L_m \varphi = 0$ FOR $m > 0$ HAS LOWEST DIMENSION (WITHIN THE FAMILY $L_m \varphi$) AND IS CALLED A <u>PRIMARY</u> FIELD.

$L_m \varphi$ WITH $m < 0$ ARE <u>DESCENDANTS</u> OF THE PRIMARY FIELD.

DESCENDANTS WITH

$$\Delta = \Delta_\varphi + l \qquad l \text{ POSITIVE INTEGER}$$

ARE CALLED <u>DESCENDANTS AT LEVEL l</u>.

NOTE: THEIR DIMENSION IS ACTUALLY BIGGER! THEY ARE LESS RELEVANT.

CONFORMAL TRANSFORMATIONS IN $d=2$

20.11.19

$$\begin{cases} z \to z + \delta z \\ \bar{z} \to \bar{z} + \delta \bar{z} \end{cases} \qquad \begin{cases} \delta z = \varepsilon(z,\bar{z}) \\ \delta \bar{z} = \bar{\varepsilon}(z,\bar{z}) \end{cases}$$

THE INFINITESIMAL SQUARE DISTANCE ELEMENT IS $dz\,d\bar{z}$, SO A TRANSFORMATION IS CONFORMAL IF

$$\delta(dz\,d\bar{z}) = f(z,\bar{z})\,dz\,d\bar{z}.$$

BUT

$$\delta(dz\,d\bar{z}) = d(\delta z)\delta\bar{z} + \delta z\,d(\delta\bar{z})$$

$$= (\partial\varepsilon\,dz + \bar{\partial}\varepsilon\,d\bar{z})d\bar{z} + dz(\partial\bar{\varepsilon}\,dz + \bar{\partial}\bar{\varepsilon}\,d\bar{z})$$

$$= dz\,dz\,\partial\bar{\varepsilon} + dz\,d\bar{z}(\partial\varepsilon + \bar{\partial}\bar{\varepsilon}) + d\bar{z}\,d\bar{z}\,\bar{\partial}\varepsilon$$

HENCE THE TRANSFORMATION IS CONFORMAL IF

$$\begin{cases} \partial\bar{\varepsilon} = 0 \;\to\; \bar{\varepsilon} = \bar{\varepsilon}(\bar{z}) \\ \bar{\partial}\varepsilon = 0 \;\to\; \varepsilon = \varepsilon(z). \end{cases}$$

AS A CONSEQUENCE, CONFORMAL SYMMETRY IN $d=2$ IS ∞-DIMENSIONAL. THIS CAN LEAD TO EXACT SOLUTIONS FOR FP'S IN $d=2$.

NOTE: I GUESS ∞-SYMMETRY $\Rightarrow$ ∞-CONSERVATION LAWS.

· SOME CONSEQUENCES OF CONFORMAL SYMMETRY IN $d=2$

·) FOR $c<1$ THERE EXIST <u>DEGENERATE PRIMARIES</u> DEFINED AS FOLLOWS: A PRIMARY IS SAID TO BE DEGENERATE AT LEVEL l IF IT ALLOWS FOR A VANISHING LINEAR COMBINATION OF DESCENDANTS AT LEVEL l.

NOTE: YOU SHOULD TAKE THESE AS FACTS (THEY COME FROM CFT).

·) THE PRIMARY DEGENERATE AT LEVEL

$l = mn$ $\qquad$ m, n INTEGERS

CAN BE DENOTED AS $\phi_{m,n}$ AND HAS

$$\Delta_{m,n} = \frac{[(p+1)m - pn]^2 - 1}{4p(p+1)} \qquad c = 1 - \frac{6}{p(p+1)}$$

WHERE $p \in \mathbb{R}$ PARAMETERIZES THE CENTRAL CHARGE AS ABOVE.

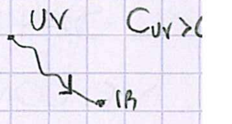
-) FOR $c<1$, REFLECTION POSITIVITY IS SATISFIED ONLY FOR

$p = 3, 4, 5, \ldots$ NOTE: i.e. $c>0$.

THIS IMPLIES THAT $c = \frac{1}{2}$ $(p=3)$ IS THE MINIMAL VALUE OF CENTRAL CHARGE FOR A REFLECTION POSITIVE FP.

THEN C-THM IMPLIES THAT IT MUST CORRESPOND TO THE ISING FP.

ONE IDENTIFIES



$$\Delta_\sigma = \Delta_{1,2} = 1/16 \qquad \overset{*}{\Rightarrow} \qquad \begin{cases} X_\sigma = 2\Delta_\sigma = \frac{1}{8} \\ X_\varepsilon = 2\Delta_\varepsilon = 1 \end{cases}$$
$$\Delta_\varepsilon = \Delta_{1,3} = 1/2$$

FROM WHICH WE GET THE CRITICAL EXPONENTS OF THE ISING MODEL IN $d=2$ (WHICH IS EXACTLY SOLVABLE ON THE LATTICE):

$$\nu = 1, \quad \alpha = 0, \quad \beta = \frac{1}{8}, \quad \gamma = \frac{7}{4}.$$

* THE EXISTENCE IN ISING FP OF $\Delta = \frac{1}{2}$ ALLOWS THE PRESENCE OF FIELDS

ψ with $(\Delta_\psi, \bar{\Delta}_\psi) = (\frac{1}{2}, 0) \rightsquigarrow s_\psi = \frac{1}{2}$

$\bar{\psi}$ with $(\Delta_{\bar\psi}, \bar{\Delta}_{\bar\psi}) = (0, \frac{1}{2}) \rightsquigarrow s_{\bar\psi} = -\frac{1}{2}$

WHICH ARE FERMIONS SATISFYING

$$\bar{\partial}\psi = \partial\bar{\psi} = 0$$

NOTE: SAYING $\bar{\Delta}_\psi = 0$ MEANS THAT ψ DOES NOT DEPEND ON $\bar{z}$, i.e. $\bar{\partial}\psi = 0$.

WHICH ARE THE EQUATIONS OF MOTION FOR THE <u>FREE FERMION</u> ACTION

*NOTE: THE ACTION IS A SCALAR, SO ALL THE FIELDS IT CONTAINS SHOULD APPEAR IN SCALAR COMBINATIONS. IN PARTICULAR, ε IS SCALAR.

$$A = \int d^d x \left[\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} \right].$$

WE CONCLUDE THAT THE ISING FP IN $d=2$ ALLOWS FOR A FREE FERMION REPRESENTATION.

IN THIS THEORY (SINCE $\Delta_\varepsilon = 1/2$ FROM ABOVE AND* $s_\varepsilon = 0$),

$$\Delta_\varepsilon = \bar{\Delta}_{\varepsilon} = \frac{1}{2} \qquad \Rightarrow \qquad \varepsilon = \psi\bar{\psi}$$

σ (with $\Delta_\sigma = \frac{1}{16}$) NON-TRIVIAL IN TERMS OF THE FERMIONS.

* THE OPE AT ISING FP ($p=3$) GIVES

$$\sigma \cdot \psi \sim \mu + \dots$$

WITH

$$\Delta_\mu = \bar{\Delta}_\mu = \frac{1}{16}.$$

BUT WE CAN'T SIMPLY IDENTIFY σ AND μ: NOTICING

$$S_\sigma + S_\psi - S_\mu = S_\psi = \frac{1}{2} \notin \mathbb{Z}$$

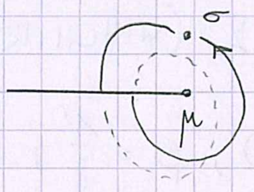
WE CONCLUDE THAT σ, ψ ARE MUTUALLY NON-LOCAL. SIMILARLY,

$$\sigma \cdot \mu \sim \psi \qquad \Rightarrow \qquad \sigma, \mu \text{ ARE MUTUALLY NON-LOCAL.}$$

HENCE $\mu \neq \sigma$, BECAUSE σ IS CLEARLY LOCAL WITH ITSELF.

THE CORRELATION FUNCTION

$$\langle \dots \sigma(x) \mu(0) \dots \rangle$$



HAS A SQUARE ROOT BRANCH CUT.

BUT SINCE THEY HAVE THE SAME SCALING DIMENSION, AT $T = T_c$

$$\langle \sigma(x) \sigma(0) \rangle = \langle \mu(x) \mu(0) \rangle = \frac{C^{I}_{\sigma\sigma}}{|x|^{1/4}}$$

$\mu \equiv$ DISORDER FIELD.

WE WILL SEE THAT IN THE OFF-CRITICAL THEORY

$$A = A_{FP} + \tau \int d^2x \, \mathcal{E}(x)$$

WE HAVE

$$\langle \sigma(x) \sigma(0) \rangle_{\tau \gtrless 0} = \langle \mu(x) \mu(0) \rangle_{\tau \lessgtr 0}.$$

FOR THE ISING MODEL, THIS IS CALLED <u>DUALITY</u> UNDER THE OPERATION

$$\begin{cases} \sigma \leftrightarrow \mu \\ \tau \to -\tau \end{cases} \rightsquigarrow \mathcal{E} \to -\mathcal{E} \Rightarrow C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}} = 0$$

($\mathcal{E}$ IS DUALITY ODD: $C^{\mathcal{E}}_{\underset{\text{ODD}}{\mathcal{E}}\underset{\text{ODD}}{\mathcal{E}}}$ CANNOT GIVE $C^{\mathcal{E} \leftarrow \text{ODD}}_{\mathcal{E}\mathcal{E}}$).

NOTE: IN THE OPE I HAVE $\underbrace{\mathcal{E}_a \cdot \mathcal{E}_b}_{\text{EVEN}} \sim C^{\mathcal{E}}_{\mathcal{E}\mathcal{E}} \underbrace{\mathcal{E}_a \delta_{ab}}_{\text{ODD}}$.

IN $d=2$, "A" CAN BE WRITTEN EXPLICITLY:

$$A = \int d^2x \left[\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + \tau \psi \bar{\psi} \right].$$

THIS IS A PARTICULAR CASE WHERE $\langle \sigma(x) \sigma(0) \rangle$ CAN BE EXACTLY COMPUTED OFF CRITICALITY.

· <u>GAUSSIAN FP IN $d=2$</u>

$$A_0 = \frac{1}{4\pi} \int d^2x \, (\nabla \varphi)^2$$

NOTE: $x=(x_1,x_2)$ OR $(z,\bar{z})=(x_1+ix_2, x_1-ix_2)$
IT FOLLOWS $\Box \varphi = 0$. USING
$\partial_1 = \frac{1}{2}(\partial_z + \partial_{\bar{z}})$, $\partial_2 = \frac{1}{2i}(\partial_{\bar{z}} - \partial_z)$
WE FIND
$\partial_1^2 + \partial_2^2 = \partial_z \partial_{\bar{z}}$.
THIS ALLOWS TO SEPARATE VARIABLES (AS IN THE WAVE EQUATION).

WHENCE THE EQUATIONS OF MOTION

$$\partial \bar{\partial} \varphi = 0 .$$

THE SOLUTION CAN THUS BE PARAMETRIZED AS

$$\varphi(x) = \phi(z) + \bar{\phi}(\bar{z}).$$

SINCE, IN $d=2$, $X_\varphi = 0$, THEN

NOTE: $|x| = \sqrt{z\bar{z}}$. $C^{I}_{\varphi\varphi} = 1$ (GAUSSIAN, ANY d)

$$\langle \varphi(x) \varphi(0) \rangle = - \ln|x| = -\frac{1}{2}(\ln z + \ln \bar{z}).$$

φ IS NOT A GOOD SCALING FIELD. CONSIDER INSTEAD

$$J(z) \equiv i \partial \phi(z)$$

NOTE: $X_\varphi = X_\phi = 0$.

$$\langle J(z) J(w) \rangle = - \partial_z \partial_w \underbrace{\langle \phi(z) \phi(w) \rangle}_{= -\frac{1}{2}\ln(z-w)} = \frac{1}{2(z-w)^2}$$

$$\Rightarrow \Delta_J = 1 .$$

* LET'S INTRODUCE

$$T_\gamma(z) \equiv \gamma J^2(z)$$

NOTE: i.e. $X_{T_\gamma} = 2$, $S_{T_\gamma} = 0$.

$$(\Delta_{T_\gamma}, \bar{\Delta}_{T_\gamma}) = (2,0)$$

$$T_\gamma(z) J(0) \underset{\text{LOOK FOR } J}{\rightsquigarrow} 2\gamma J(z) \langle J(z) J(0) \rangle = \gamma \frac{J(z)}{z^2} = \frac{\gamma}{z^2}\left[J(0) + z \partial J(0) + \dots \right].$$

$J \cdot \overset{2}{\text{—}} \cdot J$
$J \cdot$

NOTE: THE FP IS GAUSSIAN, SO WE ARE USING WICK'S THM. WE DERIVED LONG AGO

$$C^{\phi}_{T_{\mu\nu}\phi}(x) = \frac{d X_\phi}{d-1} \cdot \frac{x_\mu x_\nu - \frac{1}{d}|x|^2 \delta_{\mu\nu}}{|x|^{d+2}}$$

SPECIALIZING $C^{\phi}_{T_{\mu\nu}\phi}(x)$ TO $d=2$,

$$T(z)\phi(0) = \frac{\Delta_\phi}{z^2} \phi(0) + \dots$$

BUT TO COMPUTE IT ONE SHOULD STEP FROM $T_{\mu\nu}$ (3 g.d.l.) TO $T, \bar{T}, \Theta$. MOLLO.

(I)

WE CONCLUDE THAT

$$T_1 = T = J^2$$

IS THE STRESS TENSOR. THEN

NOTE: I'M NOT SURE IF AND WHERE HE DERIVED IT, BUT WE IDENTIFY $T_1 = T$ BECAUSE (I) IS PRECISELY THE WAY THE OPE OF T WITH A SCALING FIELD ϕ WORKS. MOREOVER,

$$TT \sim \frac{c/2}{z^4} + \ldots$$

$$T(z)T(0) = \begin{matrix} J & \overset{2}{\cdots} & J \\ J & \underset{1}{\cdots} & J \end{matrix} \, I + \ldots = 2\langle J(z)J(0)\rangle^2 I + \ldots = \frac{1}{2z^4} I + \ldots$$

BUT WE KNOW FROM THE C-THM THAT

$$C_{TT}^{I} = \frac{c}{2} \quad \Rightarrow \quad \underline{c = 1} \quad \text{FOR GAUSS FP.}$$

* WE CAN CONSIDER (ONLY IN $d=2$, WHERE ϕ IS DIMENSIONLESS)

$$V_p(z) \equiv e^{2ip\phi(z)}$$

AND COMPUTE ITS OPE WITH THE STRESS TENSOR

$$T(z)V_p(0) = \begin{matrix} i\partial \bullet \\ i\partial \bullet \end{matrix} \left(\sum_{k=0}^{\infty} \frac{(2ip)^k}{k!} \begin{matrix} \bullet 1 \\ \bullet 2 \\ \vdots \\ \bullet k \end{matrix} \right)$$

NOTE: $T(z) = J^2(z) = (i\partial\phi(z))^2$.

$$= -\sum_{k=2}^{\infty} \frac{(2ip)^k}{k!} \begin{matrix} \partial \bullet \overset{k}{\text{—}} \bullet 1 \\ \partial \bullet \underset{k-1}{\text{—}} \bullet 2 \\ \vdots \\ \bullet k \end{matrix} + \ldots = -\sum_{k=2}^{\infty} \frac{(2ip)^k}{(k-2)!} \underbrace{\left(\partial\langle\phi(z)\phi(0)\rangle\right)^2}_{(1/2z)^2} \phi^{k-2}(0) + \ldots$$

↑ WE WANT 2 CONTRACTIONS

$$= -\frac{(2ip)^2}{(2z)^2} \sum_{m=0}^{\infty} \frac{(2ip)^m}{m!} \phi^m(0) + \ldots = \frac{p^2}{z^2} V_p(0) + \ldots$$

COMPARING WITH (I),

$$\underline{\Delta_{V_p} \equiv \Delta_p = p^2} .$$

NOTE: THIS IS A PARAFERMIONIC FIELD, AS ITS SPIN IS NEITHER INTEGER, NOR HALF-INTEGER.

THIS IS A CONTINUOUS FAMILY OF "GOOD" SCALING FIELDS, i.e. A CONTINUOUS SPECTRUM OF SCALING DIMENSIONS.

* NOTICE THE OPERATION

$$\varphi \to \varphi + \alpha$$

LEAVES A_0 INVARIANT. THEN

$$\langle e^{2ip_1(\phi(z_1)+\alpha)} \cdots e^{2ip_m(\phi(z_m)+\alpha)} \rangle = \langle e^{2ip_1\phi(z_1)} \cdots e^{2ip_m\phi(z_m)} \rangle \equiv (*)$$

BUT

$$\circledast = e^{2i(p_1+\dots+p_m)\alpha}\langle e^{2ip_1\phi(z_1)}\cdot\dots\cdot e^{2ip_m\phi(z_m)}\rangle$$

WHENCE WE CONCLUDE

$$\langle V_{p_1}(z_1)\dots V_{p_m}(z_m)\rangle \neq 0 \quad \text{ONLY IF} \quad \sum_{i=1}^{m} p_i = 0.$$

THIS IS CALLED <u>NEUTRALITY CONDITION</u> (THINK OF THE p_i'S AS CHARGES).

THIS HAS TO BE FULLFILLED BY THE OPE

$$\langle \dots V_{p_i}(z_i)V_{p_{i+1}}(z_{i+1})\dots\rangle \xrightarrow[\text{OPE}]{} V_{p_i+p_{i+1}}.$$

NOTE: I THINK YOU SHOULD IMAGINE THERE ARE OTHER V_{p_j}'S IN THE CORRELATION FUNCTION, SO YOU DON'T JUST SET $p_i+p_{i+1}=0$ BUT ONLY IMPOSE THE CONSERVATION OF THE "TOTAL CHARGE".

RECAP: GAUSSIAN FP IN $d=2$ 21.11.19

$$A_0 = \frac{1}{4\pi}\int d^2x\,(\nabla\varphi)^2$$

$C=1$

$$\varphi(x) = \phi(z)+\bar\phi(\bar z)$$

** NOTE: WHY? IF φ IS AN OBSERVABLE AND WE MOVE SLIGHTLY AWAY FROM CRITICALITY, $\langle\varphi\rangle = \int\mathcal{D}\phi\,\varphi e^{-(S_{CFT}+\int\varepsilon)} \simeq \langle\varphi\rangle_{CFT} + \langle\varphi\,\varepsilon\rangle_{CFT}+\dots$

AND WE INTRODUCED THE FAMILY OF SCALING FIELDS

$$V_p(z) \equiv e^{2ip\phi(z)} \qquad \Delta_{V_p}\equiv\Delta_p = p^2$$

WHOSE OPE SATISFIES

$$V_{p_1}\cdot V_{p_2} = V_{p_1+p_2}+\dots$$

NOTE: RECALL, BY DEFINITION, $\phi_i(x)\phi_j(0) = \sum_k C_{ij}^k\, z^{\Delta_k-\Delta_i-\Delta_j}\,\bar z^{\bar\Delta_k-\bar\Delta_i-\bar\Delta_j}\,\phi(k)$

HENCE, SINCE $V_p(z)$ IS NOT A FUNCTION OF $\bar z$, IT CANNOT SCALE WITH $\bar z$ AND THUS $\bar\Delta_{V_p}\equiv\bar\Delta_p = 0$.

LET'S NOW INTRODUCE ALSO

$$\bar V_{\bar p}(\bar z)\equiv e^{2i\bar p\bar\phi(\bar z)}$$

(SAME SCALING DIMENSION)

NOTE: IN THE SENSE THAT $\bar\Delta_{\bar V_{\bar p}}\equiv\bar\Delta_{\bar p} = \bar p^2$, WHILE $\Delta_{\bar V_{\bar p}}\equiv\Delta_{\bar p}=0$.

THEN $S_{p,\bar p} = \Delta_{p,\bar p}-\bar\Delta_{p,\bar p} = (\Delta_p+\Delta_{\bar p})-(\bar\Delta_p+\bar\Delta_{\bar p})$.

HENCE THE SPIN

$$S_{p,\bar p} = \text{SPIN}\,V_p\bar V_{\bar p} = p^2-\bar p^2\,.$$

THE TWO FIELDS $V_{p_1}\bar V_{\bar p_1}$ AND $V_{p_2}\bar V_{\bar p_2}$ WILL BE MUTUALLY LOCAL IF

$$S_{p_1,\bar p_1}+S_{p_2,\bar p_2}-S_{p_1+p_2,\bar p_1+\bar p_2}\in\mathbb{Z}$$

NOTE: USE BOTH THE CONDITION OF MUTUAL LOCALITY OF ϕ_i AND ϕ_j, i.e. $S_i+S_j-S_k\in\mathbb{Z}\ \forall\phi_k$ IN THE OPE AND THE OPE PROPERTY OF V_p UP HERE.

WHICH MEANS

$$(p_1^2-\bar p_1^2)+(p_2^2-\bar p_2^2)-\left[(p_1+p_2)^2-(\bar p_1+\bar p_2)^2\right]\in\mathbb{Z}$$

$$\Rightarrow\ \underline{2(p_1p_2-\bar p_1\bar p_2)\in\mathbb{Z}}$$

(CONDITION FOR MUTUAL LOCALITY).

* THE ENERGY DENSITY $\varepsilon(x)$ MUST BE REAL, SPINLESS (SCALAR) AND (LIKE A_0) INVARIANT UNDER $\varphi\to-\varphi$. A GENERIC CHOICE IS*

$$\varepsilon\sim V_b\bar V_b+V_{-b}\bar V_{-b}\sim\cos(2b\varphi) \qquad \Delta_\varepsilon=\bar\Delta_\varepsilon = b^2.$$

PHYSICALLY INTERESTING THINGS SHOULD BE LOCAL WITH RESPECT TO ε (i.e. WE DON'T WANT BRANCH CUTS OR SO)**.

* NOTE: $\varepsilon\sim e^{2ib\phi(z)}e^{2ib\bar\phi(\bar z)}+(b\to-b)$.

$V_\rho \bar{V}_{\bar\rho}$ IS LOCAL WITH RESPECT TO ε IF

$$\underline{\rho - \bar\rho = \frac{m}{2b}, \quad m \in \mathbb{Z}}$$

(SELECTION RULE).

NOTE: i.e. $2(\rho b - \bar\rho b) \in \mathbb{Z}$.

ITS SPIN IS

$$S_{\rho,\bar\rho} = \rho^2 - \bar\rho^2 = (\rho+\bar\rho)(\rho-\bar\rho) \overset{\text{LOCAL}}{=} (\rho+\bar\rho)\frac{m}{2b}.$$

VARYING m WE CAN CONSTRUCT FERMIONS OF DIFFERENT SPIN.

THE MOST RELEVANT SPIN $\pm\frac{1}{2}$ FERMION IS OBTAINED FOR

$$\underline{m = 1, \ \rho + \bar\rho = \pm b}.$$

ITS TWO COMPONENTS ARE

$$\begin{cases} \psi = V_{\frac{1}{4b}+\frac{b}{2}} \bar{V}_{-\frac{1}{4b}+\frac{b}{2}} & (S_\psi = \frac{1}{2}) \\ \bar\psi = V_{\frac{1}{4b}-\frac{b}{2}} \bar{V}_{-\frac{1}{4b}-\frac{b}{2}} & (S_\psi = -\frac{1}{2}) \end{cases}$$

THIS IS A <u>COMPLEX (DIRAC) FERMION</u>.

ITS DECOMPOSITION

$$\psi \equiv \psi_1 + i\psi_2$$

NOTE: THE DIRAC FERMION IS $\Psi = (\psi, \bar\psi)$.

DEFINES TWO <u>REAL (MAJORANA) FERMIONS</u> ψ_1 AND ψ_2.

FOR $b^2 = \frac{1}{2}$,

$$\bar\Delta_\psi = \Delta_{\bar\psi} = \left(\frac{1}{4b} - \frac{b}{2}\right)^2 = 0$$

NOTE: RECALL
$\Delta_{V_\rho} = \rho^2$, $\bar\Delta_{\bar V_{\bar\rho}} = \bar\rho^2$, $\bar\Delta_{V_\rho} = \Delta_{\bar V_{\bar\rho}} = 0$
HENCE
$\Delta_\psi = \Delta_{V_b} + \Delta_{\bar V_b} = \Delta_{V_b} \equiv \Delta_b$
$\bar\Delta_\psi = \bar\Delta_{V_b} + \bar\Delta_{\bar V_b} = \bar\Delta_{\bar V_b} \equiv \bar\Delta_b$.

$$\Rightarrow \bar\partial\psi = \partial\bar\psi = 0$$

FREE FERMION.

FOR $b^2 \neq \frac{1}{2}$,

$$\bar\partial\psi, \ \partial\bar\psi \neq 0$$

INTERACTING FERMION.

THUS, GAUSSIAN THEORY IN $d=2$ ALSO CORRESPONDS TO A THEORY OF 2 MAJORANA FERMIONS (OR 1 DIRAC) BECOMING FREE FOR $b^2 = \frac{1}{2}$.

THIS IS NOT UNEXPECTED, BECAUSE $c_{ISING} = \frac{1}{2}$ AND

$$c_{GAUSS} = 1 = \frac{1}{2} + \frac{1}{2}.$$

NOTE: $C^{I}_{TT} = \frac{c}{2}$. IF A THEORY IS MADE OF 2 NON INTERACTING FIELDS, THEN $T = T_\psi + T_{\bar\psi}$.

THIS IS CLEAR IF $b^2 = \frac{1}{2}$: FROM THE DEFINITION OF C IN TERMS OF THE STRESS TENSOR, WE SEE IT IS MADE UP OF TWO, NON INTERACTING PARTS WHICH ADD UP. HOWEVER, SWITCHING ON $b^2 \neq \frac{1}{2}$ DOESN'T PRODUCE A FLOW (WE ARE ALWAYS AT THE FP), SO BY C-THEOREM THE CENTRAL CHARGE DOESN'T CHANGE.

BESIDES THE BOSONIC REPRESENTATION WE NOW HAVE

$$A_0 = \int d^2x \left[\sum_{i=1}^{2} \left(\psi_i \bar\partial \psi_i + \bar\psi_i \partial \bar\psi_i \right) + g(b^2)\, \psi_1 \bar\psi_1 \psi_2 \bar\psi_2 \right]$$

$$g\left(b^2 = \tfrac{1}{2}\right) = 0$$

<u>THIRRING MODEL</u>.

MARGINALITY OF $\psi_1 \bar\psi_1 \psi_2 \bar\psi_2$ PERSISTS FOR $b^2 \neq \frac{1}{2}$ (THIS COULD BE PROVED).

THE REASON FOR SUCH AN INTERACTIVE TERM IS SIMPLY THAT IT'S THE ONLY POSSIBLE TERM. IN FACT

$$\psi_i^2 = \bar\psi_i^2 = 0 \qquad \text{(GRASSMAN ALGEBRA)}.$$

<u>MORAL</u>: IN $d=2$, THE GAUSSIAN MODEL (BOSONIC) ALLOWS FOR A FERMIONIC REPRESENTATION AS WELL. THE INVERSE PROCEDURE IS CALLED BOSONIZATION.

*NOTE: THERE'S NO NEED TO GIVE $\tilde\varphi$ A PHYSICAL SIGNIFICANCE IT'S JUST SHORTHAND (AND IT WILL EVEN CANCEL OUT).

* NOW CONSIDER $\varphi(x)$ AND ITS DUAL BOSON*

$$\begin{cases} \varphi(x) = \phi(z) + \bar\phi(\bar z) \\ \tilde\varphi(x) = \phi(z) - \bar\phi(\bar z) \end{cases}$$

NOTE: RECALL THE OTHER DEFINITIONS, $V_p(z) = e^{2ip\phi(z)}$, $\bar V_{\bar p}(\bar z) = e^{2i\bar p \bar\phi(\bar z)}$ AND $\psi, \bar\psi$ ARE IN THE FORM $\sim V_\alpha \bar V_{\alpha'}$ FOR SOME PARTICULAR CHOICES OF $\alpha(b)$, $\alpha'(b)$, AND $b \in \mathbb{R}$.

WHENCE THE REAL PARTS

$$\begin{cases} \psi_1 = \frac{1}{2}(\psi + \psi^*) = \frac{1}{2}\left\{ e^{2i\left[\frac{b}{2}(\phi+\bar\phi) + \frac{1}{4b}(\phi - \bar\phi)\right]} + c.c. \right\} = \cos\Big(\underbrace{b\varphi + \frac{1}{2b}\tilde\varphi}_{\equiv \alpha}\Big) \\ \bar\psi_1 = \frac{1}{2}(\bar\psi + \bar\psi^*) = \cos\Big(\underbrace{-b\varphi + \frac{1}{2b}\tilde\varphi}_{\equiv \beta}\Big) \end{cases}$$

AND THE IMAGINARY PARTS

$$\begin{cases} \Psi_2 = \frac{1}{2i}\left(\Psi - \Psi^*\right) = \sin\left(b\varphi + \frac{1}{2b}\tilde{\varphi}\right) \\ \bar{\Psi}_2 = \frac{1}{2i}\left(\bar{\Psi} - \bar{\Psi}^*\right) = \sin\left(-b\varphi + \frac{1}{2b}\tilde{\varphi}\right) \end{cases}$$

WITH

$$\Psi_1\bar{\Psi}_1 + \Psi_2\bar{\Psi}_2 = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta) = \cos(2b\varphi)$$

$$\varepsilon_1 + \varepsilon_2 = \varepsilon(x)$$

NOTE: THE MASS TERM IN THE GAUSSIAN MODEL ($\lambda\cos$ CORRESPONDS TO THE MASS TERMS IN THE FERMIONIC ONE ($\sum_i \lambda\Psi_i\bar{\Psi}_i$).

WHICH IS THE BOSONIZATION RULE.

★ SUMMING UP,

$$\underline{\text{GAUSSIAN FP}} \quad \underset{\text{BOSONIZATION}}{\overset{\text{FERMIONIZATION}}{\rightleftarrows}} \quad \underline{\text{THIRRING MODEL}}$$

$$\downarrow +\varepsilon(x) \qquad\qquad \downarrow +\varepsilon(x)$$

$$\int d^2x \left[\frac{1}{4\pi}(\nabla\varphi)^2 + \lambda\cos(2b\varphi)\right]$$

SINE - GORDON

$$\int d^2x \left[\sum_{i=1}^{2}\left(\Psi_i\bar{\partial}\Psi_i + \bar{\Psi}_i\partial\bar{\Psi}_i + \lambda\Psi_i\bar{\Psi}_i\right) + g(b^2)\Psi_1\bar{\Psi}_1\Psi_2\bar{\Psi}_2\right]$$

MASSIVE THIRRING

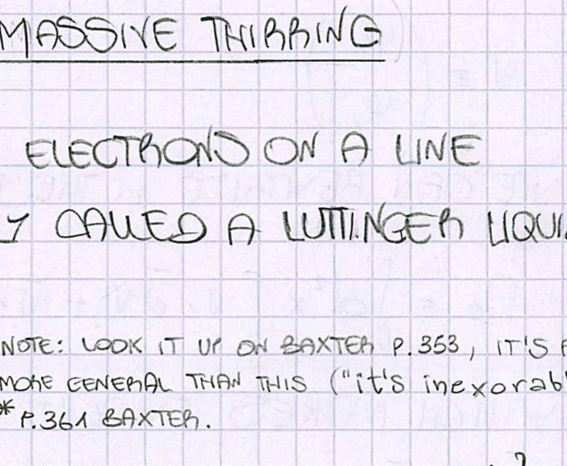
MASSIVE THIRRING DEALS BASICALLY WITH ELECTRONS ON A LINE ($d=2 \to (1+1)$), WHERE THEY'RE USUALLY CALLED A LUTTINGER LIQUID.

· ASHKIN - TELLER MODEL

NOTE: LOOK IT UP ON BAXTER P. 353, IT'S FAR MORE GENERAL THAN THIS ("it's inexorably!"
* P.361 BAXTER.

$$H = -\sum_{\langle x,y\rangle}\left\{J\left[\sigma_1(x)\sigma_1(y) + \sigma_2(x)\sigma_2(y)\right] + J_4\,\sigma_1(x)\sigma_1(y)\sigma_2(x)\sigma_2(y)\right\}$$

$\sigma_1, \sigma_2 = \pm 1$ (ISING VARIABLES)



THERE APPEARS A LINE OF FIXED POINTS*
WITH CONTINUOUSLY VARYING CRITICAL
EXPONENTS.

NOTE: $J \leftrightarrow -J$ MEANS $\sigma_i \leftrightarrow -\sigma_i$ ON ONE SUBLATTICE (FERRO ↔ ANTIFERRO). NOTICE $\sigma_1 \to -\sigma_1$, $\sigma_2 \to -\sigma_2$, $\sigma_1 \leftrightarrow \sigma$

THE CORRESPONDING ACTION IS

$$A = \sum_{i=1}^{2} \left[A^{(i)}_{ISING\,FP} + \tau \int d^2x\, \varepsilon_i(x) \right] + g \int d^dx\, \varepsilon_1 \varepsilon_2 (x)$$

AND SINCE

$$\varepsilon_i = \psi_i \bar{\psi}_i \qquad\qquad g = g(b^2)$$

THIS IS NOTHING ELSE THAN THE MASSIVE THIRRING MODEL.

THE SCALING DIMENSION OF THE ENERGY DENSITY IS

$$X_\varepsilon = 2\Delta_\varepsilon = 2b^2$$

NOTE: THIS IS X_{ε_i}. $\varepsilon_1 \varepsilon_2$ IS MARGINAL!

AND THIS ACCOUNTS FOR THE CONTINOUSLY VARYING EXPONENTS ALONG THE LINE OF FP'S.

NOTE: ALL THIS WORK WAS NOT ABOUT ISING IN $d=2$, BUT HAS MORE GENERAL IMPLICATIONS. TO SUM UP,
1) FROM CFT $\Rightarrow$ ISING $\leftrightarrow$ FREE (MAJORANA) FERMION, $c = \frac{1}{2}$.
2) GAUSSIAN BOSONIC ($\lambda = 0$, $c = 1$) $\leftrightarrow$ 1 DIRAC FERMION.
3) FOR $b^2 = \frac{1}{2}$, WE GET NON INTERACTING (DIRAC) FERMIONS, i (ISING + ISING).
4) ADDING AN ENERGY DENSITY ($\lambda\, \varepsilon(x)$) TO THE BOSONIC MODEL (SINE-GORDON) AMOUNTS TO ADDING A MASS TERM TO THE FERMIONIC ONE (MASSIVE THIRRING). WE'LL SEE THIS ACCOUNTS FOR THE BKT TRANSITION.

★ WE HAVE SEEN THAT

$$\psi_i^2 = \bar{\psi}_i^2 = 0$$

SO THE 4-POINT INTERACTION REDUCES TO

$$\psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 \propto \left(\sum_{i=1}^{2} \psi_i \bar{\psi}_i \right)^2 = (v \cdot \bar{v})^2$$

NOTE: $\Psi_1 = \begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \end{pmatrix}$ IS A MAJORANA FERMION. $v, \bar{v}$ ARE NEWLY DEFINED OBJECTS.

WHERE WE DEFINED

$$v \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad\qquad \bar{v} \equiv \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix}.$$

WE CAN REWRITE IN THESE TERMS THE THIRRING ACTION

$$A_0 = \int d^2x \left[v \cdot \bar{\partial} v + \bar{v} \cdot \partial \bar{v} + g (v \cdot \bar{v})^2 \right]$$

WHICH MAKES EVIDENT ITS INVARIANCE UNDER $O(2)$ ROTATIONS OF $v, \bar{v}$: THE GAUSSIAN FP HIDES A $O(2)$ SYMMETRY.

TO STUDY ITS IMPLICATIONS, WE GO BACK TO

$$V_p \bar{V}_{\bar{p}} = e^{2i(p\phi + \bar{p}\bar{\phi})} = e^{2i\left[\frac{p}{2}(\varphi + \tilde{\varphi}) + \frac{\bar{p}}{2}(\varphi - \tilde{\varphi})\right]} = e^{i\left[(p+\bar{p})\varphi + \underbrace{(p-\bar{p})}_{\frac{m}{2b}}\tilde{\varphi}\right]}$$

SO AS TO SEE WHAT THIS SYMMETRY LOOKS LIKE IN THE BOSONIC FORMULATION.

WE NOTICE THAT

$$\begin{cases} \Psi = e^{i\left(b\varphi + \frac{1}{2b}\tilde{\varphi}\right)} = \Psi_1 + i\Psi_2 & m = 1 \\ \bar{\Psi} = e^{i\left(-b\varphi + \frac{1}{2b}\tilde{\varphi}\right)} = \bar{\Psi}_1 + i\bar{\Psi}_2 & m = 1 \\ \varepsilon = \Psi_1\bar{\Psi}_1 + \Psi_2\bar{\Psi}_2 = v \cdot \bar{v} \sim \cos(2b\varphi) & m = 0 \end{cases}$$

HENCE ε IS A SCALAR, WHILE $\Psi, \bar{\Psi}$ ARE COMPLEX REPRESENTATIONS OF VECTORS UNDER $O(2) \sim U(1)$ SYMMETRY:

NOTE: m PLAYS THE ROLE OF $U(1)$ CHARGE. GO BACK TO $S_{p,\bar{p}} = (p+\bar{p})\frac{m}{2b}$ SO ε HAS $m=0$.

$$\begin{cases} \Psi \to e^{i\alpha}\Psi \\ \bar{\Psi} \to e^{i\alpha}\bar{\Psi} \\ \varepsilon \to \varepsilon . \end{cases}$$

AN $O(2) \sim U(1)$ ROTATION BY α AMOUNTS TO

$$\begin{cases} \frac{\tilde{\varphi}}{2b} \to \frac{\tilde{\varphi}}{2b} + \alpha \\ \varphi \to \varphi \quad \text{UNCHANGED!} \end{cases} \Rightarrow \quad \text{INVARIANCE OF } A_0$$

OR EQUIVALENTLY, GOING BACK TO $(z, \bar{z})$,

NOTE: A_0 IS THE GAUSSIAN ACTION WHICH IS THUS $O(2)$ INVARIANT.

$$\begin{cases} \frac{\phi}{2b} \to \frac{\phi}{2b} + \frac{\alpha}{2} \\ \frac{\bar{\phi}}{2b} \to \frac{\bar{\phi}}{2b} - \frac{\alpha}{2} . \end{cases}$$

* IN ASHKIN-TELLER, σ_1 AND σ_2 CANNOT BE BOSONIZED. THEY TURN OUT TO KEEP

$$X_{\sigma_i} = \frac{1}{8}$$

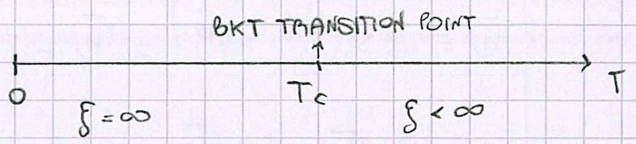
NOTE: TAKE IT AS A FACT. INDEED, WE NEVER EXPRESSED σ_1, σ_2. A SINGLE ISING CANNOT BE BOSONIZED.

ALONG THE LINE OF FP'S (AS IT CAN BE SHOWN).

IN FACT, σ_1 AND σ_2 DO NOT TRANSFORM IN A MEANINGFUL WAY UNDER THE $O(2)$ SYMMETRY.

· XY MODEL IN $d=2$

$$H = -J \sum_{\langle i,j \rangle} \underline{S}_i \cdot \underline{S}_j \qquad \underline{S}_i = (S_{1,i}, S_{2,i}) = (\cos\theta_i, \sin\theta_i)$$



OBSERVED FACTS:

1) $\langle \underline{S} \rangle = 0 \quad \forall T$

2) THERE IS A T_c S.t. FOR $T<T_c$ CORRELATIONS DECAY AS POWER LAWS DEPENDING ON T.

$\underline{S}(x)$ MUST HAVE $m=1$ (VECTOR) AND BE SCALAR $(\Delta_{\underline{S}} = \bar{\Delta}_{\underline{S}})$:

$$S_\pm \equiv S_1 + i S_2 = V_{\pm\frac{1}{4b}} \bar{V}_{\pm\frac{1}{4b}} = e^{\pm i \frac{\tilde{\varphi}}{2b}}$$

$$\Rightarrow \Delta_{\underline{S}} = \bar{\Delta}_{\underline{S}} = \frac{1}{16b^2}.$$

NOTE: $\Delta_{V_p \bar{V}_p} = \Delta_{V_p} = p^2$.

THE CONTINUUM LIMIT OF THE ANGULAR VARIABLE IS THEN

$$\theta(x) = \frac{\tilde{\varphi}(x)}{2b}.$$

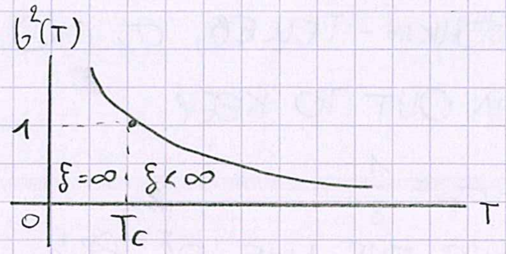
NOTE: $\underline{S}(x) = (\cos\theta(x), \sin\theta(x))$.

THE ACTION FOR THIS MODEL IS THE SINE-GORDON ONE,

$$A = \int d^2x \left[\frac{(\nabla\varphi)^2}{4\pi} + \lambda \underbrace{\cos(2b\varphi)}_{\varepsilon(x)} \right]$$

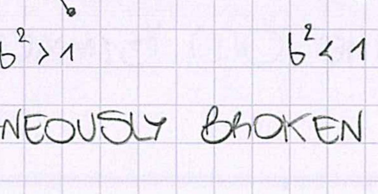
NOTE: RECALL $S_\varepsilon = 0$, HENCE $\Delta_\varepsilon = \frac{1}{2} X_\varepsilon = b^2$. REMEMBER WHAT'S IMPORTANT IS THE SCALING DIMENSION, WHICH MAY NOT COINCIDE WITH THE PHYSICAL DIMENSION.

AND IT CONTAINS 2 PARAMETERS (WHILE THERE IS ONLY ONE PHYSICAL PARAMETER): WE CONCLUDE THAT b, λ ARE FUNCTIONS OF THE TEMPERATURE T.



FOR $b^2 > 1$, ε IS IRRELEVANT AND WE OBSERVE THE GAUSSIAN FP AT LARGE DISTANCES.

FOR $b^2 < 1$, ε IS RELEVANT AND $\langle \underline{S} \rangle = 0 \ \forall T$.



CONTINUOUS SYMMETRIES ARE NOT SPONTANEOUSLY BROKEN IN $d=2$.

26.11.19

RECAP: XY MODEL IN $d=2$

$\underline{S} = (S_1, S_2) = (\cos\theta, \sin\theta)$ $\qquad d=2.$

IN THE CONTINUUM, THE ACTION IS THE SINE-GORDON MODEL:

$$A = \int d^2x \left[\frac{(\nabla\varphi)^2}{4\pi} + \lambda \cos 2b\varphi \right].$$

WE IDENTIFY

$$S_\pm = S_1 \pm i S_2 = e^{\pm i \frac{\tilde{\varphi}}{2b}} \qquad \varepsilon = \cos 2b\varphi$$

SO THAT

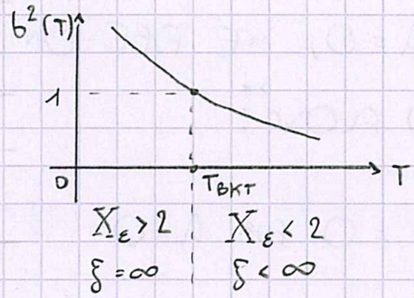
$$\theta \propto \tilde{\varphi} \qquad \begin{cases} \varphi = \phi + \bar{\phi} \\ \tilde{\varphi} = \phi - \bar{\phi} \end{cases}$$

AND WHERE

$$X_S = 2\Delta_S = \frac{1}{8b^2}, \qquad X_\varepsilon = 2\Delta_\varepsilon = 2b^2.$$

· WE ARGUED THERE IS A TRANSITION AT POINT $b^2 = 1$, WHERE

$$X_S = 1/8$$
$$X_\varepsilon = 2.$$



NOTE: A FIELD IS RELEVANT WHEN $X_\varphi < d = 2$.

* WE WANT TO CONSIDER NOW THE RG FLOW CLOSE TO THE TRANSITION POINT

$$\lambda = 0, \ b^2 = 1.$$

LET'S ADOPT THE NOTATION

$$\beta_i = l \frac{\partial g_i}{\partial l} = \frac{dg_i}{dt} \qquad l = l_0 e^t$$

NOTE: IT'S THE USUAL $\beta_i = \frac{\partial g_i}{\partial \ln l}$.

SO THAT FOR

$$\begin{cases} t \to \infty : \text{IR} \\ t \to -\infty : \text{UV}. \end{cases}$$

IN GENERAL,

$$\beta_i = (d - X_i^0) g_i - \sum_{j,k} (c^i_{jk})^0 g_j g_k + O(g^3) \qquad \text{(I)}$$

WHICH SHOULD BE SUITED TO OUR MODEL.

THIS IS MORE EASILY DONE IN THE FERMIONIC FORMALISM.

WE CALL λ THE COUPLING CONJUGATED TO

$$\varepsilon = \cos 2b\varphi = \varepsilon_1 + \varepsilon_2 = \sum_{i=1,2} \psi_i \bar{\psi}_i$$

WHILE

$$\tilde{g} \equiv g(b^2) - g(1)$$

NOTE: RECALL $g(b^2 = \frac{1}{2}) = 0$.

WILL BE THE (SMALL) COUPLING CONJUGATED TO

$$A = \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 .$$

WE KNOW THAT

$$X_A = 2 \quad \forall b^2$$

NOTE: WE NEVER PROVED THIS, BUT WE SAID THAT THE INTERACTION TERM IN THE MASSIVE THIRRING MODEL IS MARGINAL.

(THIS IS THE TRULY MARGINAL TERM), WHILE

$$X_\varepsilon = 2 \text{ AT } b^2 = 1 .$$

NOTE: ε GOES INTO THE MASS TERM IN THIRRING.

HENCE, THE LINEAR TERM WILL VANISH IN THE RG EQUATION* (I).

FOR $\lambda = 0$, WE ARE ON THE GAUSSIAN FIXED LINE, HENCE THERE IS NO FLOW**:

*NOTE: IT WILL VANISH IN BOTH THE EQUATION FOR THE FLOW OF λ AND OF $\tilde{g}$. IT DOES BECAUSE $d=2$

$$\frac{d\tilde{g}}{dt} = 0 \text{ FOR } \lambda = 0 \qquad \Rightarrow \qquad C_{AA}^{A} = 0 .$$

MOREOVER, THE FIELD ε IS ODD UNDER DUALITY: IN FACT, FOR THE ISING MODEL

$$\varepsilon_i \to -\varepsilon_i .$$

**NOTE: THE NEW COUPLING CONSTANT g APPEARED IN THE FERMIONIC THEORY IN PLACE OF b^2. IN THE OLD THEORY, FOR $\lambda = 0$ WE HAVE A LINE OF FIXED POINTS: EVEN IF WE MOVE b^2, THERE IS NO FLOW (i.e. THE β-FUNCTIONS ARE NULL). THEN $C_{AA}^{A} = 0$ BY (I

ON THE OTHER HAND,

$$A = \varepsilon_1 \varepsilon_2$$

IS EVEN; WE CONCLUDE THAT THE ONLY NONZERO OPE COEFFS ARE

$$C_{\varepsilon\varepsilon}^{A},\ C_{\varepsilon A}^{\varepsilon} \neq 0 .$$

WE CAN THEN WRITE THE KOSTERLITZ-THOULESS EQUATIONS

$$\begin{cases} \dfrac{d\lambda}{dt} = -C_{\varepsilon A}^{\varepsilon} \tilde{g} \lambda \\[2ex] \dfrac{d\tilde{g}}{dt} = -C_{\varepsilon\varepsilon}^{A} \lambda^2 . \end{cases}$$

WE CAN USE THE USUAL FORMULA TO REWRITE

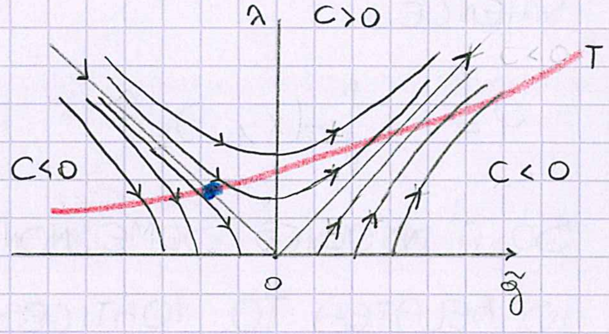
$$\rho \equiv \frac{C^{\varepsilon}_{\varepsilon A}}{C^{A}_{\varepsilon\varepsilon}} = \frac{C^{I}_{AA}}{C^{I}_{\varepsilon\varepsilon}} > 0 .$$

NOTE: $C_{123} = C^{3}_{12} C^{I}_{33}$.
REFLECTION POSITIVITY MEANS $\langle \phi(x)\phi(0) \rangle > 0$.

IN FACT, BOTH TERMS ARE POSITIVE BY REFLECTION POSITIVITY.

THEN

$$\frac{d\lambda}{d\tilde{g}} = \rho \frac{\tilde{g}}{\lambda}$$



$$d\lambda^2 = \rho\, d\tilde{g}^{\,2}$$

$$d(\lambda^2 - \rho\tilde{g}^{\,2}) = 0 \quad \Rightarrow \quad \lambda^2 - \rho\tilde{g}^{\,2} = C$$

NOTE: $(\tilde{g} < 0,\ \lambda \equiv 0)$ IS A LINE OF FP'S KNOWN AS BKT LINE.

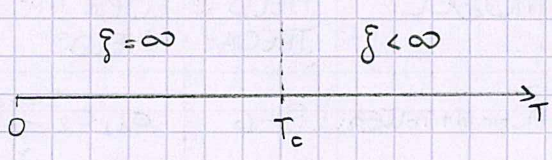
WHICH ARE HYPERBOLAE.

ON THE OTHER HAND, BOTH $\tilde{g}$ AND λ DEPEND ON THE TEMPERATURE T ALONE: CHANGING T LEADS TO THE RED LINE. WE HAVE INDICATED WITH A BLUE DOT THE CRITICAL TEMPERATURE T_c.

NOTE: $b^2(T)$ IS A NON-UNIVERSAL FUNCTION OF T.

WE NOTICE UNDER T_c

$$\nu = \frac{1}{d - X_\varepsilon} = \infty .$$



SINCE USUALLY

$$\xi \sim (T - T_c)^{-\nu}$$

HERE ξ DIVERGES EXPONENTIALLY AS $T \to T_c^{+}$. HERE, α AND γ ALSO DIVERGE, SO THAT

$$\chi \sim \xi^{\,2 - 2X_s|_{T_c}} \sim \xi^{7/4} .$$

NOTE: $X_s|_{T_c} = \frac{1}{8}$.

★ RECALL $\langle \underline{S} \rangle = 0 \quad \forall T$ (THE TRANSITION IS NOT DUE TO SSB). THE PHASE $T < T_c$ IS THEN REFFERRED TO AS QLRO (i.e. QUASI - LONG RANGE ORDER).

★ NOTICE THE OPE $\underline{S} \cdot \underline{S}$ DOES NOT PRODUCE ε. IN FACT,

$$\underline{S} \sim e^{i\tilde{\varphi}}, \quad \varepsilon \sim e^{i\varphi} .$$

IN ALL THE PREVIOUS CASES, IT HAD ALWAYS BEEN

$$\underline{S}\cdot\underline{S} \sim \varepsilon .$$

HOWEVER, BY THEIR DEFINITION IT FOLLOWS THAT

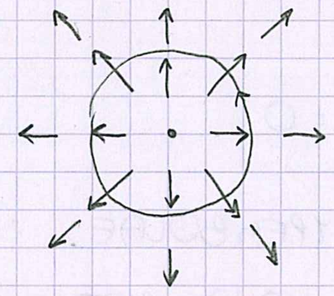
$$\partial_a \tilde{\varphi} = -i\,\varepsilon_{ab}\,\partial_b \varphi \qquad a,b = 1,2$$

WHENCE

NOTE: SEE CARDY P. 116. HERE ε_{ab} IS THE COMPLETELY ANTISYMMETRIC TENSOR.

$$\varphi = -i \int dx_1\, \partial_2 \tilde{\varphi} \qquad \varepsilon = \cos 2b\varphi = \cos\left(\bigcirc \int dx_1\, \partial_2 \theta\right)$$

SO IT INVOLVES SOME NONLOCAL, TOPOLOGICAL PROPERTIES. ε IS RELATED TO VORTICES: BY TURNING 360°, θ MAY ACQUIRE A PHASE MULTIPLE OF 2π.



THIS IS WHY BKT IS CALLED A TOPOLOGICALLY DRIVEN TRANSITION: IT IS DUE TO THE ROLE OF VORTICES BELOW T_c, WHERE THEY LOWER THE FREE ENERGY.

* TO SUMMARIZE:

MODEL	FIELD THEORY	SPIN FIELDS	SYMMETRY	TYPE OF TRANSITION	TRANSITION POINT
ASHKIN-TELLER	S-G	σ_1, σ_2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	SSB	CONTINUOUS VALUES OF b^2
XY	S-G	$S_\pm = e^{\pm i \frac{\tilde{\varphi}}{2b}}$	$O(2)$	BKT	$b^2 = 1$

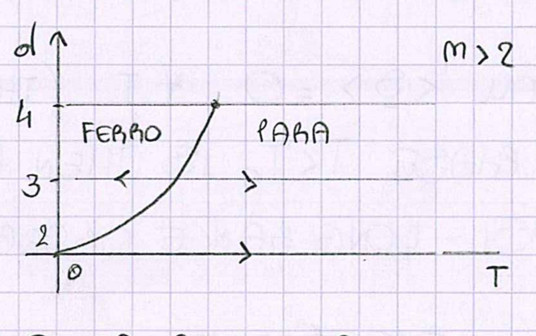
IN FACT, σ_1 AND σ_2 ARE 2 ISING MODELS:

$$\sigma_1 \leftrightarrow -\sigma_1 \;, \quad \sigma_2 \leftrightarrow \sigma_1 .$$

INSTEAD, $S_\pm$ IS A VECTOR.

· $\underline{O(m>2) \text{ IN } d\text{-DIMENSIONS}}$

WHAT HAPPENS TO THE FP WHEN WE GO DOWN TO $d=2$?



ONE OBSERVES (IN SIMULATIONS) A $T=0$ FP FOR $m>2$ WITH $X_\sigma = 0$, CONSISTENT WITH GAUSSIAN FP $\left(X_\varphi^G = \frac{d-2}{2}\right)$.

WE CAN THUS UNDERSTAND NON-TRIVIAL FP'S IN $d = 2+\varepsilon$ STARTING FROM THE GAUSSIAN FP.

$\star$ IN $d = 2+\varepsilon$, WE START WITH

NOTE: I THINK THIS ACTUALLY HAS TO BE A NONLINEAR SIGMA MODEL.

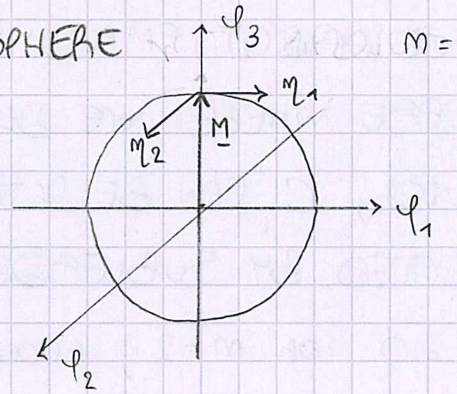
$$A = \frac{1}{\alpha}\int d^dx\, (\nabla \underline{\varphi})^2 \qquad \underline{\varphi} = (\varphi_1, \dots, \varphi_m).$$

WE ARE IN THE BROKEN PHASE WITH

$$\langle \underline{\varphi} \rangle = \underline{M}$$

SELECTED BY SSB ON THE HYPERSPHERE WITH RADIUS $|\underline{M}|$. WE CHOOSE

$$\underline{M} = (0, 0, \dots, M).$$



WE ARE INTERESTED IN FLUCTUATIONS IN THE DIRECTION TANGENT TO THE HYPERSPHERE, SO WE EXPRESS

$$\varphi_m = [M^2 - \underline{\eta}^2]^{1/2}$$

$$\underline{\eta} \equiv (\varphi_1, \dots, \varphi_{m-1})$$

AND

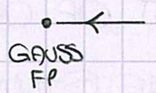
$$A \simeq \frac{1}{\alpha}\int d^dx \left\{ (\nabla \underline{\eta})^2 + \left(\nabla \sqrt{M^2 - \underline{\eta}^2}\right)^2 \right\}$$

$$\underset{\underline{\eta}\to 0}{\simeq} \frac{1}{\alpha}\int d^dx \left\{ (\nabla \underline{\eta})^2 - \frac{M^2}{4}\left(\nabla \frac{\underline{\eta}^2}{M^2}\right) + \dots \right\}$$

$$\underset{\substack{\underline{\eta}^2 \to \alpha \underline{\eta}^2 \\ M^2 \to \alpha M^2}}{\simeq} \int d^dx \left\{ (\nabla \underline{\eta})^2 - \frac{\alpha}{4M^2}(\nabla \underline{\eta}^2)^2 + \dots \right\}$$

WHICH GIVES A GAUSSIAN FIXED POINT IN $\underline{\eta}$ PLUS

$$A \equiv (\nabla \underline{\eta}^2)^2$$



$$X_A^G = 2\left(1 + 2\,\frac{d-2}{2}\right) = 2(d-1) = 2 + 2\varepsilon$$

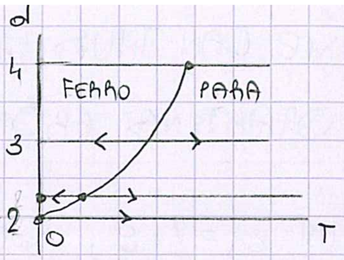
i.e. A IS SLIGHTLY IRRELEVANT.

WE FOUND

$T=0$ GAUSS FP + IRRELEVANT IN $d=2+\varepsilon$

WHENCE

$\alpha \sim T$.



*NOTE: IN XY, $T \sim b^2$, WHICH HAS NO ROLE IN PUSHING THE FLOW AWAY FROM THE GAUSSIAN FP.

ITS β-FUNCTION WILL BE

$$\beta_T = l\,\frac{\partial T}{\partial l} = \underbrace{(d - X_A^G)}_{=-\varepsilon} T + C T^2 + O(T^3, \varepsilon T^2)$$

BUT WE CANNOT EMPLOY OUR OPE FORMULA (WE ARE NOT IN THE CASE WHERE WE DERIVED IT).

NOTE: i.e. 1 SLIGHTLY RELEVANT FIELD.

HOWEVER, C CAN BE DETERMINED IN $d=2$ (IT WILL GET CORRECTED BY SUBLEADING TERMS). IN $d=2$ WE KNOW THAT

$$\begin{cases} C = 0 & \text{FOR } m=2, \text{ BECAUSE}^* \ \beta_T = 0 \text{ IN XY} \\ C > 0 & \text{FOR } m>2, \text{ BECAUSE } T \text{ IS RELEVANT FOR } m>2. \end{cases}$$

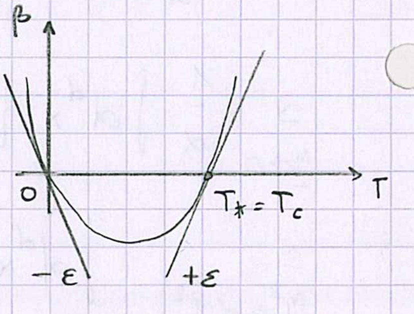
NOTE: THIS WAS AN EMPIRICAL FAC

WE DEDUCE

$$C = (m-2)\,a \qquad a > 0$$

AND WE GOT

$$\beta_T = -\varepsilon T + (m-2)\,a T^2 + O(T^3, T^2\varepsilon).$$

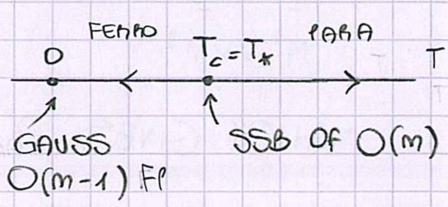


THERE IS A NONTRIVIAL FP AT

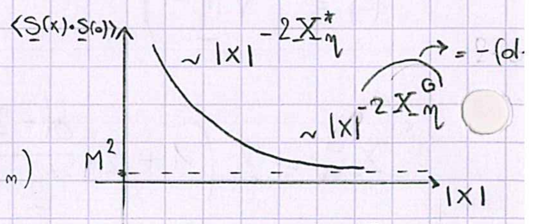
$$T_* = \frac{\varepsilon}{(m-2)\,a}$$

AND AS USUAL

$$\left.\frac{\partial \beta_i}{\partial \lambda_i}\right|_{\lambda_*} = d - X_i^*.$$



WE OBSERVE A SORT OF CROSSOVER FLOW, BUT IN THIS CASE THE SYMMETRY DOESN'T CHANGE. THIS IS DUE INSTEAD TO $\underline{\eta} = (\eta_1, \dots, \eta_m)$ (GOLDSTONE BOSONS), WHICH ARISE IN THE SB OF CONTINUOUS SYMMETRIES.



QUANTUM FIELD THEORY

28.11.19

CONSIDER, IN 2-DIMENSIONAL EUCLIDEAN SPACE,

$x = (x_1, x_2)$.

THE DISTANCE $x_1^2 + x_2^2$ IS INVARIANT UNDER ROTATIONS

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

EUCLIDEAN FIELD THEORY IN $d=2$.

NOW LET'S INTRODUCE THE ANALYTIC CONTINUATION (WICK ROTATION)

$x_2 = it$

$x \equiv (x_1, t)$

WHERE $x_1^2 - t^2$ IS INVARIANT UNDER RELATIVISTIC TRANSFORMATIONS

$$\begin{pmatrix} \mathrm{ch}\,\Lambda & \mathrm{sh}\,\Lambda \\ \mathrm{sh}\,\Lambda & \mathrm{ch}\,\Lambda \end{pmatrix} \begin{pmatrix} x_1 \\ t \end{pmatrix}$$

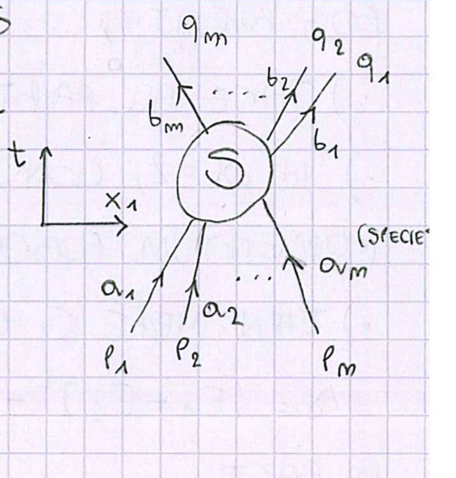
QUANTUM FIELD THEORY IN $d=1+1$

THIS IS MINKOWSKI SPACE.

THE QUANTUM VERSION NATURALLY YIELDS A PARTICLE DESCRIPTION:

- GROUND STATE $\leftrightarrow$ VACUUM STATE $|0\rangle$
- PARTICLES DESCRIBE EXCITATION MODES OVER $|0\rangle$.

WE WILL THEN CONSIDER SCATTERING EVENTS LIKE THE ONE HERE REPRESENTED PICTORIALLY.



DEFINE THE SCATTERING AMPLITUDE

$${}_{b_1 \dots b_m}\langle q_1 \dots q_m | S | p_1 \dots p_m \rangle_{a_1 \dots a_m}$$

WHICH IS AN ELEMENT OF THE S-MATRIX:

$$\mathrm{Prob}(i \to f) = |S_{i\to f}|^2$$

$$\sum_f |S_{i\to f}|^2 = 1 \quad \Rightarrow \quad SS^\dagger = 1.$$

NOTE: $\sum_f |S_{if}|^2 = \sum_f S_{if} S^*_{if} = \sum_f S_{if} (S^\dagger)_{fi}$.

PARTICLES SATISFY THE RELATIVISTIC DISPERSION RELATION

$$E^2 = p^2 + m_a^2 .$$

DEFINE ALSO

$$\begin{cases} H = \text{HAMILTONIAN OPERATOR OF QUANTUM SYSTEM IN } d = 1+1 \\ \mathbb{P} = \text{MOMENTUM OPERATOR.} \end{cases}$$

WHAT WE CALLED $p_1 .. p_n$ ARE REALLY ASYMPTOTIC STATES, FAR AWAY FROM EACH OTHER (THEY DO NOT INTERACT). THEY ARE EIGENSTATES OF BOTH

$$\begin{cases} H\,|p_1, \dots, p_n\rangle = \left(\sum_{i=1}^{n} E_i\right)|p_1, \dots, p_n\rangle \\ \mathbb{P}\,|p_1, \dots, p_n\rangle = \left(\sum_{i=1}^{n} p_i\right)|p_1, \dots, p_n\rangle . \end{cases}$$

THEY FORM A COMPLETE SET, IN THE SENSE THAT

$$I = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{dp_1}{2\pi E_1} \dots \frac{dp_n}{2\pi E_n} |p_1 .. p_n\rangle\langle p_1 ... p_n| \equiv \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|$$

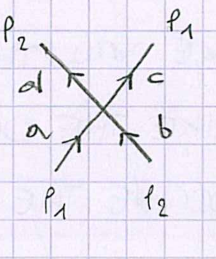
WHICH CORRESPONDS TO THE NORMALIZATION

$$\langle p_1 | p_2 \rangle = 2\pi E_1 \, \delta(p_1 - p_2) .$$

NOTE: IT COINCIDES WITH THE COVARIANT NORMALIZATION USED BY TESTA IN QFT.

* THE SIMPLEST SCATTERING EVENT IS A 2-BODY SCATTERING.

FOR BREVITY,

$S_{ab}^{cd}(s) \equiv$ 

·) TAKE ALL PARTICLES WITH MASS m.

·) IN $d = 2$, CONSERVATION OF ENERGY AND MOMENTUM FORCES FINAL MOMENTA TO COINCIDE WITH INITIAL ONES.

·) THEN THERE IS ONLY ONE RELATIVISTIC INVARIANT:

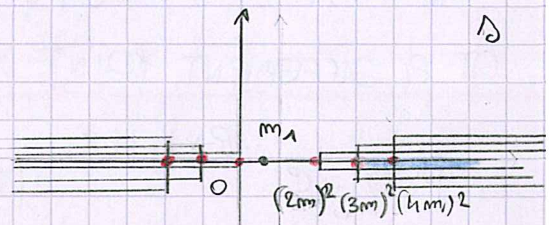
$$s = (E_1 + E_2)^2 - (p_1 + p_2)^2 \quad \Rightarrow \quad (\text{CENTRE OF MASS ENERGY})^2 .$$

IN FACT,

$\begin{pmatrix} E \\ p \end{pmatrix}$ TRANSFORMS AS $\begin{pmatrix} t \\ x \end{pmatrix}$.

· PROPERTIES OF THE AMPLITUDE

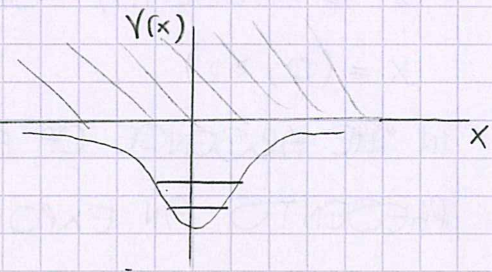
IF YOU AREN'T FAMILIAR WITH THEM IN THE NON-RELATIVISTIC LIMIT, YOU CAN TAKE THEM AS AXIOMS.



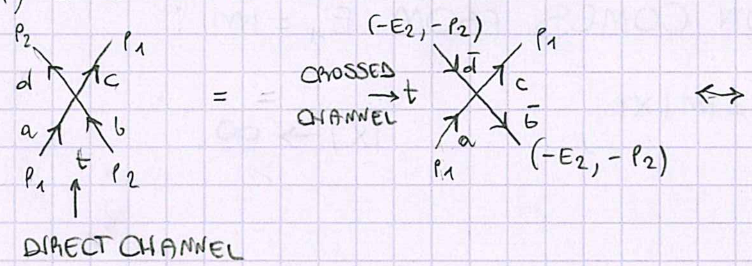
1) THE AMPLITUDE IS AN ANALYTIC FUNCTION OF THE (FORMALLY COMPLEX) VARIABLE s, UP TO SINGULARITIES THAT HAVE A PHYSICAL MEANING.

2) THE OPENING OF SCATTERING CHANNELS CORRESPONDS TO BRANCH POINTS (•). YOU NEED AN ENERGY $(2m)^2$ TO SCATTER, $(3m)^2$ TO PRODUCE A 3^{RD} PARTICLE, $(4m)^2$ TO OBTAIN 4 PARTICLES AS FINAL STATES, AND SO FORTH.

3) BOUND STATES CORRESPOND TO POLES. AS IN QM, THEY LIE BELOW THE "ZERO ENERGY" LINE.



4) CROSSING SYMMETRY:



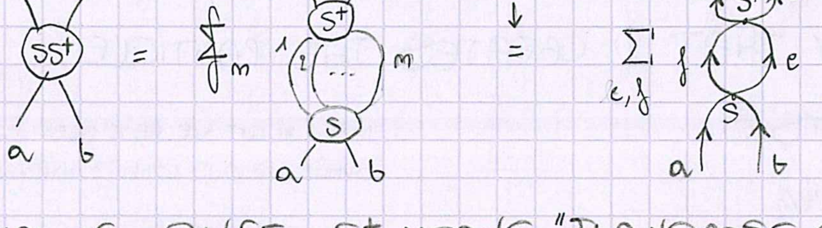
$\leftrightarrow$ NOTE: $= T^{\bar{b}c}_{\bar{d}a}(s+i\varepsilon)$.

$$S^{cd}_{ab}(s+i\varepsilon) = S^{\bar{b}c}_{\bar{d}a}(4m^2-(s+i\varepsilon$$

$s \in \mathbb{R}$

WHERE $\bar{b}$ MEANS "CHARGE CONJUGATION". THIS HAS CLEARLY A CONSEQUENCE ON THE ANALYTIC STRUCTURE OF $S(s)$.

5) UNITARITY: $SS^+ = I$. SUPPOSING ELASTIC SCATTERING,



THAT IS, SINCE S^+ MEANS "TRANSPOSE AND CONJUGATE",

$$\sum_{e,f} S^{ef}_{ab}(s+i\varepsilon) \cdot \left[S^{ef}_{cd}(s+i\varepsilon) \right]^* = \delta_{ac}\,\delta_{bd}.$$

6) REAL ANALYTICITY: (ABOVE & BELOW THE CUT) $S^{cd}_{ab}(s+i\varepsilon) = \left[S^{cd}_{ab}(s-i\varepsilon) \right]^*$.

· FORM FACTORS

$$F_m^{\phi}(p_1, \dots, p_m) \equiv \langle 0 | \phi(0) | p_1, \dots, p_m \rangle .$$

AT A DIFFERENT POINT X,

NOTE: CALLING $x_2 = it$, WE GET THE USUAL
$\phi(x) = e^{ipx}\phi(0)e^{-ipx}$
$x = (x_1, t)$, $p = (p, H)$, $(\cdot,\cdot) = \begin{pmatrix} - & \\ & + \end{pmatrix}$.

$$\phi(x) = e^{-ipx_1 + Hx_2}\,\phi(0)\,e^{ipx_1 - Hx_2}$$

(IN EUCLIDEAN COORDINATES). CORRELATION FUNCTIONS READ

$$\langle \phi(x)\phi(0) \rangle = \langle 0 | \phi(x)\phi(0) | 0 \rangle = \langle 0 | \phi(x) \sum_m |m\rangle\langle m| \phi(0) | 0 \rangle$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \int \frac{dp_1}{2\pi E_1} \dots \frac{dp_m}{2\pi E_m} |F_m^{\phi}(p_1, \dots, p_m)|^2 \, e^{-|x| \sum_{i=1}^{m} E_i} .$$

FOR ϕ SCALAR, SINCE IT ONLY DEPENDS ON $|x|$, WE HAVE CHOSEN

$x = (0, x_2)$.

IN THE ABSENCE OF ANTIPARTICLES, $E_i \geq 0 \;\forall i$ AND THE CORRELATOR PRESENTS AN EXPONENTIAL DECAY. SINCE $E_1 < E_2 < \dots < E_m$, THEN THE LEADING CONTRIBUTION COMES FROM $E_1 = m$:

$$\langle \phi(x)\phi(0) \rangle - \langle \phi \rangle^2 \sim e^{-m|x|} , \qquad |x| \to \infty .$$

THIS JUSTIFIES

$$\underline{m = 1/\xi} .$$

★ CONSIDER THE 1-PARTICLE FORM FACTOR

$$F_1^{\phi}(p) = \langle 0 | \phi(0) | p \rangle .$$

IF $F_1^{\phi}(p) \neq 0$, WE SAY THAT ϕ CREATES THE PARTICLE.

BY DIMENSIONAL ANALYSIS,

NOTE: IN QFT WE HAVE SEEN $F_1^{\phi}(p)$ FOR A SCALAR FIELD IS LORENTZ INVARIANT => NO p DEPENDS

$$F_1^{\phi}(p) = m^{X_\phi} \left(\frac{E+p}{E-p} \right)^{S_\phi/2} C_\phi$$

C_ϕ ADIMENSIONAL

(COMPARE WITH $(z/\bar{z})^{S_\phi/2}$). NOTICE THAT

$F_1^{\phi}(p) = $ CONST. FOR ϕ SCALAR $\quad (S_\phi = 0)$.

- SCATTERING AT FP'S ($d = 1+1$)

AT FP'S WE DEAL WITH MASSLESS PARTICLES:

$$E^2 = p^2 \quad \Rightarrow \quad \begin{cases} p = E > 0 & \text{RIGHT-MOVER} \\ p = -E < 0 & \text{LEFT-MOVER.} \end{cases}$$

SINCE

$$(E+p)(E-p) = m^2$$

NOTE: $\langle 0|\phi(x)|p\rangle \propto \begin{cases} e^{ipz} & \text{R-M} \\ e^{ip\bar{z}} & \text{L-M.} \end{cases}$

WE CAN REWRITE

$$\langle 0|\phi(0)|p\rangle = C_\phi (E+p)^{\Delta_\phi} (E-p)^{\bar{\Delta}_\phi}$$

WHICH IS CLEARLY NULL (FOR $m=0$) IF BOTH

$\Delta_\phi, \bar{\Delta}_\phi \neq 0$.

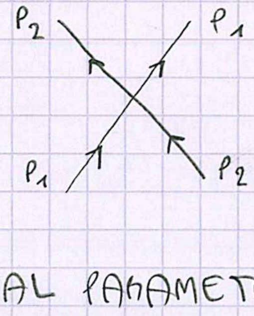
WE DEDUCE THAT PARTICLES CAN ONLY BE CREATED BY FIELDS WITH $\Delta_\phi = 0$ OR $\bar{\Delta}_\phi = 0$. IN PARTICULAR,

$$\begin{cases} \text{RIGHT-MOVERS ARE CREATED BY } \eta(z) \\ \text{LEFT-MOVERS ARE CREATED BY } \bar{\eta}(\bar{z}). \end{cases}$$

$\eta(z)$, $\bar{\eta}(\bar{z})$ ARE CALLED <u>CHIRAL FIELDS</u>, AND

$$\Delta_\eta = \bar{\Delta}_{\bar{\eta}} \qquad \bar{\Delta}_\eta = \Delta_{\bar{\eta}} = 0.$$

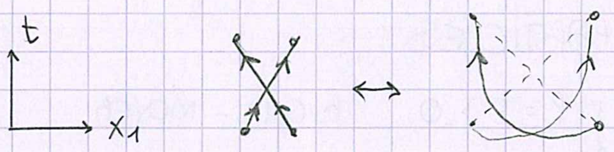
☆ ∞-DIMENSIONAL CONFORMAL SYMMETRY FORCES ELASTIC SCATTERING. THIS IS DUE TO THE ∞ CONSERVATION LAWS IT IMPLIES. THIS MEANS <u>S IS A PHASE</u>.

$S =$ 

☆ IN A CONFORMAL THEORY THERE ARE NO DIMENSIONAL PARAMETERS, BUT $\mathcal{J}$ IS DIMENSIONFUL. THIS MEANS <u>S IS A CONSTANT</u> (PHASE). THIS CANNOT REPRESENT A DYNAMICAL INTERACTION; RATHER, IT INCORPORATES INFORMATION ABOUT THE STATISTICS OF PARTICLES. S IS A STATISTICAL PHASE:

$$S = \begin{cases} 1 & \text{FOR BOSONS} \\ -1 & \text{FOR FERMIONS} \\ \text{ELSE} & \text{FOR PARAFERMIONS.} \end{cases}$$

THIS CAN BE UNDERSTOOD AS FOLLOWS:



$$S = e^{-i\pi(S_\eta - S_{\bar\eta})} = e^{-2\pi i \Delta_\eta}.$$

NOTE: WE ARE USING $\Delta_\eta = \bar\Delta_{\bar\eta}$. MOREOVER, SINCE $\Delta_{\bar\eta} = \bar\Delta_\eta = 0$, IT FOLLOWS THAT

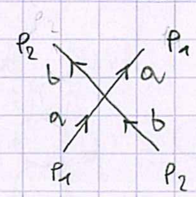
$$S_\varphi = \Delta_\varphi - \bar\Delta_\varphi \qquad \Rightarrow \qquad \Delta_\eta = S_\eta = -S_{\bar\eta}.$$

IN ABSENCE OF DYNAMICAL INTERACTIONS, THE PASSAGE FROM THE INITIAL TO THE FINAL STATE CAN ALSO BE REALIZED BY π ROTATIONS RULED BY THE EUCLIDEAN SPIN.

· RECAP: SCATTERING AT FP'S 03.12.19

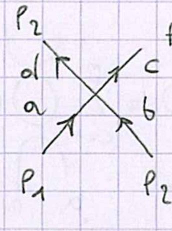
LAST TIME WE HAVE WRITTEN

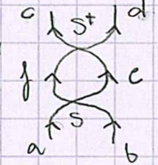
$$S = e^{-2i\pi\Delta_\eta}$$



FOR THE ELASTIC PROCESS ON THE RIGHT.

NOW WE CAN GENERALIZE TO THE CASE WHERE DIFFERENT PARTICLES ARE OBTAINED. WE GET

 $= S_{ab}^{cd}$

$\sum_{ef}$  $= \delta_{ac}\delta_{bd}$

NOTE: THIS IS THE UNITARITY EQUATION AS WE FIRST STATED IT.

$$\sum_{ef} S_{ab}^{ef}\left[S_{cd}^{ef}\right]^* = \delta_{ac}\delta_{bd}$$

UNITARITY.

NOTE: REAL ANALITICITY CANNOT BE USED ON ANY ELEMENT. KEEP TRACK OF THE $\pm i\varepsilon$ BEFORE SENDING PARTICLE MASSES TO ZERO.

SIMILARLY, USING REAL ANALITICITY,

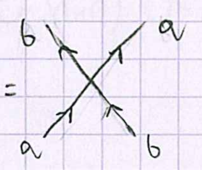
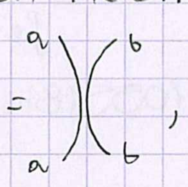
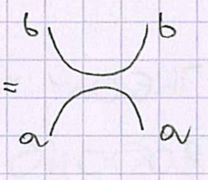
$$S_{ab}^{cd} = \left[S_{\bar{d}a}^{\bar{b}c}\right]^*$$

CROSSING.

NOTE: WE AREN'T WRITING THE ARGUMENT OF $S(\theta)$ ANYMORE, BECAUSE S=CONST. AT FP'S.

· O(m) SYMMETRY

PARTICLES IN A VECTOR MULTIPLET, LABELLED BY $a = 1, \dots, m$.

$S_2 =$  , $S_3 =$  , $S_1 =$  $(d=2)$

WE HAVE TO BUILD THESE 3 AMPLITUDES* USING THE EQUATIONS ABOVE. USING CROSSING,

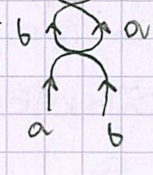
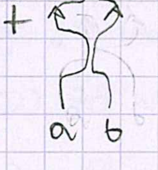
NOTE: O(m) CONTAINS NEUTRAL PARTICLES.

$$\begin{cases} S_1 = S_3^* \equiv \rho_1 e^{i\varphi} \\ S_2 = S_2^* \equiv \rho_2 \end{cases}$$

$\rho_1 \geq 0$

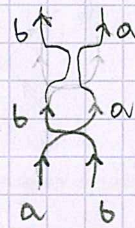
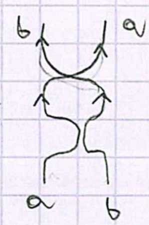
$\rho_2 \in \mathbb{R}$.

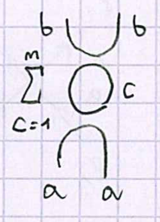
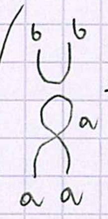
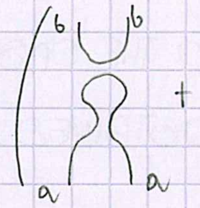
TO IMPLEMENT UNITARITY, WE HAVE FIRST OF ALL

$1 =$  $+$  $= S_2 S_2^* + S_3 S_3^* = \rho_2^2 + \rho_1^2$. (I

*NOTE: THEY ARE ANNIHILATION/CREATION (S_1), TRANSMISSION (S_2) AND REFLECTION (S_3).

SIMILARLY, CONSIDERING THE CASES WHERE $\delta_{ac}\,\delta_{bd}$ ARE NOT SATISFIED,

$0 =$  $+$  $= S_2 S_3^* + S_3 S_2^* = 2\rho_1\rho_2\cos\varphi \qquad (\mathrm{II})$

$0 = \sum_{c=1}^{m}$  $+ \Big($  $+ \text{c.c.}\Big) + \Big($  $+ \text{c.c.}\Big)$

$$= m|S_1|^2 + (S_2 S_1^* + \text{c.c.}) + (S_3 S_1^* + \text{c.c.})$$

$$= m\rho_1^2 + \underbrace{2\rho_1\rho_2\cos\varphi}_{=0 \text{ BY (II)}} + 2\rho_1^2\cos 2\varphi . \qquad (\mathrm{III})$$

NOTE: $= \rho_1^2\,[m - 2 + 4\cos^2\varphi]$.

WE NOTICE A FACTOR m IS ASSOCIATED WITH EACH LOOP.

★ THERE ARE 3 WAYS OF SATISFYING EQUATION (II):

i) $\cos\varphi = 0 \quad \overset{(\mathrm{III})}{\Longrightarrow} \quad m = 2$.

THIS WILL BE RECONSIDERED IN A MOMENT.

ii) $\rho_1 = 0 \quad \overset{(\mathrm{I})}{\Longrightarrow} \quad \rho_2 = S_2 = \pm 1$. (AND $S_1 = S_3 = 0$).

THIS IS A FREE THEORY WITH 2 POSSIBILITIES:

$S_2 = 1$: FREE BOSONS

$S_2 = -1$: FREE FERMIONS.

$\forall m$

(iii) $\rho_2 = 0 \quad \overset{(\mathrm{I})}{\Longrightarrow} \quad \rho_1 = 1$.

THEN, BY (III),

$$m = -2\cos 2\varphi \in (-2, 2).$$

HERE WE EXPECT TO FIND SELF-AVOIDING WALKS AT $m = 0$. MOREOVER,

$$S_2 = 0$$

MEANING NO INTERSECTION OF PATHS.

*NOTE: "it is not difficult to check that the superposition of two-particle states $\sum_{a_i}' a_i a_i$ scatters into itself with amplitude S" (DELFINO J. STAT. MECH. (2013) 024001).

★ IN THE PRESENCE OF MANY SPECIES, A STATE CAN BE CONSTRUCTED* s.t.

$$\sum_a aa \to S \sum_a aa \qquad S = mS_1 + S_2 + S_3$$

i.e. IT SCATTERS INTO ITSELF: BY UNITARITY, S MUST BE A PHASE.

FOR SOLUTION (iii),

$$S = m S_1 + S_1^*$$

NOTE: $S_3 = S_1^*$ AND $S_2 = \rho_2 = 0$.

$$= -2\cos 2\varphi \, e^{i\varphi} + e^{-i\varphi} = -e^{3i\varphi}.$$

SINCE WE HAVE BASICALLY DIAGONALIZED THE PROCESS, WE IDENTIFY

$$S = e^{-3i\varphi} = e^{-2i\pi\Delta_\eta} \qquad (\eta \text{ CHIRAL FIELD}).$$

THIS GIVES Δ_η AS A FUNCTION OF m.

FOR $m = 1$ IN (iii),

NOTE: $m = -2\cos 2\varphi$.

$$\varphi = \frac{2\pi k}{3}, \; k = 1,2 \pmod 3 \quad \Rightarrow \quad S = -1$$

(WHICH WE EXPECTED, BECAUSE $m = 1$ IS THE ISING MODEL: THIS SHOWS THAT IN $d = 2$ ISING IS A FREE FERMION.

* LET'S GO BACK TO (i), WHERE $m = 2$ AND

$$\begin{cases} \rho_1^2 + \rho_2^2 = 1 \\ 1 + \cos 2\varphi = 0 \\ \cos\varphi = 0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} \rho_1 = \sin\alpha \\ \rho_2 = \cos\alpha \\ \varphi = -\frac{\pi}{2} \end{cases}$$

WHICH AGAIN IS THE LINE OF FIXED POINTS PARAMETERIZED BY α (AT $m = 2$). WE GET IN FACT

NOTE: $S = \cos\alpha - i\sin\alpha$.

$$S = 2S_1 + S_2 + S_3 = e^{-i\alpha} \equiv e^{-2i\pi\Delta_\eta}$$

WHERE η MUST BE THE MOST RELEVANT CHIRAL FIELD LOCAL WITH RESPECT TO

$$\varepsilon = \cos 2b\varphi \quad \Rightarrow \quad \Delta_\eta = \frac{1}{4b^2}$$

WHENCE

$$\underline{\alpha = \frac{\pi}{2b^2}}.$$

NOTE: $\Delta_\varepsilon = b^2$ AND WE CAN USE THE USU[AL] ARGUMENTS OF LOCALITY, WITH $\bar{p} = 0$ BECAUS[E] η IS CHIRAL.

INDEED, FOR $b^2 = 1$ WE GET $\alpha = \frac{\pi}{2}$, WHICH MEANS FREE FERMIONS.

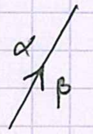
WE THEN RECOVER THE FP'S WE KNOW FOR $O(m)$ LATTICE MODELS IN $d=2$:

- ·) NON-TRIVIAL FP FOR $m<2$ (WHICH INCLUDES SAW'S FOR $m=0$) FROM SOLUTION (iii).
- ·) $T=0$ GAUSSIAN FP FOR $m=2$ FROM SOLUTION (ii). (MI SA $m\leq 2$)
- ·) BKT LINE OF FP'S FOR $m=2$ FROM SOLUTION (i).

· q-STATE POTTS MODEL

WE HAVE SEEN IT ENJOYS S_q SYMMETRY (EXCHANGE OF q COLORS). HOW DO WE IMPLEMENT IT IN THIS LANGUAGE?

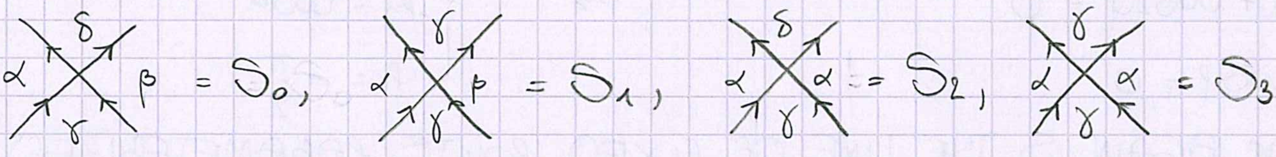
FIRST OF ALL, WHAT IS A GOOD PARTICLE BASIS? IT TURNS OUT ONE SHOULD CONSIDER



$\alpha,\beta=1,2,\dots,q$

$\alpha\neq\beta$.

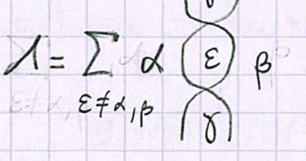
IN FACT, A SCATTERING PROCESS LIKE ONE OF THESE



SEPARATES THE PLANE IN 4 DIFFERENT REGIONS: WE ASSOCIATE A COLOR TO EACH OF THEM (CONFINING REGIONS MUST HAVE DIFFERENT COLORS). CROSSING EQUATIONS THEN READ

$$\begin{cases} S_0 = S_0^* \equiv \rho_0 \in \mathbb{R} \\ S_1 = S_2^* \equiv \rho\, e^{i\varphi} \\ S_3 = S_3^* \equiv \rho_3 \in \mathbb{R} \end{cases}$$

ON WHICH WE CAN APPLY THE UNITARITY EQUATION:



$$1=\sum_{\varepsilon\neq\alpha,\beta} \; = \underset{(\varepsilon=\gamma)}{|S_1|^2} + \underset{(\varepsilon\neq\gamma)}{(q-3)|S_0|^2} = \rho^2+(q-3)\rho_0^2 .$$

NOTE: THE SECOND CASE IS REALLY $\varepsilon\neq\gamma,\alpha,\beta$. THERE ARE $(q-3)$ SUCH CHOICES OF ε.

SIMILARLY, SINCE HERE KRONECKER δ'S ARE NOT SATISFIED,

$$0 = \sum_{\alpha \neq \alpha, \beta}' \alpha \begin{vmatrix} \delta \\ \varepsilon \\ \gamma \end{vmatrix} \beta = \underset{(\varepsilon = \gamma)}{S_1 S_0^*} + \underset{(\varepsilon = \delta)}{S_0 S_1^*} + \underset{(\varepsilon \neq \gamma, \delta)}{(q-4)|S_0|^2}$$

$$= 2\rho_0 \rho_1 \cos\varphi + (q-4)\rho_0^2$$

$$1 = \sum_{\varepsilon \neq \alpha} \alpha \begin{vmatrix} \gamma \\ \varepsilon \\ \gamma \end{vmatrix} \alpha = \rho_3^2 + (q-2)\rho^2$$

$$0 = \sum_{\varepsilon \neq \alpha} \alpha \begin{vmatrix} \delta \\ \varepsilon \\ \delta \end{vmatrix} \alpha = (q-3)\rho^2 + 2\rho\rho_3 \cos\varphi .$$

THERE ARE SEVERAL SOLUTIONS, AMONG WHICH

$$\rho_0 = -1, \quad \rho = \sqrt{4-q}, \quad 2\cos\varphi = -\sqrt{4-q}, \quad \rho_3 = q-3 \qquad (\text{IV})$$

WHICH ARE FIXED POINTS DEFINED FOR $q \in (0,4)$.

THIS IS THE ONLY SOLUTION WHICH INCLUDES $q = 2,3,4$:

$q=2$: ISING MODEL

$q=4$: ASHKIN-TELLER MODEL (IN A SPECIFIC CASE).

IN A-T, WE HAVE SEEN σ_1, σ_2 HAVE

$$X_{\sigma_1} = X_{\sigma_2} = \frac{1}{8} \quad \forall b^2$$

BUT THEIR PRODUCT CAN BE BOSONIZED:

$$\sigma_1 \sigma_2 = \sin b\varphi \qquad X_{\sigma_1\sigma_2} = \frac{b^2}{4\,(\text{o } 2?)} \propto b^2$$

SO THERE WILL BE A VALUE OF b^2 S.t.

$$X_{\sigma_1\sigma_2} = \frac{1}{8}$$

AND FOR THIS PARTICULAR VALUE WE GET THE 4-STATES POTTS MODEL.

NOTE: SEE FOCUS IN 2 PAGES.

NOTICE SOLUTION (IV) HINTS AT THE FACT THAT THE TRANSITION BECOMES 1ST ORDER (i.e. NO FIXED POINTS) FOR $q > 4$: THIS HAS BEEN RIGOROUSLY PROVED BY BAXTER, BUT INDEED IT IS THE SIMPLEST THING WHICH CAN HAPPEN (ANYTHING ELSE WOULD BE ARTIFICIAL).

* AGAIN, WE CAN CONSTRUCT A STATE WHICH SCATTERS INTO ITSELF,

$$\sum_{\gamma\neq\alpha}' (\alpha\gamma)(\gamma\alpha) \rightarrow S \sum_{\gamma\neq\alpha} (\alpha\gamma)(\gamma\alpha)$$

WHERE AGAIN S MUST BE A PHASE, AND IT TURNS OUT TO BE

$$S = (q-2) S_2 + S_3 = e^{-4i\varphi} = e^{-2i\pi\Delta_\eta}$$

WHICH GIVES Δ_η AS A FUNCTION OF q.

* WE LOOK FOR $O(m)$ WITH $m \leq 2$, AND POTTS WITH $q \leq 4$: BOTH ARE GROUPS OF THEORIES WITH CENTRAL CHARGE

$c \leq 1$

($c=1$ WAS THE GAUSSIAN THEORY, AT $m=2$ OR $q=4$).

BUT THEN WE KNOW THAT THERE ARE DEGENERATE FIELDS WITH

$$\Delta_{m,n} = \frac{[(p+1)m - pn]^2 - 1}{4p(p+1)} \qquad c = 1 - \frac{6}{p(p+1)}, \quad p \in \mathbb{R}.$$

SINCE

$\Delta_\varepsilon < 1$

$\Delta_\varepsilon = \frac{1}{2}$ AT $c = \frac{1}{2}$ $(p=3)$

THERE ARE TWO CANDIDATES:

$$\Delta_\varepsilon = \begin{cases} \Delta_{2,1} & \text{POTTS} \\ \Delta_{1,3} & O(m) \end{cases}$$

BY DUALITY ARGUMENTS, ONE CAN IN FACT DECIDE WHICH ONE IS WHICH.

WE LOOK FOR η AS THE MOST RELEVANT CHIRAL FIELD LOCAL WITH RESPECT TO ε, USING CONFORMAL OPE: ONE WOULD FIND

$$\Delta_\eta = \begin{cases} \Delta_{1,3} & \text{POTTS} \\ \Delta_{2,1} & O(m) \end{cases} \rightsquigarrow \begin{cases} c(q) \\ c(m). \end{cases}$$

THIS IS THE ONLY HUMAN WAY OF SOLVING THIS PROBLEM.

WE JUST SUMMARIZE THE RESULTS OF SUCH A CALCULATION:

SYMMETRY	c	Δ_ε	Δ_η	Δ_σ	
$O(n),\ n = 2\cos\frac{\pi}{p}$	$1 - \frac{6}{p(p+1)}$	$\Delta_{1,3}$	$\Delta_{2,1}$	$\Delta_{\frac{p-1}{2},\frac{p+1}{2}}$	
$S_q,\ q = 4\sin^2\frac{\pi(p-1)}{2(p+1)}$	$1 - \frac{6}{p(p+1)}$	$\Delta_{2,1}$	$\Delta_{1,3}$	$\Delta_{\frac{p-1}{2},\frac{p+1}{2}}$	⊛

(WE DIDN'T SHOW THE WAY Δ_σ CAN BE CALCULATED). IN PARTICULAR, THE EXPONENTS FOR PERCOLATION IN $d=2$ ARE FOUND IN ⊛:

$$X_\varepsilon = \frac{5}{4}, \quad X_\sigma = \frac{5}{48}$$

$$\nu = \frac{4}{3}, \quad \alpha = -\frac{2}{3}, \quad \beta = \frac{5}{36}, \quad \gamma = \frac{43}{18}.$$

· FOCUS: A-T AND 4-POTTS

$$H_{A-T} = -\sum_{\langle xy\rangle}\left\{J\left[\sigma_1(x)\sigma_1(y) + \sigma_2(x)\sigma_2(y)\right] + J_4\,\sigma_1(x)\sigma_1(y)\sigma_2(x)\sigma_2(y)\right\}.$$

FOR $J = J_4$,

$$H = -J\sum_{\langle x,y\rangle}\left\{\underline{S}(x)\cdot\underline{S}(y) + \frac{1}{2}\left(\underline{S}(x)\cdot\underline{S}(y)\right)^2 - 1\right\} \equiv \sum_{\langle x,y\rangle} H_{xy}$$

WHERE WE DEFINED THE VARIABLE

$$\underline{S}(x) \equiv \left(\sigma_1(x), \sigma_2(x)\right).$$

THIS CAN ASSUME FOUR POSSIBLE VALUES $(1,1), (1,-1), (-1,1), (-1,-1)$. BY INSPECTION

$$H_{xy} = -J\cdot(A+B+AB) = \begin{cases} -3J, & S_x = S_y \\ J, & S_x \neq S_y \end{cases}$$

$$\begin{array}{l} S_x = (\sigma_1(x), \sigma_2(x)) \\ S_y = (\sigma_1(y), \sigma_2(y)) \\ \qquad\ \ A \qquad\quad B \end{array}$$

WHICH IS, INDEED, A 4-STATES POTTS.

A SIMILAR GENERALIZATION HOLDS FOR THE NON-ISOTROPIC ASHKIN-TELLER (SEE BAXTER P. 353 OR C. FAN 1972).

EXACT SOLVABILITY AWAY FROM FP'S IN $d=2$

05.12.19

(INTEGRABILITY)

NOTE: $\partial_\mu T_{\mu\nu} \to \begin{cases} \bar{\partial} T + \frac{1}{4}\partial\Theta = 0 \\ \partial\bar{T} + \frac{1}{4}\bar{\partial}\Theta = 0 . \end{cases}$
BUT UCCI SAYS $T, \bar{T}$ DON'T TALK TO EACH OTHER.

WE ALWAYS HAVE CONSERVATION OF ENERGY AND MOMENTUM, i.e.

$$\bar{\partial} T \propto \partial\Theta .$$

ADDITIONAL CONSERVATION LAWS SHOULD BE OF THE FORM

$$\bar{\partial} T_{s+1} = \partial\Theta_{s-1} \qquad (\mathrm{I})$$

WHERE $s \pm 1$ ARE THE SPINS $(s = \Delta - \bar{\Delta})$ OF SUCH FIELDS.

IN FACT, $\bar{\partial}$ HAS SPIN -1 AND ∂ HAS SPIN 1. WE CAN REGARD

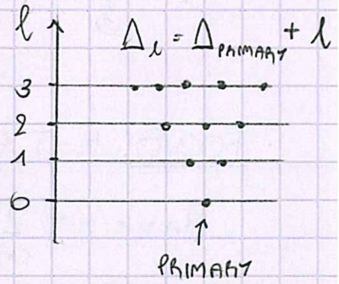
$$s = 1$$

NOTE: $\partial \to (1,0)$, $\bar{\partial} \to (0,1)$.

AS THE TRIVIAL CONSERVATION LAW, WHICH IS ALWAYS PRESENT.

★ AT FP'S, THERE ALWAY EXIST T_{s+1} WITH

$$(\Delta, \bar{\Delta}) = (s+1, 0)$$



BECAUSE THEY ARE DESCENDANTS OF THE IDENTITY.

SINCE $\bar{\Delta} = 0$, THEN

$$\bar{\partial} T_{s+1} = 0 . \qquad (\mathrm{II})$$

NOTE: $\Delta_{\mathbb{1}} = \bar{\Delta}_{\mathbb{1}} = 0$.

NOTE: WE ARE TAKING ϕ SCALAR, SO $\Delta_\phi = \bar{\Delta}_\phi$. BUT $X_\phi = \Delta_\phi + \bar{\Delta}_\phi = 2\Delta_\phi$, AND WE WANT $X_\phi < d$ IN ORDER FOR ϕ TO BE RELEVANT.

$$\Delta_\phi = \bar{\Delta}_\phi < 1 .$$

★ AWAY FROM A FP, CONSIDER

$$A = A_{FP} + g \int d^2x \, \phi(x)$$

IN GENERAL, (II) DOESN'T HOLD ANYMORE AND WE HAVE INSTEAD

$$\bar{\partial} T_{s+1} = \sum_{m=1}^{N} g^m A_s^{(m)}$$

NOTE: A HAS $(0,0)$, AND d^2x HAS $(-1,-1)$. IN FACT z HAS $(-1,0)$ AND $\bar{z}$ HAS $(0,-1)$. JESUS, I WILL NEVER LEARN THEM.

WHERE g HAS $(1-\Delta_\phi, 1-\Delta_\phi)$, BY COMPARISON WITH A.

THIS IMPLIES THAT $A_s^{(m)}$ HAS

$$\left(s+1-m(1-\Delta_\phi),\ 1-m(1-\Delta_\phi)\right).$$

NOTE: CORRELATION FUNCTIONS SCALE AS $\sim z^{-\Delta} \bar{z}^{-\bar{\Delta}}$ SO IF $\Delta, \bar{\Delta} < 0$ THEY DIVERGE!

WE DEDUCE THAT N SHOULD BE FINITE TO AVOID NEGATIVE DIMENSIONS, AND THAT $\Delta_{A_s^{(m)}}$ CAN BE IN THE SPECTRUM OF A GENERIC THEORY ONLY FOR $m=1$.

NOTE: IN FACT, $\Delta_\phi < 1$.

INDEED, $A_s^{(1)} \equiv A_s$ HAS $(\Delta_{\phi+s}, \Delta_\phi)$ AND IS DESCENDANT OF ϕ (IT IS IN THE SPECTRUM).

ALL OF THIS LEADS US TO

$$\bar{\partial} T_{s+1} = g A_s .$$

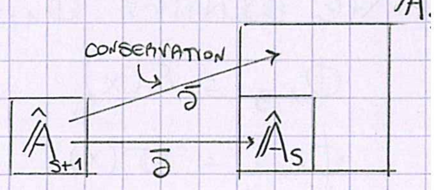
THEN, COMPARING WITH (I), WE HAVE CONSERVATION IF

$$A_s = \partial B_{s-1}$$

(WE EXCLUDE THE TRIVIAL* CASE $T_{s+1} = \partial C_s$).

*NOTE: I CAN ALWAYS PRODUCE FIELDS OF THIS FORM. FOR INSTANCE, $T \rightsquigarrow (2,0)$, $\partial T \rightsquigarrow (3,0)$.

★ LET'S INTRODUCE THE NOTATION:



$\hat{/\!\!A}_{s+1}$ = SPACE OF FIELDS $T_{s+1} \neq \partial C_s$.

$/\!\!A_s$ = SPACE OF FIELDS A_s (DESCENDANTS OF ϕ)

$\hat{/\!\!A}_s$ = SPACE OF FIELDS $A_s \neq \partial B_{s-1}$.

WE HAVE CONSERVATION IF

$$\underline{\dim \hat{/\!\!A}_s < \dim \hat{/\!\!A}_{s+1}}$$

NOTE: THIS IS A SUFFICIENT CONDITION.

(COUNTING ARGUMENT)

WE KNOW THAT SUCH INEQUALITY IS IN GENERAL

i) SATISFIED FOR $s = 1$

ii) NOT ALWAYS SATISFIED FOR $s > 1$.

NOTE: A FIELD IS DEGENERATE IF IT ADMITS A NULL COMBINATION OF DESCENDANTS AT LEVEL l.

IF ϕ IS DEGENERATE, THEN $\dim /\!\!A_s$ IS SMALLER THAN USUAL AND THE INEQUALITY MAY BE SATISFIED ALSO FOR $s > 1$.

A SYSTEMATIC ANALYSIS SHOWS THAT NON-TRIVIAL CONSERVATION LAWS EXIST FOR $\phi = \phi_{1,2}$ OR $\phi_{1,3}$ OR $\phi_{2,1}$; ONE CAN ARGUE THAT THEY ARE ∞-LY MANY.

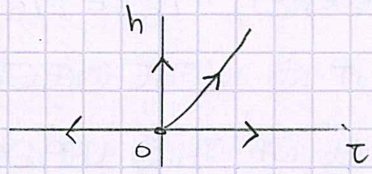
NOTE: THE CONSERVATION LAWS.

LAST TIME WE IDENTIFIED THE ENERGY DENSITY FIELDS

$$\varepsilon = \begin{cases} \phi_{1,3} & \text{IN } O(n) \\ \phi_{2,1} & \text{IN } q\text{-STATE POTTS} \end{cases}$$

AND THIS IMPLIES INTEGRABILITY.

⋆ CONSIDER THE GOOD OLD ISING MODEL,



$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i$$

$$A = A^{FP}_{ISING} + \tau \int d^2x \, \varepsilon(x) + h \int d^2x \, \sigma(x), \qquad \begin{cases} \tau \sim J - J_c \\ h \sim H \end{cases}.$$

BUT ISING IS $O(m)$ WITH $m=2$ AND POTTS WITH $q=2$, SO WE IDENTIFY $\phi_{1,3} = \phi_{2,1}$. WE WOULD ALSO FIND

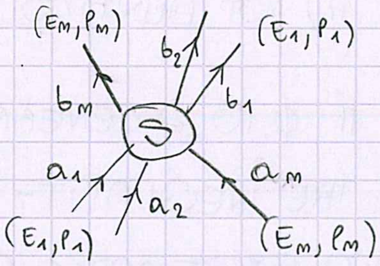
$$\phi_{1,3} = \varepsilon(x)$$
$$\phi_{1,2} = \sigma(x).$$

WE HAVE INTEGRABILITY FOR

$$\begin{cases} h = 0 & \text{(FREE MASSIVE FERMIONS, TRIVIAL)} \\ \tau = 0. \end{cases}$$

SO ISING IN $d=2$ WITH NONZERO MAGNETIC FIELD BECOMES SOLVABLE AT $T=T_c$ (IN THE CONTINUUM: IT HAS NEVER BEEN SOLVED ON THE LATTICE). THIS WAS DISCOVERED BY ZAMOLODCHIKOV BY THIS COUNTING ARGUMENT.

• S-MATRIX OF INTEGRABLE THEORIES

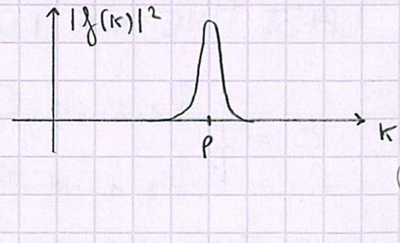


1. ∞-LY MANY CONSERVATION EQUATIONS $\Rightarrow$ COMPLETE ELASTICITY.
2. LET P_s BE SPIN s CONSERVED QUANTITY:

$$P_s |p\rangle = \gamma_s (E+p)^s |p\rangle.$$

CONSIDER A WAVE PACKET CENTERED AT $x_1 \equiv x = 0$ AT $t=0$,

$$\psi_p(x) = \int dk \, e^{ikx} f(k).$$



HEURISTICALLY, NEGLECTING TIME (i.e. KINEMATICS),

$$e^{iaP_s} \psi_p(x) \sim \int dk \, f(k) \, e^{i(kx + \gamma k^s a)}.$$

USING A STATIONARY PHASE APPROXIMATION FOR LARGE a,

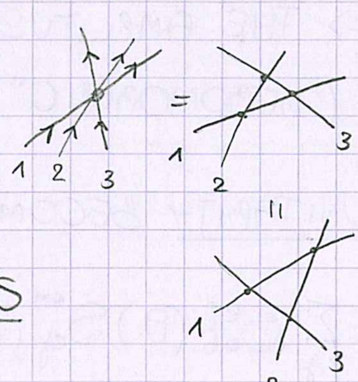
$$0 \equiv \partial_k \left(kx + \gamma_s k^s a\right)\Big|_{k=p} = x + s\gamma_s a p^{s-1}$$

GIVING A DISPLACEMENT

$$\Delta x \propto a p^{s-1}.$$

THIS MEANS PARTICLES WITH DIFFERENT MOMENTA ARE MOVED DIFFERENTLY IF $s \neq 1$.

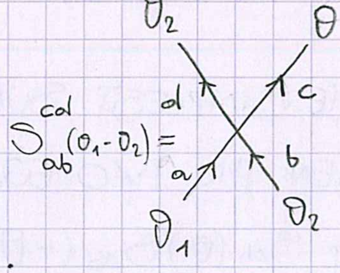
THIS AMOUNTS TO THE PICTURE ON THE RIGHT.
SINCE P_s COMMUTES WITH H, THEN THE SCATTERING AMPLITUDE CANNOT CHANGE:
THIS IMPLIES THE <u>FACTORIZATION EQUATIONS</u>



$$S_{123} = S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}.$$

MATHEMATICALLY, THEY ARE THE SAME AS THE YANG-BAXTER EQUATIONS WHICH RELATE BOLTZMANN WEIGHTS IN LATTICE MODELS (WHICH IS WHY THEY'RE SOMETIMES CALLED BY THIS NAME).

★ LET'S NOW CONSIDER THE SCATTERING PROCESS ON THE RIGHT.
THE DISPERSION RELATION IS AUTOMATICALLY ENFORCED IF WE ADOPT THE RAPIDITY θ:



$$(E, p) = (m\cosh\theta, m\sinh\theta)$$

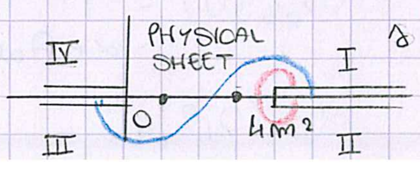
$\theta \to \theta + \Lambda$ UNDER RELATIVISTIC TRANSFORMATIONS.

RELATIVISTIC INVARIANT QUANTITIES CAN ONLY DEPEND ON THE DIFFERENCE OF RAPIDITIES. IN FACT,

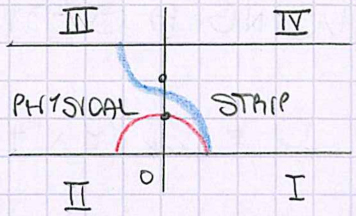
$$s = (E_1+E_2)^2 - (p_1+p_2)^2 = \left(2m \,\mathrm{ch}\, \frac{\theta_1-\theta_2}{2}\right)^2$$

NOTE: RECALL THE BRANCH CUT FOR $\mathrm{Re}(s) < 0$ IS DUE TO CROSSING, $S_{ab}^{cd}(s+i\varepsilon) = S_{da}^{bc}(4m^2-(s+i\varepsilon))$

FOR PARTICLES OF EQUAL MASS m. THERE'S A SINGLE BRANCH POINT AT $4m^2$, BECAUSE THERE CANNOT BE INELASTIC SCATTERING.



DUE TO THE PERIODICITY OF COSH, THE s PLANE CAN BE MAPPED ON THE $(\theta_1-\theta_2)$ PLANE. THE PHYSICAL SHEET IS MAPPED IN A.



PHYSICAL STRIP: IN THE PLANE $(\theta_1-\theta_2)$, THERE ARE NO CUTS ANYMORE, THE ONLY REMAINING SINGULARITIES ARE THE POLES, WHICH COME IN PAIRS BECAUSE OF CROSSING SYMMETRY.

$\Rightarrow$ THE AMPLITUDE HAS ONLY POLES IN $(\theta_1-\theta_2)$ COMPLEX PLANE ("MEROMORPHIC").

UNITARITY BECOMES, IN TERMS OF RAPIDITIES,

$$\sum_{e,f} S_{ab}^{ef}(\theta)\, S_{ef}^{cd}(-\theta) = \delta_{ac}\delta_{bd}$$

AND CROSSING BECOMES

$$S_{ab}^{cd}(\theta) = S_{d\bar{a}}^{\bar{b}c}(i\pi-\theta).$$

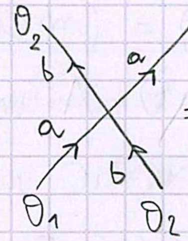
RECALL THEY RELATED POINTS ON DIFFERENT SIDES OF THE CUTS:

— : UNITARITY

— : CROSSING.

★ THE SIMPLEST SUBCLASS IS THAT OF PURE TRANSMISSION. THEN THE TWO EQUATIONS REDUCE TO

$$\begin{cases} S_{ab}(\theta)\, S_{ab}(-\theta) = 1 \\ S_{ab}(\theta) = S_{\bar{b}a}(i\pi-\theta). \end{cases}$$



$$= S_{ab}^{ab}(\theta_1-\theta_2) \equiv S_{ab}(\theta_1-\theta_2)$$

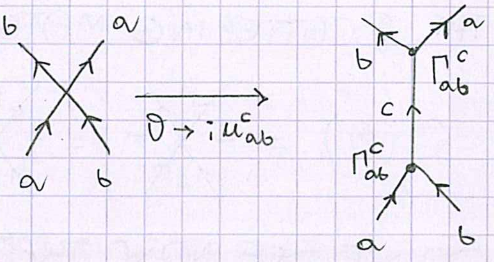
YOU CAN CHECK THAT ITS GENERAL SOLUTION FOR S_{ab} MEROMORPHIC, REAL ANALYTIC AND EXPONENTIALLY BOUNDED IN θ IS

$$S_{ab}(\theta) = \prod_{\alpha\in A_{ab}} f_\alpha(\theta), \qquad f_\alpha(\theta) = \frac{\mathrm{sh}\,\frac{1}{2}(\theta+i\pi\alpha)}{\mathrm{sh}\,\frac{1}{2}(\theta-i\pi\alpha)}$$

$\alpha \in (0,2)$.

· <u>BOUND STATES</u>

$$S_{ab}(\theta) \underset{\theta \simeq i u_{ab}^c}{\simeq} i \frac{(\Gamma_{ab}^c)^2}{\theta - i u_{ab}^c}$$



WHERE

$$m_c = 2m \cos \frac{u_{ab}^c}{2} .$$

NOTE: THIS IS THE ADAPTATION OF THE TRIANGULAR RULE
$$m_c^2 = m_a^2 + m_b^2 + 2 m_a m_b \cos(u_{ab}^c)$$
TO THE SIMPLE CASE WHERE $m_a = m_b = m$.

IF $\Gamma_{ab}^c \in \mathbb{R}$, THE RESIDUE IS

$\left\{\begin{array}{l}\text{POSITIVE IN THE DIRECT CHANNEL}\\ \text{NEGATIVE IN THE CROSSED CHANNEL.}\end{array}\right.$

· <u>EXAMPLE</u>: ISING

$$A = A_{FF} + \tau \int d^2x \, \varepsilon(x)$$

$$= \int d^2x \left[\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + \tau \psi \bar{\psi} \right]$$

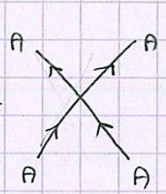
(FREE FERMIONIC WITH MASS $\sim |\tau|$). THEN

$S(\theta) = -1 \quad \forall T$.

HOW CAN THE SAME SCATTERING AMPLITUDE DESCRIBE PHYSICAL SITUATIONS WHICH ARE ACTUALLY DIFFERENT?

FOR $T > T_c$,

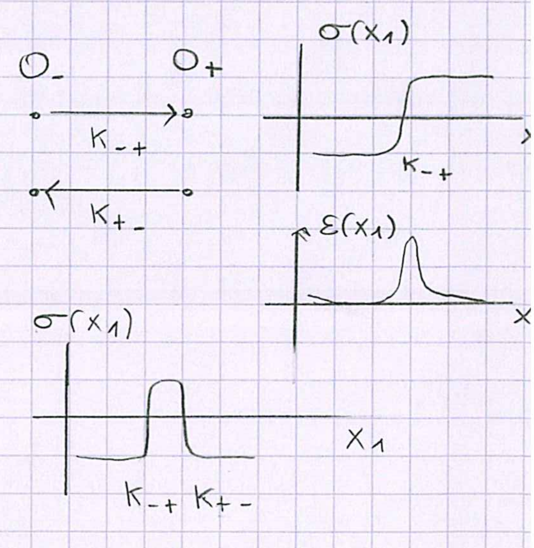
$|0\rangle, |A(\theta_1) \dots A(\theta_m)\rangle,$

$S(\theta) =$  $= -1$.

FOR $T < T_c$, SSB GIVES $|0_+\rangle, |0_-\rangle$.

THEN THE FUNDAMENTAL EXCITATIONS ARE KINKS AND ANTIKINKS CONNECTING THE TWO VACUA ("TOPOLOGICAL").

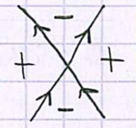
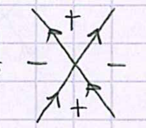
COMPOSING KINK AND ANTIKINK YOU CAN GET EXCITATIONS WHICH ARE LOCALIZED IN SPACE.



A GENERIC STATE IS THEN

$|\dots K_{+-}(\theta_i) K_{-+}(\theta_{i+1}) K_{+-}(\theta_{i+2}) \dots \rangle$.

THE SCATTERING MATRIX IS NOW

$S(\theta) =$  $=$  $= -1.$

SO WE SEE NOW THAT THE AMPLITUDES HAVE A DIFFERENT INTERPRETATION BELOW T_c.

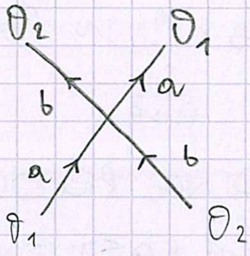
NOTE: YOU CAN ACTUALLY INTERPRET THE DIAGRAMS UP HERE AS THE SPREADING OF DOMAIN WALLS.

· S-MATRIX: RELOADED

LAST TIME WE DERIVED, FOR PURE TRANSMISSION,

$$\begin{cases} S_{ab}(\theta) S_{ab}(-\theta) = 1 \\ S_{ab}(\theta) = S_{\bar{b}a}(i\pi - \theta) \end{cases}$$

$$S_{ab}(\theta_1 - \theta_2) =$$



WHENCE

$$S_{ab}(\theta) = \prod_{\alpha \in A_{ab}} f_\alpha(\theta) \qquad f_\alpha(\theta) = \frac{\sinh \frac{1}{2}(\theta + i\pi\alpha)}{\sinh \frac{1}{2}(\theta - i\pi\alpha)}.$$

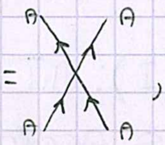
★ LET'S SEE AN EXAMPLE OF AN INTERACTING THEORY.

CONSIDER A DOUBLET $A, \bar{A}$ WITH MASS m AND CHARGE

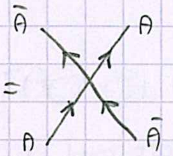
$$\begin{cases} q & \text{FOR } A \\ -q & \text{FOR } \bar{A}. \end{cases}$$

WE HAVE THE AMPLITUDES

$$S_{AA}(\theta) = S_{\bar{A}\bar{A}}(\theta) =$$



$$S_{A\bar{A}}(\theta) = S_{\bar{A}A}(\theta) =$$



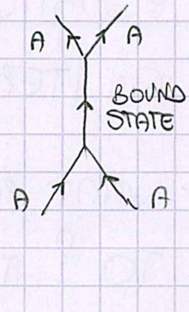
CROSSING EQUATION READS

$$S_{AA}(\theta) = S_{\bar{A}A}(i\pi - \theta)$$

AND YOU CAN CHECK THAT THIS IS SOLVED BY

$$S_{AA}(\theta) = f_{2/3}(\theta) \qquad S_{A\bar{A}}(\theta) = -f_{1/3}(\theta). \qquad (\mathrm{I})$$

WHAT THEORY IS THIS? S_{AA} HAS A POLE IN $\theta = \frac{2\pi i}{3}$:



THEN THE BOUND STATE OF AA HAS MASS

$$m_c = 2m \cos \frac{u^c_{ab}}{2} \underset{u = 2\pi/3}{=} m.$$

THIS BOUND STATE IS EITHER A OR $\bar{A}$. IF

$AA \to A \quad \Rightarrow \quad 2q = q \to q = 0$ DISCARDED.

$AA \to \bar{A} \quad \Rightarrow \quad 2q = -q \to 3q = 0.$

THIS MIGHT MEAN THAT THE STATE AAA HAS TOTAL CHARGE 0.

THIS IS CONSISTENT FOR SYMMETRY MA ENCÜLET!

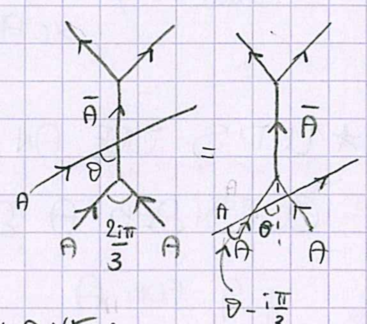
$$\mathbb{Z}_3 : A \to e^{2i\pi/3} A, \quad \bar{A} \to e^{-2i\pi/3} \bar{A}.$$

THE TOTAL SYMMETRY IS THEN

$$\mathbb{Z}_3 \times \underset{\substack{\uparrow \\ A \to \bar{A}}}{\mathbb{Z}_2} \sim S_3 \quad \text{(PERMUTATIONS OF 3 OBJECTS)}$$

AND WE RECOGNIZE THE <u>3-STATE POTTS</u> MODEL. WE HAVE SEEN LAST TIME THAT IT IS INTEGRABLE AWAY FROM CRITICALITY.

* LET'S DERIVE A GENERAL PROPERTY.

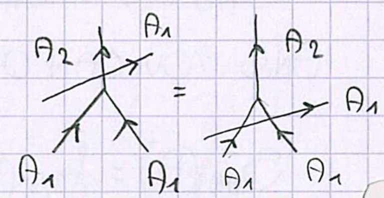


TRANSLATING THE "A" LINE BEFORE AND AFTER THE FUSION, AND USING THE PARAMETERIZATION
$(E, p) = (m \operatorname{ch}\theta, m \operatorname{sh}\theta)$
WE CAN REGARD RAPIDITIES AS ANGLES ON A PLANE:

$$S_{A\bar{A}}(\theta) = S_{AA}\left(\theta - i\frac{\pi}{3}\right) S_{AA}\left(\theta + i\frac{\pi}{3}\right) \qquad \underline{\text{BOOTSTRAP EQUATION}}.$$

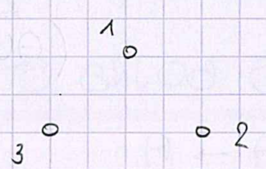
THIS IS SATISFIED BY SOLUTION (I): IT IS NOT REALLY A NEW EQUATION IN THIS CASE, BUT THIS IS ACCIDENTAL ($\frac{\pi}{3}$, PARTICLE WHICH IS BOUND STATE OF ITSELF). SUPPOSE INSTEAD

$$A_1 A_1 \to A_2 .$$

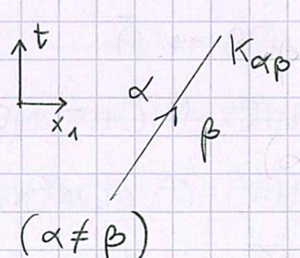


KNOWING THE AMPLITUDE FOR A_1, THE EQUATION GIVES ME THAT OF A_2 (AND SO ON ITERATIVELY, WHENCE THE TERM BOOTSTRAP).

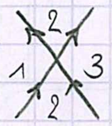
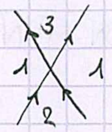
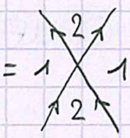
* WHAT PHASE OF THE 3-STATE POTTS MODEL DOES THIS REFER TO? IT HAS TO BE $T > T_c$. IN FACT, FOR $T < T_c$ SSB GIVES
$|0_\alpha\rangle$, $\alpha = 1, 2, 3$.



THE EXCITATIONS ARE KINKS CONNECTING THESE 3 VACUA. A KINK IS A PARTICLE WHOSE TRAJECTORY IN SPACETIME IS A DOMAIN WALL SEPARATING TWO DIFFERENTLY COLORED REGIONS.



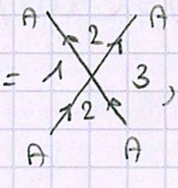
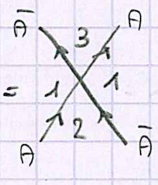
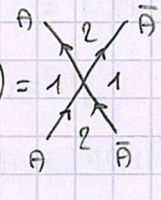
S_3 SYMMETRY GIVES 3 POSSIBLE AMPLITUDES:

$S_1(\theta) =$  , $S_2(\theta) =$  , $S_3(\theta) =$  .

THERE IS CORRESPONDENCE WITH $T > T_c$ IF

$$\begin{cases} K_{\alpha, \alpha+1} \pmod 3 = A \\ K_{\alpha, \alpha-1} \pmod 3 = \bar{A} \end{cases}.$$

WITH THESE ASSIGNMENT,

$S_1(\theta) =$  , $S_2(\theta) =$  , $S_3(\theta) =$  ← REFLECTION.

IF WE SET $S_3 \equiv 0$, WE ALREADY KNOW THE SOLUTION:

$$\begin{cases} S_1 = S_{AA} \\ S_2 = S_{A\bar{A}} \end{cases}.$$

THIS MANIFESTS DUALITY (ABOVE AND BELOW T_c).

- A RELEVANT EXAMPLE WITH REFLECTION: SINE-GORDON

$$A_{SG} = \int d^d x \left[\frac{(\nabla\varphi)^2}{4\pi} + \lambda \cos 2b\varphi \right].$$

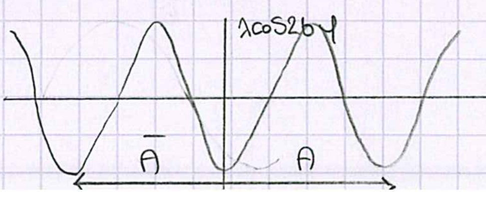
WE HAVE SEEN THIS CAN BE FERMIONIZED AND WE GET THE MASSIVE THIRRING MODEL

$$A_{SG} = \int d^2 x \left[\sum_{i=1}^{2} \left(\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i + \lambda \psi_i \bar{\psi}_i \right) + g(b^2) \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 \right].$$

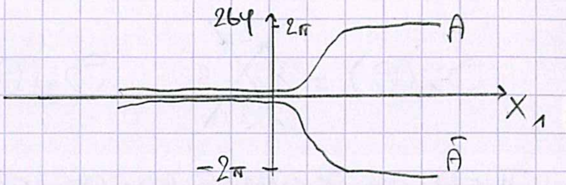
IT HAS LONG SINCE BEEN KNOWN THAT THIS THEORY IS INTEGRABLE. IT CONTAINS 2 MAJORANA FERMIONS WHICH CAN BE COMBINED INTO A DIRAC FERMION

$$\begin{cases} \psi = \psi_1 + i\psi_2 & \rightarrow \quad A = A_1 + iA_2 \\ \psi^* = \psi_1 - i\psi_2 & \rightarrow \quad \bar{A} = A_1 - iA_2 \end{cases}.$$

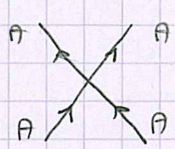
THE POTENTIAL SHOWS SEVERAL MINIMA.

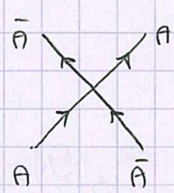


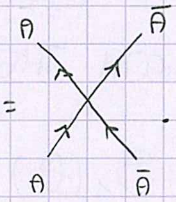
THE TOPOLOGICAL EXCITATIONS WHICH BRING FROM ONE TO THE OTHER ARE CALLED SOLITON AND ANTISOLITON.



CHARGE CONSERVATION LEADS TO

$$S^{AA}_{AA}(\theta) \equiv S_0(\theta) =$$ 

$$S^{A\bar{A}}_{A\bar{A}}(\theta) \equiv S_T(\theta) =$$ 

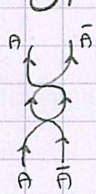
$$S^{\bar{A}A}_{A\bar{A}}(\theta) \equiv S_R(\theta) =$$ .

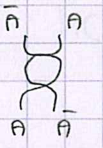
CROSSING EQUATIONS IMPLY

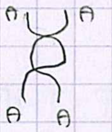
$$\begin{cases} S_0(\theta) = S_T(i\pi - \theta) \\ S_R(\theta) = S_R(i\pi - \theta) \end{cases}$$

WHILE BY UNITARITY

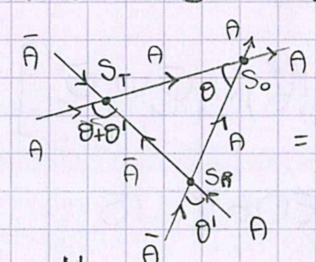
NOTE: A SUM OVER THE POSSIBLE INTERMEDIATE PARTICLES IS UNDERSTOOD.

$1 =$  $= S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta)$

$0 =$  $= S_T(\theta)S_R(-\theta) + S_R(\theta)S_T(-\theta)$

$1 =$  $= S_0(\theta)S_0(-\theta)$.

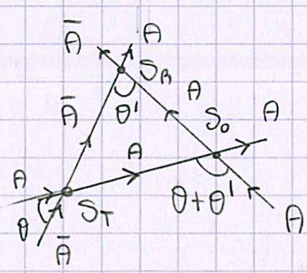
THE FACT THAT $S_R \neq 0$ GIVES NONTRIVIAL FACTORIZATION EQS:



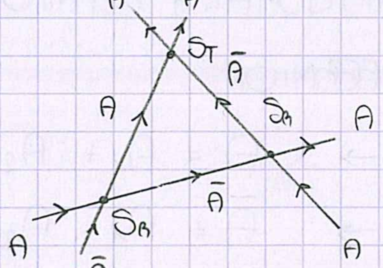
$$= S_R(\theta')S_T(\theta+\theta')S_0(\theta)$$

NOTE: THESE ARE AGAIN BOOTSTRAP EQUATIONS, JUST DRAWN DIFFERENTLY.

$\|$



$+$



$$= S_T(\theta)S_0(\theta+\theta')S_R(\theta') + S_R(\theta)S_R(\theta+\theta')S_T(\theta').$$

MEROMORPHICITY ALLOWS TO WRITE SOLUTIONS AS PRODUCTS (IN GENERAL INFINITE) OF Γ FUNCTIONS.

HOWEVER, INTEGRAL REPRESENTATIONS ARE THEN OBTAINED USING

$$\Gamma(z) = \exp\left\{ \int_0^\infty \frac{dt}{t} \left[\frac{e^{-tz} - e^{-t}}{1-e^{-t}} + (z-1)e^{-t} \right] \right\}.$$

SINE-GORDON SOLUTION:

$$\begin{cases} S_T(\theta) = -\dfrac{\operatorname{sh}\frac{\pi}{\xi}\theta}{\operatorname{sh}\frac{\pi}{\xi}(\theta - i\pi)} S_0(\theta) \\ S_R(\theta) = -\dfrac{\operatorname{sh} i\pi^2/\xi}{\operatorname{sh}\frac{\pi}{\xi}(\theta - i\pi)} S_0(\theta) \end{cases}.$$

THIS SOLVES THE FACTORIZATION EQUATIONS. THEN ONE USES UNITARITY AND CROSSING TO DETERMINE

$$S_0(\theta) = -\exp\left\{ -i \int_0^\infty \frac{dt}{t} \frac{\operatorname{sh}\frac{t}{2}(1-\xi/\pi)}{\operatorname{sh}\frac{t\xi}{2\pi} \operatorname{ch} t/2} \sin\frac{\theta t}{\pi} \right\}.$$

* WHAT ARE THE PHYSICAL IMPLICATIONS OF THIS RESULT?

ξ MUST BE RELATED TO b^2. FOR $\xi = \pi$,

$S_T = S_0 = -1, \; S_R = 0 \quad \Rightarrow \quad$ FREE FERMION $\left(b^2 = \frac{1}{2}\right)$.

MOREOVER,

$$\lim_{\theta\to\infty} S_T(\theta) = e^{-i\frac{\pi}{2}\left(1+\frac{\pi}{\xi}\right)}.$$

THIS IDENTIFIES THE UV LIMIT OF THIS THEORY: INDEED, WE ARE ASSUMING $b^2 < 1$, SO THAT $\cos 2b\varphi$ IS RELEVANT AND IN THE UV LIMIT WE GET THE GAUSSIAN FP. THEN THIS MUST COINCIDE WITH

GAUSS FP

$\cos 2b\varphi$

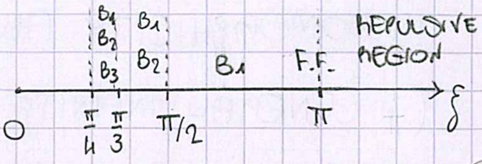
$$e^{-i\pi/2b^2}$$

NOTE: ON 03.12.19 WE GOT $S = e^{-2\pi i \Delta_\eta}$, $\Delta_\eta = 1/4b^2$.

OBTAINED IN $O(2)$ FP SCATTERING: THIS IMPLIES

$$\xi = \frac{\pi b^2}{1-b^2} \in (0,\infty) \qquad (b^2 \in (0,1)).$$

NOTICE S_T, S_R HAVE POLES AT

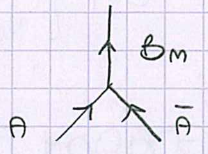


$$\theta = i(\pi - m\xi)$$

WHICH ARE LOCATED IN THE PHYSICAL STRIP $\mathrm{Im}\,\theta \in (0,\pi)$

WHEN $m\xi < \pi$. SUCH POLES CORRESPOND TO

NEUTRAL BOUND STATES B_m WITH MASSES



$$m_m = 2M \sin\frac{m\xi}{2} \qquad\qquad 1 \le m \le \frac{\pi}{\xi}$$

$M \equiv$ SOLITON MASS.

WE SEE m_{MAX} INCREASES AS ξ DECREASES.

NOW WE SEE THAT THE PARTICLE CREATED BY φ IS IDENTIFIED WITH B_1. IT DISAPPEARS FOR $\xi > \pi$: IT OFTEN HAPPENS IN QFT THAT, FOR LARGE ENOUGH COUPLINGS, A PARTICLE DISAPPEARS FROM THE SPECTRUM.

NOTICE FOR $b^2 \to 0$

$\cos 2b\varphi \to \text{const.} + \alpha\varphi^2$ FREE BOSONIC LIMIT.

INDEED, THE SOLITON MASS INCREASES IN THIS LIMIT:

$$\frac{M}{m_1} = \frac{\text{SOLITON MASS}}{\text{BOSON MASS}} \to \infty .$$

NOTE: $\xi = \frac{\pi b^2}{1-b^2}$, $b^2 \in (0,1)$. $\frac{M}{m_1} = \frac{1}{2}\frac{1}{\sin \xi/2}$.

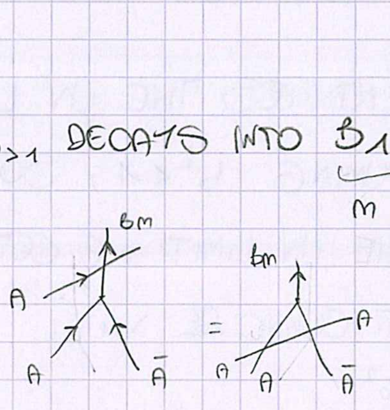
THE SOLITON IS SAID TO DECOUPLE FROM THE THEORY.

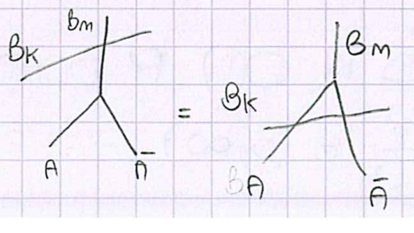
WHAT HAPPENS TO THE BOUND STATE?

$$\frac{m_m}{m_1} \to m .$$

$B_{m>1}$ DECAYS INTO $\underbrace{B_1 \ldots B_1}_{m \text{ TIMES}}$.

FINALLY, THE BOOTSTRAP EQUATION CAN BE USED TO DETERMINE S_{AB_m} AND $S_{B_m B_k}$.





INTEGRABILITY

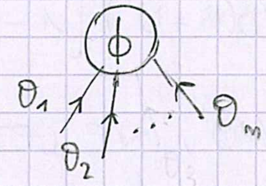
17.12.19

S-MATRIX $\xrightarrow{\quad ? \quad}$ FORM FACTORS $\longrightarrow$ CORRELATORS

(NON OBSERVABLE)

$\langle \phi(x)\phi(0)\rangle \quad \sum\!\!\!\!\!\!\int_m |m\rangle\langle m|$

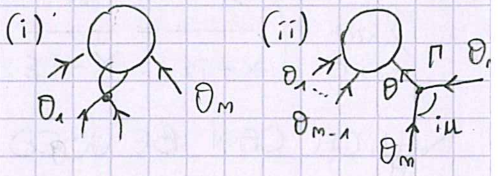
FORM FACTORS IN INTEGRABLE THEORIES



$$F_m^\phi(\theta_1,\dots,\theta_m) = \langle 0|\phi(0)|\theta_1,\dots,\theta_m\rangle.$$

THEY ARE DETERMINED VIA 4 EQUATIONS:

i) $F_m^\phi(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_m) = F_m^\phi(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_m)\,S(\theta_i-\theta_{i+1})$



ii) $\underset{\theta_m-\theta_{m+1}=iu}{\mathrm{Res}} \quad F_{m+1}^\phi(\theta_1,\dots,\theta_m,\theta_{m+1})$

$= i\Gamma\, F_m^\phi(\theta_1,\dots,\theta_{m-1},\theta)$

(Γ IS THE RESIDUAL, θ IS DETERMINED BY ENERGY AND MOMENTUM CONSERVATION, u IS THE BOUND STATE POLE).

iii) $F_m^\phi(\theta_1+2i\pi,\theta_2,\dots,\theta_m) = e^{-2i\pi\gamma_\phi}\,F_m^\phi(\theta_2,\dots,\theta_m,\theta_1)$

iv) $\underset{\theta'-\theta=i\pi}{\mathrm{Res}} \quad F_{m+2}^\phi(\theta',\theta,\theta_1,\dots,\theta_m) = i\left[1-e^{2i\pi\gamma_\phi}\prod_{k=1}^m S(\theta-\theta_k)\right]F_m^\phi(\theta_1,\dots,\theta_m).$

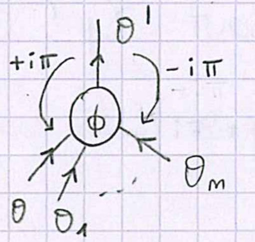
WE HAVE TO UNDERSTAND THE LAST 2 EQUATIONS. CONSIDER ($\gamma_\phi=0$

$$\langle\theta'|\phi(0)|\theta,\theta_1,\dots,\theta_m\rangle \qquad \theta'\geq\theta>\theta_1>\dots>\theta_m.$$

BY CROSSING, WE CAN MOVE

$(E',p')\to(-E',-p')$

$\theta'\to\theta'\pm i\pi$



i.e. WE CAN DO IT IN TWO WAYS:

$$\langle\theta'|\phi(0)|\theta,\theta_1,\dots,\theta_m\rangle = \begin{cases} F_{m+2}^\phi(\theta'+i\pi,\theta,\theta_1,\dots,\theta_m)+2\pi\delta(\theta-\theta')F_m^\phi(\theta_1,\dots,\theta_m) \\ F_{m+2}^\phi(\theta,\theta_1,\dots,\theta_m,\theta'-i\pi)+2\pi\delta(\theta-\theta')F_m^\phi(\theta_1,\dots,\theta_m)\prod_{k=1}^m S(\theta-\theta \end{cases}$$

WHERE THE $\delta(\theta-\theta')$ ACCOUNTS FOR THE POSSIBILITY OF AN ANNIHILATION. IN THE SECOND CASE, HOWEVER, θ HAS TO CROSS ALL THE OTHER PARTICLES IN ORDER TO ANNIHILATE θ', AND THEN

BY EQUATION (i) WE GET SOME $S(\theta-\theta_k)$ FACTORS.

SUBTRACTING THE TWO WE FIND

$$F^{\phi}_{m+2}(\theta'+\pi, \theta, \theta_1, \dots, \theta_m) - F^{\phi}_{m+2}(\theta, \theta_1, \dots, \theta_m, \theta' - i\pi)$$

$$= -2\pi\delta(\theta-\theta')\left[1 - \prod_{k=1}^{m} S(\theta-\theta_k)\right] F^{\phi}_m(\theta_1, \dots, \theta_m).$$

FOR $\theta \neq \theta'$, WE GET EQUATION (iii). NOTE: WE HAD SET $\gamma_\phi = 0$.

FOR $\theta \to \theta'$, THERE IS A POLE FOR $\theta = \theta'$. RECALL

$$\lim_{\varepsilon \to 0}\left(\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right) = 2\pi i\,\delta(x)$$

WHICH CAN BE USED TO FIND THE RESIDUE: THIS GIVES (iv).

NOTE: THIS IS SOKHOTSKY'S FORMULA

$$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i\varepsilon} = PV\frac{1}{x} \mp i\pi\delta(x)$$

THAT IS

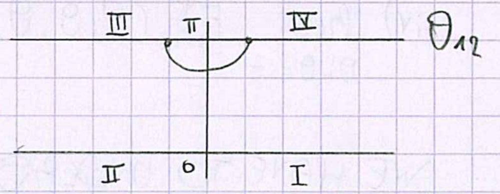
$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x \pm i\varepsilon} = \mp i\pi f(0) + \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} dx \frac{f(x)}{x}.$$

* EQUATION (iii) GIVES IN PARTICULAR

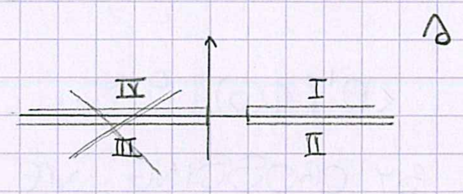
$$F^{\phi}_2(\theta_1 + i\pi, \theta_2) = F^{\phi}_2(\theta_2, \theta_1 - i\pi).$$

FOR ϕ SCALAR, $F^{\phi}_2(\theta_1, \theta_2)$ DEPENDS ONLY ON $\theta_{12} = \theta_1 - \theta_2$, SO

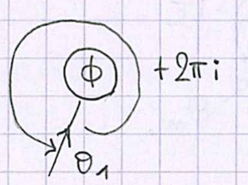
$$F^{\phi}_2(\theta_{12} + i\pi) = F^{\phi}_2(i\pi - \theta_{12}).$$

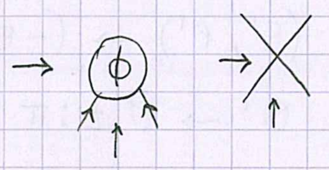


WE CONCLUDE THAT THERE IS NO CROSSING CUT IN F^{ϕ}_2.



* WHAT ABOUT γ_ϕ? CONSIDER





i.e. $e^{-2\pi i \gamma_\phi}$ IS THE MUTUAL LOCALITY PHASE BETWEEN ϕ AND THE FIELD WHICH CREATES THE PARTICLE.

· EXAMPLE: ISING

$S(\theta) = -1$

NO BOUND STATES $\to$ $\Gamma = 0$ IN EQUATION (ii).

LET $T > T_c$. FIRST, IDENTIFY THE FIELD THAT CREATES THE PARTICLE, WHICH HERE IS σ (AND THE PARTICLE IS $\mathbb{Z}_2$-ODD).

THEN, LET'S LOOK FOR THE MUTUAL LOCALITY PHASES AND F'S:

$$\phi \equiv \sigma : \quad \gamma_\sigma = 0, \quad F^\sigma_{2k} = 0 \quad (\mathbb{Z}_2 \text{ PARITY}).$$

BUT THERE ARE ∞-LY MANY SUCH FIELDS (ALL THE DESCENDANTS).

CONSIDER THE DERIVATIVES OF THE FIELD σ (i.e. TAKE THE DERIVATIVES OF THE FORM FACTOR); AMONG THEM:

$$F_m^{(\partial\bar\partial)^k\sigma}(\theta_1, \dots, \theta_m) \propto \left[\sum_{i=1}^m (E_i + p_i) \sum_{i=1}^m (E_i - p_i) \right]^k F_m^\sigma(\theta_1, \dots, \theta_m)$$

$$\sim e^{k\theta_i} F_m^\sigma \qquad \text{FOR } \theta_i \to \infty.$$

THE PRIMARY IS THE ONE WITH THE MILDEST BEHAVIOR AT LARGE ENERGY. THIS GIVES A UNIQUE SOLUTION:

$$F^\sigma_{2k+1}(\theta_1, \dots, \theta_{2k+1}) = i^k F_1^\sigma \prod_{i<j} \text{TH} \frac{\theta_i - \theta_j}{2}$$

(THE RESULT ABOUT THE MILDEST BEHAVIOR HOLDS FOR ALL THE DESCENDANTS, NOT ONLY THE DERIVATIVES, EVEN THOUGH WE DIDN'T SHOW IT).

* TAKE NOW $\phi = \mu$: BY DUALITY,

$$F_m^\mu \big|_{T > T_c} = F_m^\sigma \big|_{T < T_c}.$$

BY OPE (AND SINCE $\langle \psi \rangle = 0$)

$$\sigma \cdot \mu \sim \psi \qquad \Rightarrow \qquad \langle \sigma(x) \mu(0) \rangle = 0.$$

BUT THEN, FOR ALL ODD NUMBERS OF PARTICLES

$$F^\mu_{2k+1} = 0.$$

MOREOVER, SINCE WE PRODUCED A FERMION AND ITS SPIN IS $\frac{1}{2}$

$$\gamma_\mu = \frac{1}{2}.$$

YOU CAN CHECK THAT

$$F^\mu_{2k}(\theta_1, \dots, \theta_{2k}) = i^k F_0^\mu \prod_{i<j} \text{TH} \frac{\theta_i - \theta_j}{2}.$$

* TAKE $\phi = \varepsilon$. SINCE ε IS $\mathbb{Z}_2$-EVEN,

$$\gamma_\varepsilon = 0$$

AND THEN

$$F_m^\varepsilon = \text{CONST.} \cdot \delta_{m,2} \, \sinh \frac{\theta_1 - \theta_2}{2}.$$

THIS IS CONSISTENT WITH $\varepsilon = \psi\bar{\psi}$.

* CONSIDER THE SUSCEPTIBILITY

$$\chi \simeq \Gamma_\pm |T - T_c|^{-\gamma} \qquad T \to T_c^\pm$$

WITH

$$\frac{\Gamma_+}{\Gamma_-} = \frac{\int d^2x \, \langle \sigma(x) \sigma(0) \rangle_{T > T_c}}{\int d^2x \, \langle \sigma(x) \sigma(0) \rangle_{T < T_c}} \overset{\text{ISING}}{\underset{\text{2PA}}{\simeq}} 37.699.$$

2 PARTICLE APPROXIMATION: FORM FACTOR EXPANSION OF $\langle \sigma\sigma \rangle$ WITH CONTRIBUTIONS WITH $m \leq 2$.

$\sum_m |m\rangle\langle m|$

COMPARE WITH THE EXACT RESULT 37.6936.

* IS IT A MIRACLE THAT THIS HAPPENS FOR THE ISING MODEL?

CONSIDER q-STATE POTTS AND AGAIN THE RATIO Γ_+/Γ_-:

q	FF (2PA)	SIMULATIONS
1	160.2	162.5 ± 2
2	37.699	37.6936
3	13.85	13.83(8)
4	4.01	3.9(1)

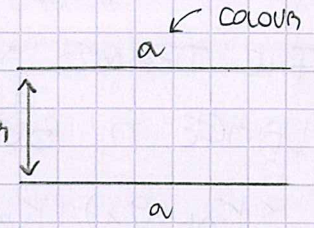
TAKE HOME MESSAGES:

1) IF YOU WANT TO CALCULATE A CORRELATOR IN $d = 2$, KNOW THAT PROBABLY IT'S INTEGRABLE.
2) IF SO, FORM FACTOR THEORY IS QUITE POWERFUL.

- INTERFACES

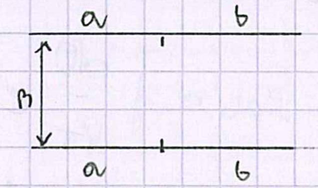
LET $d=2$. FOR $T<T_c$ THERE WILL BE COEXISTING PHASES. BOUNDARY CONDITIONS CAN SELECT A PHASE.

FOR $R\to\infty$, THE MAGNETIZATION TENDS TO



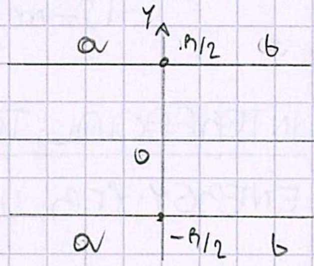
$$\langle\sigma\rangle_a \equiv \langle 0_a|\sigma(x)|0_a\rangle .$$

BUT WE COULD ALSO CHOOSE DIFFERENT COLORS ON EACH OF THE BOUNDARIES. THEN, AS $R\to\infty$, WE EXPECT A PHASE a ON THE FAR LEFT AND A PHASE b ON THE FAR RIGHT, WITH AN INTERFACIAL REGION IN BETWEEN.



WE WANT TO DESCRIBE IT WITH FIELD THEORY. LET y BE THE IMAGINARY TIME AXIS,

$$t = iy .$$



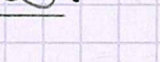
DEFINE

$$\langle\sigma(x,y)\rangle_{ab} \equiv \frac{1}{Z_{ab}}\langle B_{ab}(\tfrac{R}{2})|\sigma(x,y)|B_{ab}(-\tfrac{R}{2})\rangle$$

$$Z_{ab} = \langle B_{ab}(\tfrac{R}{2})|B_{ab}(-\tfrac{R}{2})\rangle .$$

BOUNDARY STATES:

$$|B_{ab}(\pm\tfrac{R}{2})\rangle =$$

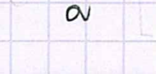


WE CAN EXPAND THEM OVER THE BASIS OF BULK PARTICLE STATES:

$$|B_{ab}(\pm\tfrac{R}{2})\rangle = e^{\pm\frac{R}{2}H}\left[\int\frac{d\theta}{2\pi}\underset{\text{AMPLITUDE}}{f(\theta)}\,|\underset{\text{KINK}}{K_{ab}}(\underset{\text{RAPIDITY}}{\theta})\rangle + \underset{\text{e.g. }\sum_c |K_{ac}K_{cb}\rangle}{(\text{HEAVIER STATES})}\right].$$

NOTICE INSTEAD

$$|B_a(\pm\tfrac{R}{2})\rangle =$$



$$= e^{\pm\frac{R}{2}H}\left[|0_a\rangle + \underset{\text{e.g. }\sum_c |K_{ac}K_{ca}\rangle}{(\text{HEAVIER})}\right] = |0_a\rangle + \dots$$

THIS IMPLIES* (m = MASS OF THE KINK)

$$\mathcal{Z}_{ab} \underset{R \gg \xi \sim \frac{1}{m}}{\sim} \int \frac{d\theta_1}{2\pi} \cdot \frac{d\theta_2}{2\pi} f^*(\theta_1) f(\theta_2) \langle K_{ab}(\theta_1) | K_{ab}(\theta_2) \rangle e^{-\frac{R}{2} m (\cosh\theta_1 + \cosh\theta_2)} + \dots$$

THE TERMS WE ARE DISCARDING ARE SUBLEADING FOR LARGE R. BUT

$$\langle K_{ab}(\theta_1) | K_{ab}(\theta_2) \rangle = 2\pi \delta(\theta_1 - \theta_2)$$

AND WE ALSO SEE THAT THE INTEGRAND IS DOMINATED BY LOW ENERGY (i.e. SMALL θ_1, θ_2) CONTRIBUTIONS, HENCE

$$\mathcal{Z}_{ab} \sim \int \frac{d\theta}{2\pi} e^{-mR\left(1 + \frac{\theta^2}{2}\right)} |f(0)|^2$$

$$= \frac{|f(0)|^2}{\sqrt{2\pi m R}} e^{-mR} .$$

* INTERFACIAL TENSION : INTERFACIAL CONTRIBUTION TO FREE ENERGY PER UNIT LENGTH,

$$\Sigma_{ab} = -\lim_{R \to \infty} \frac{1}{R} \ln \frac{\mathcal{Z}_{ab}}{\mathcal{Z}_a} = m$$

WHERE WE SUBTRACTED

$$\mathcal{Z}_a \sim \langle 0_a | 0_a \rangle = 1 .$$

*NOTE: RECALL THAT, USING RAPIDITIES, $H|K_{ab}(\theta_1)\rangle = m \cosh\theta_1 |K_{ab}(\theta_1)\rangle$

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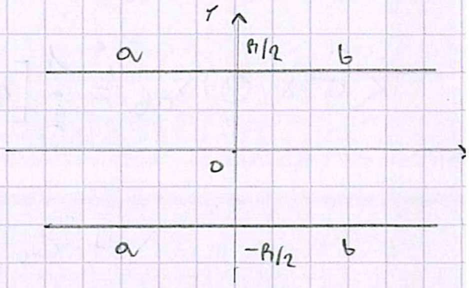
· RECAP: INTERFACES

IN FIELD THEORY, WE HAVE SEEN THE PROBLEM CAN BE RECAST AS



$$\langle \sigma(x,y) \rangle_{ab} = \frac{1}{Z_{ab}} \langle B_{ab}(\tfrac{\beta}{2}) | \sigma(x,y) | B_{ab}(-\tfrac{\beta}{2}) \rangle .$$

WE CALCULATED, FOR LARGE β,

$$Z_{ab} \sim \frac{|f(0)|^2}{\sqrt{2\pi m\beta}} e^{-m\beta}$$

AND IN THIS LIMIT WE CAN EVALUATE THE INTERFACE TENSION

$$\Sigma_{ab} = - \lim_{\beta \to \infty} \frac{1}{\beta} \ln Z_{ab} = m .$$

THE MAGNETIZATION AT THE INTERFACE WILL BE

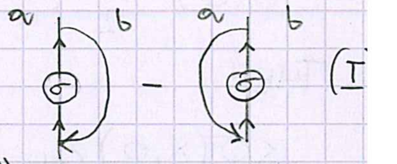
$$\langle \sigma(x,0) \rangle_{ab} \sim \frac{|f(0)|^2}{Z_{ab}} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \underbrace{\langle K_{ab}(\theta_1) | \sigma(0,0) | K_{ab}(\theta_2) \rangle}_{f^{\sigma}_{ab}(\theta_1,\theta_2)} e^{-m\left[\beta\left(1+\frac{\theta_1^2}{4}+\frac{\theta_2^2}{4}\right)-i\theta_{12}\right.}$$

$\theta_{12} = \theta_1 - \theta_2$

AND WE HAVE ALREADY SEEN THAT (CROSSING)

$$f^{\sigma}_{ab}(\theta_1,\theta_2) = \langle 0_b | \sigma(0,0) | K_{ba}(\theta_1 + i\pi) K_{ab}(\theta_2) \rangle$$

$$= i \, \frac{\langle \sigma \rangle_a - \langle \sigma \rangle_b}{\theta_{12}} + \sum_{n=0}^{\infty} c_n \theta_{12}^n , \qquad \theta_1, \theta_2 \to 0$$



WHERE THE LEADING TERM HAS THE "ANNIHILATION POLE".

THIS HAS TO BE REGULARIZED: USE THE TRICK

(i) TAKE ∂_x TO CANCEL THE POLE

(ii) INTEGRATE IN θ_1, θ_2

(iii) INTEGRATE BACK IN x WITH THE CONDITION

$\langle \sigma(+\infty, 0) \rangle_{ab} = \langle \sigma \rangle_b$

WHICH FIXES THE INTEGRATION CONSTANT.

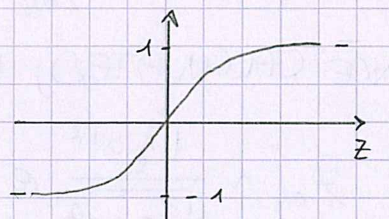
BY DOING THIS, YOU GET

$$\langle \sigma(x,0)\rangle_{ab} = \frac{1}{2}\left[\langle\sigma\rangle_a + \langle\sigma\rangle_b\right] - \frac{1}{2}\left[\langle\sigma\rangle_a - \langle\sigma\rangle_b\right] \mathrm{Erf}\left(x\sqrt{\frac{2m}{R}}\right)$$

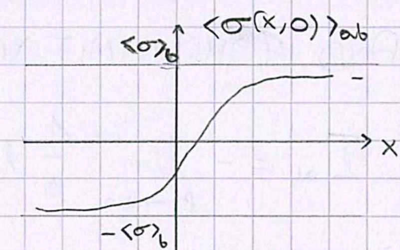
$$+ C_0 \sqrt{\frac{2}{\pi m R}}\, e^{-2m\frac{x^2}{R}} + \dots \qquad (\mathrm{II})$$

WHERE

$$\mathrm{Erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z dt\, e^{-t^2}.$$



SO THE PHASE SEPARATION IS DETERMINED BY THE ANNIHILATION POLE.

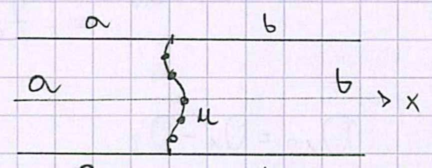


· <u>INTERFACE</u>

WHAT IS IT?

(i) THINK OF IT AS A SIMPLE CURVE SEPARATING PURE PHASES.

(ii) ADD "STRUCTURE" ON THE CURVE.



THE MAGNETIZATION AT X FOR INTERSECTION AT u IS

$$\sigma_{ab}(x|u) = \theta(u-x)\langle\sigma\rangle_a + \theta(x-u)\langle\sigma\rangle_b + A_0\,\delta(x-u) + A_1\,\delta'(x-u) + \dots$$

$$\theta(0) = \frac{1}{2}.$$

THEN

$$\langle\sigma(x,0)\rangle_{ab} = \int_{-\infty}^{+\infty} du\, \sigma_{ab}(x|u)\, p(u)$$

WHERE

$p(u)\,du$ = PROBABILITY OF INTERSECTION IN $(u, u+du)$

$p(u) \equiv$ PASSAGE PROBABILITY DENSITY.

WE FIND

$$\langle\sigma(x,0)\rangle_{ab} = \langle\sigma\rangle_a \int_x^\infty du\, p(u) + \langle\sigma\rangle_b \int_{-\infty}^x du\, p(u) + A_0\, p(x) + \dots$$

SINCE $p(u) = p(-u)$ IF THE TWO PHASES HAVE A SYMMETRIC ROLE,

$$\int_0^\infty du\, p(u) = \frac{1}{2}.$$

THEN WE GET

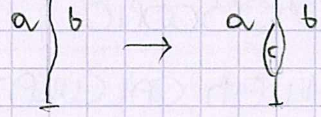
$$\langle \sigma(x,0) \rangle_{ab} = \frac{1}{2}\left[\langle\sigma\rangle_a + \langle\sigma\rangle_b\right] - \left[\langle\sigma\rangle_a - \langle\sigma\rangle_b\right]\int_0^x du\, \rho(u) + A_0 \rho'(x) + \dots$$

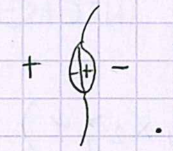
THIS REPRODUCES THE RESULT OF FIELD THEORY WE DERIVED FOR

$$\rho(x) = \sqrt{\frac{2m}{\pi R}}\, e^{-2m\frac{x^2}{R}} \qquad A_0 = \frac{C_0}{m}$$

SO WE SEE THAT DEVIATIONS FROM A "SHARP" SEPARATION COME FROM THE C_0 TERM IN (I).

IN ORDER TO CREATE A "BUBBLE", YOU NEED A 3RD PHASE (c). THIS IS ABSENT IN ISING, WHERE INDEED $C_0 = 0$. THE FIRST CORRECTION TO ISING IS THEN THIS ⟶

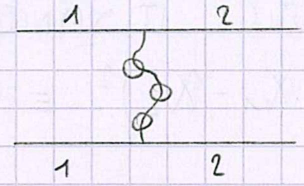




IN 3-STATES POTTS, ON THE OTHER HAND,

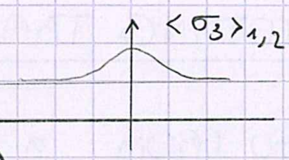
σ_α , $\alpha = 1,2,3$

$\sum_\alpha \sigma_\alpha = 0$ ← WTF?



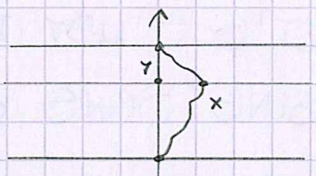
SO THE TERM $C_0 \neq 0$. HERE WE DRAW

$\langle\sigma_3\rangle_1 = \langle\sigma_3\rangle_2$.



INDEED, THIS EFFECT IS SUBLEADING IN R (EQ. (II)).

★ INCLUSION OF y-DEPENDENCE GIVES

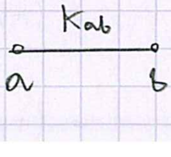


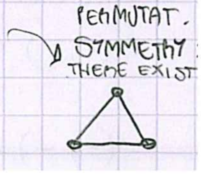
$$\rho(x;y) = \frac{1}{k}\left[\frac{2m}{\pi R}\right]^{\frac{1}{2}} e^{-\chi^2}$$

$$k \equiv \left[1 - \frac{4y^2}{R^2}\right]^{\frac{1}{2}} \qquad \chi \equiv \left[\frac{2m}{R}\right]^{\frac{1}{2}} \frac{x}{k}.$$

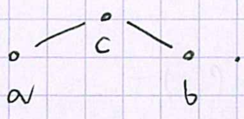
★ THIS WHOLE STORY IS GENERAL UP TO THE ASSUMPTION WE TACITLY MADE THAT THERE IS IN OUR THEORY A KINK K_{ab} CONNECTING THE TWO PHASES a AND b.

THIS IS NOT ALWAYS THE CASE.





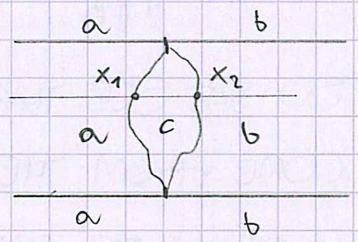
IT CAN BE THAT YOU HAVE TO PASS THROUGH A 3RD PHASE,



IN THIS CASE THE BOUNDARY STATE WILL TAKE THE FORM

$$|B_{ab}(\pm \tfrac{R}{2})\rangle = e^{\pm \frac{R}{2} H}\left[\sum_c \int d\theta_1 d\theta_2 \, f_{acb}(\theta_1,\theta_2)|K_{ac}(\theta_1)K_{cb}(\theta_2)\rangle + \ldots\right]$$

AND A DOUBLE INTERFACE SETS IN, NOT AS A SUBLEADING CORRECTION BUT AS SOMETHING MACROSCOPIC.

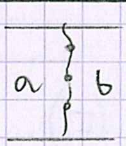
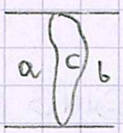


SIMILAR CALCULATIONS TO THOSE OF THE PREVIOUS CASE GIVE US THE INTERSECTION PROBABILITY

$$P(x_1, x_2; y) = \frac{2m}{\pi K^2 R}(x_1 - x_2)^2 e^{-(x_1^2 + x_2^2)}.$$

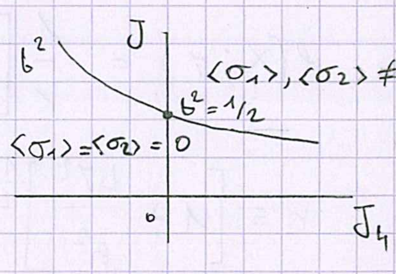
NOTICE IT VANISHES IF x_1 AND x_2 GETS TOO CLOSE: SO $(x_1 - x_2)^2$ = REPULSION FACTOR WHICH INFLATES THE BUBBLE.

· WETTING TRANSITION

GO FROM  TO  CHANGING A PARAMETER.

LET'S STUDY IT IN THE ASHKIN-TELLER MODEL, i.e. TWO ISING SPINS σ_1, σ_2 COUPLED BY

$$J_4 \sum_{\langle x,y\rangle} \sigma_1(x)\sigma_1(y)\sigma_2(x)\sigma_2(y)$$



WE KNOW IT RENORMALIZES ON SINE-GORDON FOR

$$\xi = \frac{\pi b^2}{1 - b^2}.$$

NOTE: SO ξ, b^2, J_4 ARE ESSENTIALLY THE SAME.

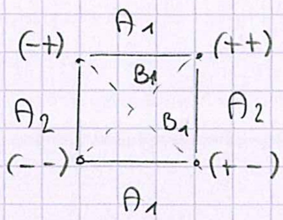
THERE ARE 4 ORDERED PHASES LABELLED BY $(\mathrm{sign}\langle\sigma_1\rangle, \mathrm{sign}\langle\sigma_2\rangle)$:

(−,+) •   • (+,+)

(−,−) •   • (+,−)

FOR $\xi > \pi$ ($J_4 < 0$): REPULSIVE REGIME FOR SOLITONS ($\to$ DIRAC ψ)
THEN WE HAVE ONLY (MAJORANA FERMIONS)

$$\begin{cases} A = A_1 + i A_2 \\ \bar{A} = A_1 - i A_2 \end{cases}$$

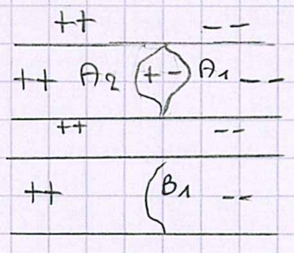


FOR $\xi < \pi$ ($J_4 > 0$): ATTRACTIVE REGIME FOR SOLITONS.
THERE IS B_1 BOUND STATE OF $A\bar{A}$ OR $A_1 A_2$.

IN TERMS OF INTERFACES,

$$\Sigma_{(++)(+-)} = m \qquad \forall \xi$$

$$\Sigma_{(++)(--)} = \begin{cases} 2m & \xi > \pi \\ m_1 = 2m \sin\frac{\xi}{2} & \xi < \pi \end{cases}$$

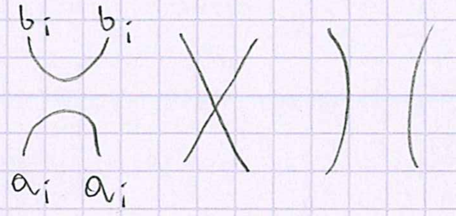


SO WE SEE THAT TUNING ξ (i.e. J_4) WE
GET A WETTING TRANSITION AT $\xi = \pi$, WHERE $m_1 = 2m$.

THIS CAN ALSO BE EXTENDED TO $d=3$.

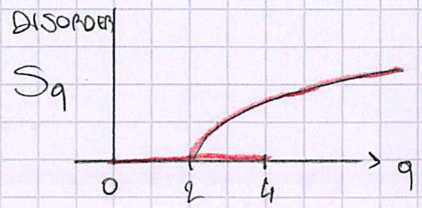
* QUENCHED DISORDER

WE CONSIDER MANY REPLICAS OF $O(n)$:



$a = 1, 2, \dots, m$
$i = 1, \dots, N$

YOU REPEAT THE ABOVE CALCULATIONS AND THEN YOU SEND $N \to 0$
IN THE RESULT.



FOR q-POTTS, WE HAVE SEEN THERE ARE
TRIVIAL DISORDER FIXED POINTS FOR $q < 2$,
BUT NONTRIVIAL ONES AS WELL FOR $q > 2$.

NOTE: S_q-INVARIANT MEANS PERMUTATIONALLY-INVARIANT.

IN THE S_q-INVARIANT SECTOR, THIS NEW LINE LOSES THE
DEPENDENCE ON q. NUMERICALLY, THEY FOUND ν INDEPENDENT
OF q (THE SYMMETRY): HERE, INDEED, $\nu = (d - X_\varepsilon)^{-1}$ SUPER-UNIVERSA