RANDOM MATRIX THEORY

Lecture notes by
Davide Venturelli

davide.venturelli@sissa.it

Prof. P. Vivo
2019-2020
SISSA (Trieste)

RANDOM MATRICES - LECTURE 1

RMT = LINEAR ALGEBRA + PROBABILITY THEORY

GIVEN A MATRIX

$$X = \begin{pmatrix} x_{11} & \dots & x_{1N} \\ \vdots & & \\ x_{N1} & \dots & x_{NN} \end{pmatrix} + \underset{\text{JOINT PROBABILITY DENSITY}}{P(x_{11}, \dots, x_{NN})}$$

WE WANT TO SAY SOMETHING ABOUT ITS EIGENVALUES.

2x2 RM, "SPACING DISTRIBUTION"

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \qquad \text{REAL, SYMMETRIC} \qquad (\text{i.e. REAL } \lambda_i)$$

$$x_1, x_2 \sim \mathcal{N}(0,1) \qquad\qquad \rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$x_3 \sim \mathcal{N}\left(0, \frac{1}{2}\right) \qquad\qquad \rho(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

WE WANT TO COMPUTE THE PDF OF THE SPACING

$$s = \lambda_2 - \lambda_1$$

(INDEED, THE EIGENVALUES ARE RANDOM VARIABLES).

THE CHARACTERISTIC EQUATION IS

$$\lambda^2 - (\operatorname{Tr} X)\lambda + \det X = 0$$

$$\lambda^2 - (x_1 + x_2)\lambda + x_1 x_2 - x_3^2 = 0$$

WHENCE

$$\lambda_{1,2} = \frac{1}{2}\left\{ x_1 + x_2 \pm \left[(x_1+x_2)^2 - 4(x_1 x_2 - x_3)^2 \right]^{1/2} \right\}$$

$$s = \lambda_2 - \lambda_1 = \left[(x_1 - x_2)^2 + 4x_3^2 \right]^{1/2}.$$

NOW WE MIGHT AS WELL FORGET WHERE 'S' CAME FROM AND JUST COMPUTE ITS PDF:

$$P(s) = \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{2\pi}} \frac{dx_2}{\sqrt{2\pi}} \frac{dx_3}{\sqrt{\pi}} e^{-\frac{1}{2}(x_1^2 + x_2^2)} e^{-x_3^2} \delta\left(s - \left[(x_1 - x_2)^2 + 4x_3^2\right]^{1/2}\right).$$

GOING TO POLAR COORDS,

$$\begin{cases} x_1 - x_2 = r\cos\theta \\ 2x_3 = r\sin\theta \\ x_1 + x_2 = \psi \end{cases} \rightarrow \begin{cases} x_1 = \frac{1}{2}(r\cos\theta + \psi) \\ x_2 = \frac{1}{2}(\psi - r\cos\theta) \\ x_3 = \frac{1}{2} r\sin\theta \end{cases}$$

WHOSE JACOBIAN IS

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \psi} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\theta & -\frac{1}{2} r\sin\theta & \frac{1}{2} \\ -\frac{1}{2}\cos\theta & \frac{1}{2} r\sin\theta & \frac{1}{2} \\ \frac{1}{2}\sin\theta & \frac{1}{2} r\cos\theta & 0 \end{pmatrix}$$

$$\det J = -\frac{r}{4}.$$

THEN

$$P(s) = \frac{1}{2\pi\sqrt{\pi}} \cdot \frac{1}{4} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\psi \; \overset{|\det J|}{r} \; \delta(s-r) \times$$

$$\times \exp\left\{ -\frac{1}{2}\left[\left(\frac{r\cos\theta + \psi}{2}\right)^2 + \left(\frac{\psi - r\cos\theta}{2}\right)^2 + 2\cdot\frac{r^2}{4}\sin^2\theta \right] \right\}$$

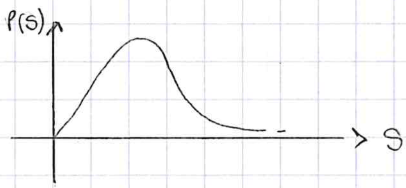
$$= \frac{1}{(2\pi)\sqrt{\pi}} \cdot \frac{1}{4} s \cdot \theta(s) \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\psi \; \times$$

$$\times \exp\left\{ -\frac{1}{2}\left[\frac{s^2\cos^2\theta + \psi^2 + 2\psi s\cos\theta}{4} + \frac{\psi^2 + s^2\cos^2\theta - 2s\psi\cos\theta}{4} + \frac{s^2}{2}\sin^2\theta \right] \right\}$$

$$= \frac{1}{(2\pi)\sqrt{\pi}} \cdot \frac{1}{4} s \cdot \theta(s)\, e^{-\frac{s^2}{4}} \int_0^{2\pi} d\theta \; \underbrace{\int_{-\infty}^{+\infty} d\psi \, e^{-\psi^2/4}}_{=\sqrt{4\pi}} .$$

NOTICE THE θ - DEPENDENCE ONLY DISAPPEARS BECAUSE WE HAVE CHOSEN APPROPRIATELY THE VARIANCES OF x_1, x_2, x_3, OTHERWISE A BESSEL FUNCTION WOULD APPEAR. WE FOUND

$$\underline{P(s) = \theta(s)\, \frac{s}{2}\, e^{-\frac{s^2}{4}}}.$$



THIS IS CALLED "<u>WIGNER'S SURMISE</u>". NOTICE "TO SURMISE" MEANS "TO THINK OR INFER WITHOUT CERTAIN OR STRONG EVIDENCE" (BUT THIS IS HISTORICAL, AS HE SURMISED IT IN A CONFERENCE

ON NEUTRON SCATTERING).

NOTE: W.S. IS ONLY APPROXIMATE IN N×N CASE

NOTICE EIGENVALUES REPEL EACH OTHER, EVEN THOUGH THE ENTRIES OF THE MATRIX WERE INDEPENDENT ("LEVEL REPULSION"). THIS IS SIMILAR TO WHAT IS MEASURED FOR PARKED CARS AND PERCHED BIRDS. EIGENVALUES ARE NOT INDEPENDENT R.V..

· SPACING OF i.i.d. RANDOM VARIABLES

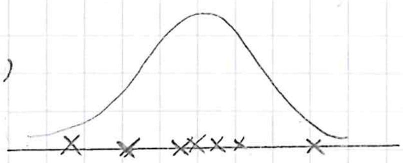
$$\{X_1, \dots, X_N\}, \quad P(x_1, \dots, x_N) = p_X(x_1) \cdot p_X(x_2) \cdot \ldots \cdot p_X(x_N).$$

LET'S INTRODUCE THE CUMULATIVE DISTRIBUTION FUNCTION (CDF) ASSOCIATED TO THE PDF $p_X(x_i)$,

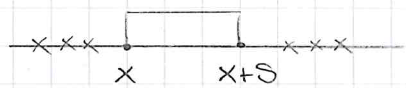
$$F(x) = \int^{x} p_X(x')\, dx' = \text{PROB}[X \leq x].$$

WHAT IS THE DISTRIBUTION OF THE SPACINGS?

IF THE ORIGINAL PDF IS PEAKED ANYWHERE, WE EXPECT THE AVG SPACING TO BE SMALL. NOW CONSIDER



$p_N(s \mid X_j = x)$ = {CONDITIONAL PDF GIVEN THAT ONE OF THE R.V. X_j TAKES THE VALUE x, THAT THERE IS ANOTHER X_k $(k \neq j)$ AT THE POSITION $(x+s)$ AND NO OTHER VARIABLES IN BETWEEN}



$$= p_X(x+s)\,\underbrace{\left[F(x) + (1 - F(x+s))\right]}_{\text{PROB. THAT A RV IS SMALLER THAN } x \text{ OR LARGER THAN } (x+s)}{}^{N-2}.$$

THIS IS AN EXACT RESULT. THEN (BY THE LAW OF TOTAL PROBABILITY)

$$p_N(s, \text{ANY } X = x) = \sum_{j=1}^{N} p_N(s \mid X_j = x) \cdot \underbrace{\text{PROB}(X_j = x)}_{= p_X(x)}$$

$$= N\, p_N(s \mid X_j = x)\, p_X(x).$$

BUT THIS IS STILL A CONDITIONAL PROBABILITY, SO WE

FINALLY INTEGRATE OVER x ("THE INITIAL POINT CAN BE ANYWHERE"):

$$P_N(s) = \int_\sigma P_N(s, \text{ANY } X = x)\, dx = N \int_\sigma P_N(s \mid X_j = x)\, P_X(x)\, dx$$

$\sigma = \text{SUPP}[P_X(x)]$.

THIS IS EXACT, BUT IT DEPENDS ON $P_X(x)$. CHECK THAT

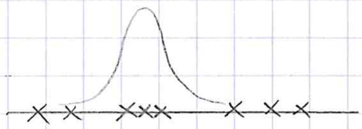
$$\int P_N(s)\, ds = 1.$$

NOTE: $\int ds\, P_N(s \mid X_j = x) = 1$.

* CAN WE EXTRACT SOME UNIVERSAL FEATURE IN THE LARGE N LIMIT?

LET'S APPLY THE "LOCAL CHANGE OF VARIABLES"

$$s = \frac{\hat{s}}{N P_X(x)} \to \text{TYPICAL SPACING AROUND } x.$$



WHY? IMAGINE WE TOSS A LOT OF DARTS: THE TYPICAL SPACING WILL GET SMALLER BOTH IF N GETS LARGE AND IF $P_X(x)$ GETS LARGE, SO WE WANT TO TAKE AWAY SPURIOUS EFFECTS. THIS CAN BE ACHIEVED BY

$$P_N\left(s = \frac{\hat{s}}{N P_X(x)} \,\middle|\, X_j = x\right) = P_X\left(x + \frac{\hat{s}}{N P_X(x)}\right)\left[1 + F(x) - F\left(x + \frac{\hat{s}}{N P_X(x)}\right)\right]^{N-2}.$$

BUT

$$F\left(x + \frac{\hat{s}}{N P_X(x)}\right) = F(x) + \frac{\hat{s}}{N P_X(x)} \overbrace{F'(x)}^{P_X(x)} + \dots$$

SO THE SQUARE BRACKET BECOMES JUST

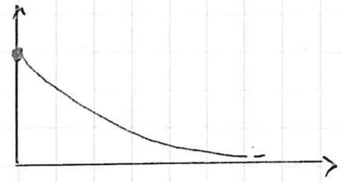
$$\left[1 - \frac{\hat{s}}{N}\right]^{N-2} \xrightarrow[N \to \infty]{} e^{-\hat{s}}.$$

NOTE: $\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$.

KEEPING JUST THE LEADING ORDER IN THE OTHER TERM,

$$P_N\left(s = \frac{\hat{s}}{N P_X(x)} \,\middle|\, X_j = x\right) \underset{N \gg 1}{\simeq} P_X(x)\, e^{-\hat{s}}.$$

FINALLY,



$$\hat{P}_N(\hat{s}) = P_N\left(s = \frac{\hat{s}}{N P_X(x)}\right)\frac{ds}{d\hat{s}}$$

$$= N \cdot \frac{1}{N}\int dx\, P_X(x) \cdot \frac{1}{P_X(x)}\, P_X(x)\, e^{-\hat{s}} \underset{N\gg 1}{=} e^{-\hat{s}}.$$

NOTICE THIS IS COMPLETELY DIFFERENT FROM WIGNER'S SURMISE ("WIGNER-DYSON" STATISTICS). IT IS CALLED "POISSON STATISTICS", WHICH DOESN'T MEAN IT'S A POISSONIAN, BUT IT JUST LABELS "THINGS THAT ATTRACT EACH OTHER".

- "LAYMAN" CLASSIFICATION OF RM MODELS

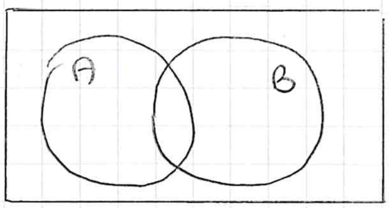
$$X = \begin{pmatrix} x_{11} & \cdots & x_{1N} \\ \vdots & & \vdots \\ x_{N1} & \cdots & x_{NN} \end{pmatrix}, \quad \mathcal{P}[X] = P(x_{11}, \dots, x_{NN})$$ JOINT PROB. DENSITY OF THE ENTRIES.

REQUIREMENT: X HAS REAL SPECTRUM $\{\lambda_1 \dots \lambda_N\}$.

TO BE EVEN MORE RESTRICTIVE,

$$X = \begin{cases} \text{REAL SYMMETRIC} & \beta = 1 \\ \text{COMPLEX HERMITIAN} & \beta = 2 \\ \text{QUATERNION SELF-DUAL} & \beta = 4 \end{cases}$$

WHERE β IS THE "DYSON INDEX". FROBENIUS THM OF DIVISION ALGEBRA FORBIDS TERNIONS.



WE CAN THEN DISTINGUISH:

A) INDEPENDENT ENTRIES

$$P(x_{11}, \dots, x_{NN}) = P_{11}(x_{11}) \dots P_{NN}(x_{NN})$$

(MODULO ADDITIONAL SYMMETRIES, WHICH CAN REDUCE THE P_{ij} NEEDED).

B) ROTATIONAL INVARIANCE

$$\mathcal{P}[X]\, dx_{11} \dots dx_{NN} = \mathcal{P}[X']\, dx'_{11} \dots dx'_{NN} \qquad X' = U X U^{-1}.$$

ROTATION LEAVES THE STATISTICAL WEIGHT UNCHANGED.

NOTICE IF X IS

- REAL SYMMETRIC → U ORTHOGONAL
- COMPLEX HERMITIAN → U UNITARY
- QUATERNION SELF-DUAL → U SYMPLECTIC.

WHAT DOES THIS MEAN FOR THE EIGENVECTORS? THEY CAN'T BE IMPORTANT, BECAUSE THEY CAN BE ROTATED.
ON THE CONTRARY, THE FACTORIZATION PROPERTY IN (A) ONLY HOLDS IN A PARTICULAR BASIS.

· <u>EXAMPLE</u>: WHAT HAPPENS IN THE INTERSECTION $A \cap B$?

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

$$P[X] = P(x_1, x_2, x_3) \overset{\text{INDEPENDENT ENTRIES}}{=} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(x_1^2 + x_2^2) - x_3^2}$$

$$= \frac{1}{(2\pi)\sqrt{\pi}} e^{-\frac{1}{2}[x_1^2 + x_2^2 + 2x_3^2]} = \frac{1}{(2\pi)\sqrt{\pi}} e^{-\frac{1}{2}\operatorname{Tr} X^2}.$$

INDEED (AND THIS IS POSSIBLE BECAUSE OF FACTORIZATION),

$$X^2 = \begin{pmatrix} x_1^2 + x_3^2 & x_1x_3 + x_3x_2 \\ x_1x_3 + x_3x_2 & x_2^2 + x_3^2 \end{pmatrix}.$$

BY PERFORMING A ROTATION, THE CYCLIC PROPERTY GIVES

$$P[X'] \propto e^{-\frac{1}{2}\operatorname{Tr}(UXU^T)^2} = e^{-\frac{1}{2}\operatorname{Tr} X^2}$$

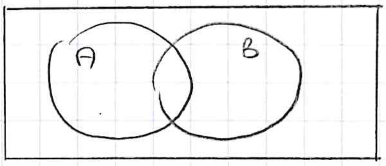
NOTE: $P[X]$ HAS THIS FORM FOR ALL GAUSSIAN MATRICES.

* IS THIS POSSIBLE FOR OTHER MODELS? NO: THE ONLY ENSEMBLE WITH INDEPENDENT ENTRIES AND ROTATIONAL INVARIANCE IS THE GAUSSIAN ENSEMBLE (IT'S A THEOREM BY PORTER AND ROSENZWEIG).

LECTURE 2

A: MODELS WITH INDEPENDENT ENTRIES

B: MODELS WITH ROTATIONAL INVARIANCE



RM MODELS WITH REAL SPECTRUM

$A \cap B$: GAUSSIAN ENSEMBLES,

$$P[H] \propto e^{-\alpha \operatorname{Tr} X^2}.$$

THIS CONTAINS

GOE (GAUSSIAN ORTHOGONAL ENSEMBLE) $\beta = 1$

GUE (GAUSSIAN UNITARY ENSEMBLE) $\beta = 2$

GSE (GAUSSIAN SYMPLECTIC ENSEMBLE) $\beta = 4$.

NOTICE THE NAME IS MEASLEADING: GOE DOES NOT CONTAIN ORTHOGONAL MATRICES, BUT SYMMETRIC MATRICES DIAGONALIZED BY AN ORTHOGONAL ONE.

* WE HAVE SEEN THE EXAMPLE OF A 2×2 GOE MATRIX

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \qquad P(x_1, x_2, x_3) \propto e^{-\frac{1}{2}\operatorname{Tr} X^2}.$$

CAN WE COMPUTE $P(\lambda_1, \lambda_2)$? THE LONG WAY IS

$$P(\lambda_1, \lambda_2) = \int dx_1\, dx_2\, dx_3\, P(x_1, x_2, x_3)\, \delta(\lambda_1 - \dots)\, \delta(\lambda_2 - \dots)$$

AND PLUG IN THE EXPRESSIONS WE HAD FOUND. ONE FINDS

$$P(\lambda_1, \lambda_2) \propto e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} |\lambda_1 - \lambda_2|$$

WHERE THE SECOND FACTOR ACCOUNTS FOR LEVEL REPULSION (IT GIVES RISE TO THE 'S' FACTOR IN THE DISTRIBUTION $P(S)$ OF LEVEL SPACINGS).

- GAUSSIAN MATRICES (GOE, GUE, GSE) $N \times N$

IT CAN BE SHOWN THAT

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) \propto e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}.$$

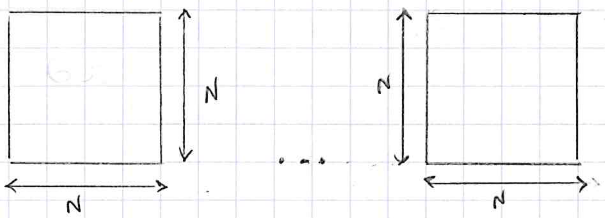
$\rightarrow$ DYSON INDEX.

HENCE, THIS IS A SYSTEM OF STRONGLY CORRELATED RANDOM VARIABLES (LONG-RANGE CORRELATIONS).
THIS FORMULA IS A COROLLARY OF THM 2 IN A PAPER BY Hsu IN Annals of Eugenics (1939)!

IN GENERAL, ONE HAS $O(N^2)$ ENTRIES BUT ONLY $O(N)$ EIGENVALUES. THE PRICE WE PAY IS THAT THEY GET CORRELATED.

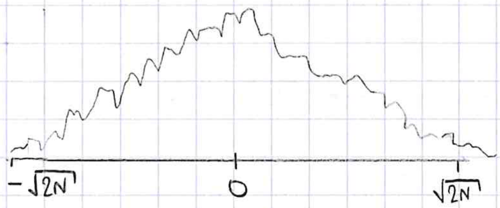
· QUANTITIES OF INTEREST

IMAGINE IN A NUMERICAL EXPERIMENT WE GENERATE GOE MATRICES



$\{\lambda_1^{(1)}, \dots, \lambda_N^{(1)}\}$ ... $\{\lambda_1^{(M)}, \dots, \lambda_N^{(M)}\}$

$\Rightarrow \{\lambda_1^{(1)}, \lambda_2^{(1)}, \dots, \lambda_1^{(M)}, \dots, \lambda_N^{(M)}\}$



THAT IS, WE COLLECTED INTO A VECTOR THE $N \times M$ REAL EIGENVALUES OF THE M SAMPLES AND PLOTTED A NORMALIZED HISTOGRAM.
WE FIND IT IS SYMMETRIC AROUND ZERO (ON AVG, POSITIVE AND NEGATIVE λ_i ARE EQUIPROBABLE). MOREOVER, IT GOES TO ZERO AROUND THE EDGE POINT $\sim \sqrt{2N}$, i.e. THE EXTENT OF λ_i's (AND NOT ONLY THEIR NUMBER) GROWS WITH N.
THE SHAPE OF THIS HISTOGRAM FOR $M \gg 1$ IS THE (AVERAGE) SPECTRAL DENSITY

$\rho_N(\lambda)$ = PROB. OF FINDING AN EIGENVALUE (REGARDLESS OF WHICH SAMPLE IT CAME FROM, OR WHICH PLACE IT OCCUPIED IN THE SORTED VERSION) AROUND POINT λ ON THE REAL AXIS.

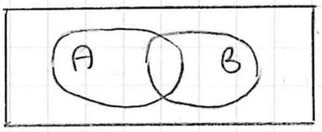
NOTICE THE AREA UNDER THE HISTOGRAM MUST BE 1, WHENCE

$$\int_{\sigma_N} d\lambda \, \rho_N(\lambda) = 1.$$

- REMARK:

THIS $\rho_N(\lambda)$ IS A HIGHLY NON-UNIVERSAL QUANTITY.

- PROBLEM: GIVEN $P(x_{11}, \dots, x_{NN})$, CAN WE COMPUTE $\rho_N(\lambda)$?



A) INDEPENDENT ENTRIES

$\rho_N(\lambda)$ CANNOT (YET) BE COMPUTED, BUT ONLY SOMETIMES "$\rho_{N\to\infty}(\lambda)$" USING REPLICAS.

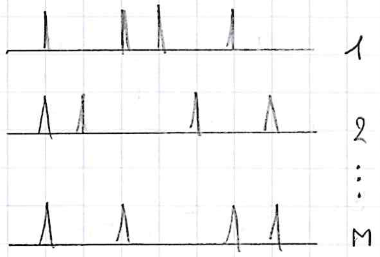
B) ROTATIONAL INVARIANCE

$\rho_N(\lambda)$ CAN BE COMPUTED BY THE METHOD OF ORTHOGONAL POLYNOMIALS, AND FOR $N\to\infty$ WE GET $\rho(\lambda)$.

THE EASIEST CASES ARE USUALLY $N=2$, $N\to\infty$.

★ A MORE FORMAL DEFINITION OF THE AVERAGE SPECTRAL DENSITY IS

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle .$$



GIVEN M SAMPLES, ONE CAN TAKE THIS AVERAGE.

- $N\to\infty$: RESOLVENT METHOD

LET'S DEFINE

$$G_N(z) = \frac{1}{N} \operatorname{Tr} \frac{1}{z\mathbb{1} - X} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z-\lambda_i} \qquad z \in \mathbb{C} \setminus \{\lambda_i\}.$$

NOTE: $G_N(z)$ IS A RANDOM COMPLEX FUNCTION WITH SIMPLE POLES AT THE LOCATION OF EACH EIGENVALUE.

BEING THIS A RANDOM OBJECT, WE CAN TAKE ITS AVERAGE OVER THE ENSEMBLE AND DEFINE THE RESOLVENT GREEN'S FUNCTION

$$\mathcal{G}(z) = \lim_{N\to\infty} \langle G_N(z) \rangle$$

$$= \int_\sigma d\lambda \, \frac{\rho(\lambda)}{z-\lambda} \qquad z \in \mathbb{C} \setminus \sigma .$$

NOTE: $\mathcal{G}(z)$ IS ALSO CALLED THE STIELTJES TRANSFORM OF $\rho(\lambda)$.

IF WE ARE ABLE TO INVERT THIS RELATION, WE HAVE $\rho(\lambda)$.

CONSISTENCY CHECK:

$$\int d\lambda \, \frac{\rho_N(\lambda)}{z-\lambda} = \int \frac{d\lambda}{z-\lambda} \langle \frac{1}{N} \sum_i \delta(\lambda-\lambda_i) \rangle = \langle \frac{1}{N} \sum_i \int \frac{d\lambda \, \delta(\lambda-\lambda_i)}{z-\lambda} \rangle$$

WHENCE, TAKING THE LIMIT FOR $N \to \infty$,

$$\int d\lambda \, \frac{\rho(\lambda)}{z-\lambda} = \lim_{N\to\infty} \langle \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z-\lambda_i} \rangle .$$

· PROPERTIES OF GREEN'S FUNCTION

1) FOR $|z| \to \infty$,

$$G(z) = \int d\lambda \, \frac{\rho(\lambda)}{z-\lambda} = \frac{1}{z} \int d\lambda \, \rho(\lambda) + \ldots = \frac{1}{z} + \ldots$$

2) IT IS THE GENERATING FUNCTION OF (TRACE) MOMENTS:

$$\int d\lambda \, \frac{\rho(\lambda)}{z-\lambda} = \frac{1}{z} \int d\lambda \, \frac{\rho(\lambda)}{1-\lambda/z} = \frac{1}{z} \int d\lambda \, \rho(\lambda) \sum_{k=0}^{\infty} \left(\frac{\lambda}{z}\right)^k$$

WHENCE

$$G(z) = \sum_{k=0}^{\infty} \frac{\mu_k}{z^{k+1}}$$

NOTE: $= \lim_{N\to\infty} \langle \frac{1}{N} \int d\lambda \sum_i \delta(\lambda-\lambda_i) \lambda^k \rangle = \lim_{N\to\infty} \langle \frac{1}{N} \sum_i \lambda_i^k \rangle .$

$$\mu_k = \int d\lambda \, \rho(\lambda) \, \lambda^k = \lim_{N\to\infty} \langle \frac{1}{N} \operatorname{Tr} X^k \rangle .$$

3) HOW TO RECONSTRUCT $\rho(\lambda)$ FROM $G(z)$?

WE NEED <u>SOKHOTSKI - PLEMELJ FORMULA</u>

$$\lim_{\varepsilon\to 0^+} \frac{1}{y \pm i\varepsilon} = PV\left(\frac{1}{y}\right) \mp i\pi \delta(y) .$$

PROOF:

LET φ BE A COMPLEX-VALUED FUNCTION DEFINED AND CONTINUOUS ON THE REAL LINE. LET $a<0<b$ BE REAL CONSTANTS AND TAKE

$$\int_a^b \frac{\varphi(y)\,dy}{y+i\varepsilon} = \int_a^b \frac{\varphi(y)\,dy}{y+i\varepsilon} \cdot \frac{y-i\varepsilon}{y-i\varepsilon} = \int_a^b \frac{\varphi(y)\,dy}{y^2+\varepsilon^2} (y-i\varepsilon)$$

$$= \int_a^b \frac{\varphi(y)\, y \, dy}{y^2+\varepsilon^2} - i \int_a^b \frac{\varphi(y)\, \varepsilon \, dy}{y^2+\varepsilon^2} .$$

BUT

$$\delta(y) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \frac{\varepsilon}{y^2+\varepsilon^2} ,$$

CAUCHY PRINCIPAL VALUE.

$$\lim_{\varepsilon\to 0^+} \int_a^b \frac{\varphi(y)\, y}{y^2+\varepsilon^2} \, dy = PV \int_a^b \frac{\varphi(y)}{y} \, dy .$$

★ NOW, IF WE COMPUTE THE RESOLVENT IN

$$\mathcal{G}(x-i\varepsilon)=\int d\lambda \frac{\rho(\lambda)}{x-i\varepsilon-\lambda}$$

WE CAN USE THE FORMULA ABOVE TO REWRITE

$$\lim_{\varepsilon\to 0^+} \operatorname{Im} \mathcal{G}(x-i\varepsilon) = \int d\lambda\, \rho(\lambda)\, \pi\, \delta(x-\lambda) = \pi \rho(x)$$

WHENCE THE INVERSION FORMULA

$$\rho(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \operatorname{Im} \mathcal{G}(x-i\varepsilon).$$

NOTICE WE NEED TO KNOW $\mathcal{G}(x)$ ON THE COMPLEX PLANE AS WELL.

- $\mathcal{G}(z)$ FOR GAUSSIAN ENSEMBLE

$$P(\lambda_1,\dots,\lambda_N) = \frac{1}{Z_N(\beta)} e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j-\lambda_k|^\beta$$

WHERE THE NORMALIZATION FACTOR IS

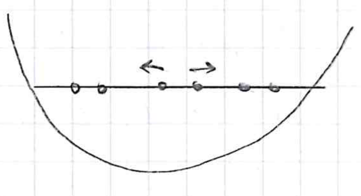
$$Z_N(\beta) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} d\lambda_1 \dots d\lambda_N\, e^{-\frac{1}{2}\sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j-\lambda_k|^\beta$$

$$= C_N(\beta) \int_{-\infty}^{+\infty} d\lambda_1 \dots d\lambda_N\, e^{-\beta N \mathcal{H}[\lambda_1,\dots,\lambda_N]} \qquad (\mathrm{I})$$

WHERE WE RESCALED $\lambda_i \to \sqrt{\beta N}\,\lambda_i$ AND CALLED

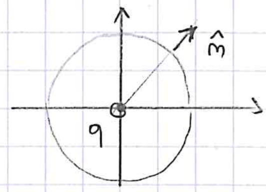
$$\mathcal{H}[\lambda_1,\dots,\lambda_N] = \frac{1}{2}\sum_{i=1}^N \lambda_i^2 - \frac{1}{2N}\sum_{i\neq j} \ln|\lambda_i-\lambda_j|.$$

THEN WE SEE $Z_N(\beta)$ CAN BE INTERPRETED AS THE CANONICAL PARTITION FUNCTION OF AN ASSOCIATED THERMODYNAMICAL SYSTEM. THE EIGENVALUES BEHAVE AS A GAS OF CHARGED PARTICLES, CONFINED INTO A WELL, WHICH REPEL EACH OTHER (BUT NOTICE THIS IS A LONG-RANGE INTERACTION).



THIS IS CALLED A "2D COULOMB GAS": INDEED, COULOMB

INTERACTION IS ONLY LOGARITHMIC IN 2D, SO WE PRETEND TO HAVE PARTICLES ON A PLANE CONFINED TO MOVE ON A LINE (AND WITH UNIT CHARGE). NOTICE IN FACT



$$\int_{\gamma} \underline{E} \cdot \hat{m} \propto q \quad \Rightarrow$$

$$E(r) \cdot 2\pi r \propto q \quad \Rightarrow \quad E(r) \propto \frac{q}{r}.$$

· <u>QUESTION</u>: FOR $N \gg 1$, WHAT IS THE MOST LIKELY CONFIGURATION OF OUR PARTICLES AT EQUILIBRIUM?

WE TURNED OUR PROBLEM INTO ONE OF CLASSICAL STATISTICAL MECHANICS. FOR $N \to \infty$, NO MATTER WHAT VALUE β TAKES, THE SYSTEM IN (I) IS BROUGHT TO ITS MINIMAL ENERGY CONFIGURATION:

$$\frac{\partial \mathcal{H}}{\partial \lambda_j} = 0 \quad \forall i = 1, \dots, N \qquad \text{(ZERO T LIMIT).}$$

THIS LEADS TO THE STABILITY CONDITION

$$\lambda_i = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}. \qquad \text{(II)}$$

NOTE: WE USE THIS CONDITION TO DERIVE AN EQUATION FOR THE RESOLVENT $G_N(z)$.

MULTIPLYING IT BY $[N(z-\lambda_i)]^{-1}$ AND SUMMING OVER i, THE LHS OF (II) BECOMES

$$\sum_{i=1}^{N} \frac{\lambda_i \quad (+z-z)}{N(z-\lambda_i)} = -1 + z\, G_N(z) \qquad G_N(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z-\lambda_i}$$

AND DOING THE SAME ON THE RHS WE HAVE

$$\beta = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z-\lambda_i} \cdot \frac{1}{\lambda_i - \lambda_j} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z-\lambda_j} \left[\frac{1}{z-\lambda_i} + \frac{1}{\lambda_i - \lambda_j} \right].$$

THIS IS MADE OF THE 2 CONTRIBUTIONS

$$\alpha = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z-\lambda_j} \cdot \frac{1}{z-\lambda_i} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{z-\lambda_j} \cdot \frac{1}{z-\lambda_i} - \frac{1}{N^2} \sum_{i=1}^{N} \frac{1}{(z-\lambda_i)^2}$$

$$= G_N^2(z) + \frac{1}{N} G_N'(z)$$

AND

$$b = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z-\lambda_j} \cdot \frac{1}{\lambda_i - \lambda_j} .$$

*NOTE: THIS DOESN'T MEAN THAT THE FULL RICCATI EQUATION GIVES THE CORRECT SOLUTION FOR FINITE N.

THIS IS A SYMMETRIC FUNCTION IN $\{\lambda_1, \dots, \lambda_N\}$. FOR INSTANCE, FOR $N=2$ WE GET

$$\frac{1}{z-\lambda_2} \cdot \frac{1}{\lambda_1-\lambda_2} + \frac{1}{z-\lambda_1} \cdot \frac{1}{\lambda_2-\lambda_1}$$

WHILE

$$B(N=2) = \frac{1}{z-\lambda_1} \cdot \frac{1}{\lambda_1-\lambda_2} + \frac{1}{z-\lambda_2} \cdot \frac{1}{\lambda_2-\lambda_1} .$$

IN GENERAL, IT REMAINS TRUE FOR ANY N THAT

$$b = -B$$

WHICH ALLOWS US TO REWRITE

$$B = G_N^2(z) + \frac{1}{N} G_N'(z) - B \quad \Rightarrow \quad B = \frac{1}{2} G_N^2(z) + \frac{1}{2N} G_N'(z) .$$

THERE REMAINS TO EQUATE THIS WITH THE LHS AND SOLVE

$$-1 + z\, G_N(z) = \frac{1}{2} G_N^2(z) + \frac{1}{2N} G_N'(z) .$$

THIS IS A RICCATI EQUATION, BUT FOR $N \to \infty$ WE CAN DISCARD* THE SUBLEADING TERM AND WE FIND THE ALGEBRAIC EQUATION

$$\frac{1}{2} G^2(z) - z\, G(z) + 1 = 0 \quad \Rightarrow \quad G(z) = z \pm \sqrt{z^2-2} .$$

★ TO FIND THE SPECTRAL DENSITY, WE COMPUTE

$$G(x - i\varepsilon) = x - i\varepsilon \pm \left[(x^2 - \varepsilon^2 - 2) + i(-2x\varepsilon) \right]^{1/2} . \qquad \text{(III)}$$

- LEMMA

THE PRINCIPAL SQUARE ROOT OF THE COMPLEX NUMBER $a+ib$ IS

$$\sqrt{a+ib} = p + iq$$

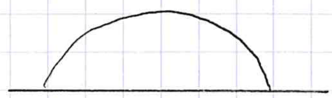
$$p = \frac{1}{\sqrt{2}} \sqrt{\sqrt{a^2+b^2} + a} \qquad q = \frac{\operatorname{sgn}(b)}{\sqrt{2}} \sqrt{\sqrt{a^2+b^2} - a} .$$

USING THIS ON (III) WE FIND

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} G(x - i\varepsilon) = \pm \frac{\operatorname{sgn}(-x)}{\pi \sqrt{2}} \sqrt{|x^2 - 2| - x^2 + 2}$$

SO THAT IF

$$\begin{cases} |x| > \sqrt{2} & \rightarrow \rho(x) = 0 \\ |x| < \sqrt{2} & \rightarrow \rho(x) = \frac{1}{\pi}\sqrt{2 - x^2} \end{cases}$$



WHICH IS WIGNER'S SEMICIRCLE LAW (IT'S ACTUALLY A SEMI-ELLIPSE).

LECTURE 3

ORTHOGONAL POLYNOMIALS TECHNIQUE

CONSIDER A REAL SYMMETRIC H, $N \times N$. THEN

$H = O \Lambda O^T$ $\qquad\qquad$ $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$

$O O^T = O^T O = \mathbb{1}$ $\qquad\qquad$ O CONTAINS THE EIGENVECTORS.

WHAT IS THE EFFECT OF THIS TRANSFORMATION ON $P(H_{11}, \dots, H_{NN})$?

$$\underbrace{P(H_{11}, \dots, H_{NN})}_{\frac{N(N+1)}{2} \text{ ENTRIES IN THE UPPER TRIANGLE}} dH_{11} \dots dH_{NN}$$

$$= \hat{P}(\lambda_1, \dots, \lambda_N, \{O\})\, d\lambda_1 \dots d\lambda_N \, d\{O\}$$

$\frac{N(N-1)}{2}$ PARAMETERS TO CHARACTERIZE O (THE EIGENVECTORS)

WHERE

$$\hat{P}(\lambda_1, \dots, \lambda_N, \{O\}) = P\left(H_{11}(\underline{\lambda}, \{O\}), H_{21}(\underline{\lambda}, \{O\}), \dots, H_{NN}(\underline{\lambda}, \{O\})\right) \cdot |J(H \to \{\underline{\lambda}, O\})|$$

AND THE JACOBIAN COULD IN GENERAL DEPEND ON $\underline{\lambda}$ AND $\{O\}$.
HOWEVER, IT TURNS OUT FOR RANDOM MATRICES IT DOES NOT:

$$J(H \to \{\underline{\lambda}, O\}) = J(\underline{\lambda}) = \left[\prod_{j<k} (\lambda_j - \lambda_k)\right]^{\beta}.$$

NOTICE THE ONLY REQUIREMENT ON H IS THAT IT BE SIMMETRIC (NO FURTHER RESTRICTIONS ON THE MEASURE).

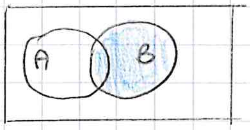
★ THIS $\hat{P}$ IS THE JOINT PROBABILITY DENSITY OF THE EIGENVALUES AND THE EIGENVECTOR "ENTRIES". IF WE WANT THE jPDF OF THE EIGENVALUES ALONE,

$$\hat{\hat{P}}(\lambda_1, \dots, \lambda_N) = \int dO \, \hat{P}(\lambda_1, \dots, \lambda_N, \{O\})$$

$$= \underbrace{\int dO \, P(H_{11}(\underline{\lambda}, \{O\}), \dots, H_{NN}(\underline{\lambda}, \{O\}))}\, \left|\prod_{j<k} (\lambda_j - \lambda_k)\right|^{\beta}. \qquad (\mathrm{I})$$

WE MAY OR MAY NOT BE ABLE TO COMPUTE THE INTEGRAL OVER $\{O\}$.

★ IT CAN ALWAYS BE PERFORMED IF $H \in$ B (ROTATIONALLY INVARIANT):



$$P[H] = P(H_{11}, \dots, H_{NN}) = \varphi(\mathrm{Tr}H, \mathrm{Tr}H^2, \dots, \mathrm{Tr}H^N).$$

THIS IS <u>WEYL'S LEMMA</u>.

USING IT ON (I), FOR ROTATIONALLY INVARIANT ENSEMBLES

$$\hat{\hat{P}}(\lambda_1, \dots, \lambda_N) = \underbrace{\int dO}_{\text{THIS IS A NUMBER}^*} \varphi\Big(\sum_i \lambda_i, \sum_i \lambda_i^2, \dots, \sum_i \lambda_i^N\Big) \cdot \Big|\prod_{j<k} (\lambda_j - \lambda_k)\Big|^\beta$$

$$= C_N \Big|\prod_{j<k} (\lambda_j - \lambda_k)\Big|^\beta \varphi\Big(\sum_i \lambda_i, \sum_i \lambda_i^2, \dots, \sum_i \lambda_i^N\Big).$$

FOR GAUSSIAN ENSEMBLES, FOR INSTANCE,

$$\varphi(\dots) = e^{-\frac{1}{2}\sum_i \lambda_i^2}.$$

*NOTE: $U(N)$, $O(N)$ ARE COMPACT GROUPS. SINCE $O^T O = \mathbb{1}$, ITS ROWS/COLUMNS ARE ORTHONORMAL, WHENCE $|O_{ij}| \leq 1$.

★ THERE ARE OTHER MODELS WHERE THIS INTEGRATION CAN BE PERFORMED, BUT MODELS IN $A \setminus B$ (INDEPENDENT ENTRIES, NOT ROTATIONALLY INVARIANT) ARE NOT YET AMONG THOSE.

★ THE ORTHOGONAL POLYNOMIAL TECHNIQUE APPLIES ON B, i.e. ON ENSEMBLES FOR WHICH $\hat{P}(\lambda_1, \dots, \lambda_N)$ IS KNOWN.
LET'S CONSIDER THE SIMPLEST CASE <u>$\beta = 2$</u>.
LAST TIME WE INTRODUCED THE AVERAGE SPECTRAL DENSITY

$$\rho_N(\lambda) = \Big\langle \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \Big\rangle$$

WHICH CONTAINS LESS INFORMATION THAN $\hat{\hat{P}}(\lambda_1, \dots, \lambda_N)$: INDEED,

$$\rho_N(\lambda) = \int d\lambda_1 \dots d\lambda_N \, \hat{\hat{P}}(\lambda_1, \dots, \lambda_N) \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i).$$

IF $\hat{\hat{P}}$ IS A SYMMETRIC FUNCTION OF THE EIGENVALUES, YOU CAN PROVE THAT

$$\rho_N(\lambda) = \int d\lambda_2 \ldots d\lambda_N \; \hat{\hat{P}}(\lambda, \lambda_2, \ldots, \lambda_N)$$

*NOTE: THIS IS NOT THE MOST GENERAL FORM, BUT IT COVERS MANY INTERESTING CASES.

I.E. IT IS THE "MARGINAL PDF".

HOWEVER, THIS INTEGRAL IS STILL VERY COMPLICATED: $\hat{\hat{P}}$ IS STRONGLY CORRELATED, SO IT HAS BAD FACTORIZATION PROPERTIES. DOUBLE HAT UNDERSTOOD, WE CAN EXPRESS*

$$P(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \prod_{j<k} (\lambda_j - \lambda_k)^2 \exp\left\{ -\sum_{i=1}^{N} V(\lambda_i) \right\}$$

FOR $\beta = 2$ AND $V(\lambda_i)$ SOME MODEL SPECIFIC FUNCTION (IN THE CONTEXT OF ROTATIONALLY INVARIANT MODELS).

- <u>TASK</u>: EVALUATE THE $(N-1)$-FOLD INTEGRAL

$$\rho_N(\lambda_1) = \int d\lambda_2 \ldots d\lambda_N \, \frac{1}{Z_N} \prod_{j<k} (\lambda_j - \lambda_k)^2 \exp\left\{ -\sum_{i=1}^{N} V(\lambda_i) \right\}.$$

LET'S CALL

$$\Delta_N(\underline{\lambda}) \equiv \prod_{j>k} (\lambda_j - \lambda_k) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & & \lambda_N^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{pmatrix}$$

(PROVE IT BY EXERCISE).

NOTE: GAUSSIAN ELIMINATION + INDUCTION.

THIS IS CALLED "<u>VANDERMONDE DETERMINANT</u>" (EVEN THOUGH IT WAS CAUCHY WHO WROTE IT). NOTICE THE NICE PROPERTY

$$\det \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \lambda_2 - \lambda_1$$

$$\det \begin{pmatrix} 1 & 1 \\ 3\lambda_1 + 17 & 3\lambda_2 + 17 \end{pmatrix} = 3(\lambda_2 - \lambda_1).$$

THIS SUGGESTS THAT WE CAN REPLACE EVERY ROW BY A POLYNOMIAL OF DEGREE k,

NOTE: IN GENERAL, ONLY THE HIGHEST COEFFICIENT COUNTS (SEE MATHEMATICA)

$$\pi_k(x) = a_k x^k + \ldots$$

THIS ALLOWS US TO REWRITE

$$\Delta_N(\underline{\lambda}) = \frac{1}{a_0 a_1 \cdots a_{N-1}} \det \begin{pmatrix} \pi_0(\lambda_1) & \cdots & \pi_0(\lambda_N) \\ \pi_1(\lambda_1) & & \pi_1(\lambda_N) \\ \vdots & & \\ \pi_{N-1}(\lambda_1) & & \pi_{N-1}(\lambda_N) \end{pmatrix} . \qquad (1)$$

THIS WAY WE HAVE INCREASED ENORMOUSLY THE NUMBER OF PARAMETERS WITHOUT CHANGING THE DETERMINANT. THIS IS TRUE FOR ANY ARBITRARY CHOICE OF POLYNOMIALS.
CAN WE CHOOSE THEM IN A CLEVER WAY?

* NOTICE, MOREOVER, THAT WE NEED

$$\rho(\lambda_1, \dots, \lambda_N) = \dots \prod_{j<k} (\lambda_j - \lambda_k)^2 e^{-\sum_i V(\lambda_i)} .$$

BUT RECALL

$$(\det A)^2 = \det(A^T A) = \det\left(\sum_{j=1}^{N} A_{ji} A_{jk}\right)$$

WHENCE

$$\Delta_N^2(\underline{\lambda}) = \frac{1}{\left(\prod_{j=0}^{N-1} a_j\right)^2} \det\left(\sum_{j=1}^{N} \pi_{j-1}(\lambda_i)\pi_{j-1}(\lambda_k)\right). \qquad (2)$$

* FINALLY, WE CAN PUT THE WEIGHT FUNCTION $e^{-\sum_i V(\lambda_i)}$ INSIDE THE DETERMINANT VIA

$$\left(\prod_l \gamma_l\right) \det\left(f(i,j)\right) = \det\left(\sqrt{\gamma_i \gamma_j}\, f(i,j)\right)$$

NOTE: JUST TRY TO SKETCH IT. MULTIPLYING EITHER A FULL ROW OR A FULL COLUMN BY A FACTOR PUTS THAT FACTOR IN FRONT OF THE DETERMINANT.

WHICH GIVES

$$e^{-\sum_i V(\lambda_i)} \Delta_N^2(\underline{\lambda}) = \left(\prod_{j=0}^{N-1} a_j\right)^{-2} \det\left(\sum_{j=0}^{N-1} \phi_j(\lambda_i)\phi_j(\lambda_k)\right) \qquad (3)$$

WHERE

$$\phi_i(x) \equiv e^{-\frac{1}{2}V(x)} \pi_i(x) .$$

THIS MAKES CLEAR THAT THE EIGENVALUES ARE NOT INDEPENDENT IN $\rho(\lambda_1, \dots, \lambda_N)$: THEY GET ENTANGLED BY A det.

★ SUMMARIZING, WE FOUND

$$\rho(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N \left(\prod_{j=0}^{N-1} a_j\right)^2} \det\left(K_N(\lambda_i, \lambda_k)\right)$$

WHERE WE DEFINED THE KERNEL

$$K_N(x, x') = e^{-\frac{1}{2}[V(x)+V(x')]} \sum_{j=0}^{N-1} \pi_j(x)\, \pi_j(x') \qquad \text{(II)}$$

WHICH DEPENDS ON BOTH THE POTENTIAL $V(x)$ AND THE CHOICE OF THE POLYNOMIALS $\pi_i(x)$.

IT TURNS OUT THE BEST CHOICE FOR $\pi_j(x)$ IS TO TAKE THEM ORTHONORMAL WRT $V(x)$,

$$\int dx\, e^{-V(x)}\, \pi_i(x)\, \pi_j(x) = \delta_{ij}.$$

NOTE: ORTHONORMAL POLYNOMIALS ARE UNIQUE. SEE FOR INSTANCE Koornwinder 1303.2825, (2.1 (UNIQUE FOR ANY GIVEN $V(x)$).

· EXAMPLE : GAUSSIAN CASE

$$\pi_j(x) = \frac{H_j(x/\sqrt{2})}{\sqrt{\sqrt{2\pi}\, 2^j\, j!}}$$

HERMITE POLYNOMIALS.

NOTE: THINK OF THE EIGENFUNCTIONS OF THE HARMONIC OSCILLATOR.

★ IF THE POLYNOMIALS π_k ARE CHOSEN THIS WAY, THEN THE KERNEL SATISFIES A CURIOUS "REPRODUCING" PROPERTY:

$$\int dy\, K_N(x, y)\, K_N(y, z) = K_N(x, z)$$

(PROVE IT BY EXERCISE).

NOTE:

$$= e^{-\frac{1}{2}(V(x)+V(z))} \sum_{j,k=0}^{N-1} \pi_j(x)\, \pi_k(z) \underbrace{\int dy\, e^{-V(y)}\, \pi_j(y)\, \pi_k(y)}_{=\delta_{jk}}.$$

WE HAVE TO COMPUTE

$$\int d\lambda_2 \dots d\lambda_N \det\left(K_N(\lambda_i, \lambda_k)\right) = ?$$

· EXAMPLE : 2×2

$$J_2(\underline{x}) = \begin{pmatrix} f(x_1, x_1) & f(x_1, x_2) \\ f(x_2, x_1) & f(x_2, x_2) \end{pmatrix}, \qquad \underline{x} = (x_1, x_2).$$

LET f SATISFY

$$\int f(x,y)\, f(y,z)\, dy = f(x,z).$$

THEN

$$\int dx_2 \det J_2(\underline{x}) = \int dx_2 \left[f(x_1,x_1) f(x_2,x_2) - f(x_1,x_2) f(x_2,x_1) \right]$$

$$= f(x_1,x_1) \cdot q - \int dx_2\, f(x_1,x_2) f(x_2,x_1) = (q-1) f(x_1,x_1)$$

WHERE WE CALLED

$$q = \int dx_2\, f(x_2,x_2).$$

BUT NOTICE

$$f(x_1,x_1) = \det J_1(\underline{x}).$$

IF THIS PROPERTY CARRIES OVER FOR LARGER MATRICES, WE ARE BASICALLY DONE!

· <u>THEOREM</u> (DYSON - GAUDIN INTEGRATION LEMMA)

LET $J_N(\underline{x})$ BE A $N \times N$ MATRIX WHOSE ENTRIES DEPEND ON A REAL VECTOR $\underline{x} = (x_1, \dots, x_N)$ AND HAVE THE FORM

$$J_{N\,ij} = f(x_i, x_j)$$

WHERE f SATISFIES THE REPRODUCING PROPERTY

$$\int dy\, f(x,y) f(y,z) = f(x,z).$$

THEN

$$\int \det [J_N(\underline{x})]\, dx_N = [q - (N-1)] \det(J_{N-1}(\underline{\tilde{x}}))$$

WHERE

$$\underline{\tilde{x}} = (x_1, \dots, x_{N-1}) \qquad q = \int dx\, f(x,x).$$

★ USING THIS RESULT, WE FIND

$$\int \det\left(K_N(\lambda_i,\lambda_j)\right)_{i,j=1,\dots,N} d\lambda_N = \det\left(K_N(\lambda_i,\lambda_j)\right)_{i,j=1,\dots,N-1}$$

WHERE WE USED THE FACT (PROVE IT!) THAT

$$q = \int dx\, K_N(x,x) = N.$$

NOTE: TRIVIAL.

• "<u>DOMINO EFFECT</u>"

$$\int \dots \int \det\left(K_N(\lambda_i,\lambda_j)\right)_{i,j=1,\dots,N} d\lambda_{K+1} \dots d\lambda_N$$

$$= (N-K)!\, \det\left(K_N(\lambda_i,\lambda_j)\right)_{1\le i,j\le K}.$$

IN PARTICULAR, SETTING $K=0$,

$$\int \dots \int \det(\dots)\, d\lambda_1 \dots d\lambda_N = N!.$$

BY NORMALIZATION,

$$1 = \int d\lambda_1 \dots d\lambda_N\, \rho(\lambda_1,\dots,\lambda_N)$$

$$= \int d\lambda_1 \dots d\lambda_N \frac{1}{Z_N \left(\prod_{j=0}^{N-1} a_j\right)^2} \det\left(K_N(\dots)\right)$$

WHENCE

$$Z_N = \frac{N!}{\left(\prod_{j=0}^{N-1} a_j\right)^2}.$$

★ WE HAVE NOW ALL THE INGREDIENTS TO COMPUTE

$$\rho_N(\lambda_1) = \frac{1}{Z_N} \int d\lambda_2 \dots d\lambda_N\, e^{-\sum_i V(\lambda_i)} \prod_{j<k} (\lambda_j-\lambda_k)^2$$

$$= \frac{1}{N!} \underbrace{\int d\lambda_2 \dots d\lambda_N \det\left(K_N(\lambda_i,\lambda_k)\right)}_{\to\ (N-1)!\, K_N(\lambda_1,\lambda_1)}$$

THE FINAL RESULT IS THAT

$$\rho_N(\lambda) = \frac{1}{N} K_N(\lambda, \lambda).$$

NOTE: $= \frac{1}{N} e^{-V(\lambda)} \sum_{j=0}^{N-1} \pi_j^2(\lambda)$

(THE KERNEL WAS DEFINED IN (II)).

- EXAMPLE: GUE ($\beta = 2$)

$$\rho_N(\lambda) = \frac{1}{N} K_N(\lambda,\lambda) = \frac{1}{N} e^{-\frac{1}{2}\left(\frac{1}{2}\lambda^2 + \frac{1}{2}\lambda^2\right)} \sum_{j=0}^{N-1} \frac{H_j^2(\lambda/\sqrt{2})}{\sqrt{2\pi}\, 2^j\, j!}$$

$$= \frac{1}{N\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} \sum_{j=0}^{N-1} \frac{H_j^2(\lambda/\sqrt{2})}{2^j\, j!}.$$

THIS IS AN EXACT RESULT FOR THE SHAPE OF THE HISTOGRAM. WE EXPECT TO RECOVER, IN THE $N \to \infty$ LIMIT, THE SEMICIRCLE LAW. AS WE DO SO, THE FREQUENCY OF "WIGGLES" IN THE HISTOGRAM INCREASES WITH N AND ITS SHAPE GRADUALLY TENDS TO A SEMICIRCLE.

LECTURE 4

• LARGEST EIGENVALUE (EXTREME VALUE THEORY)

ONE STEP BACK. CONSIDER A SET $\{X_1, \dots, X_N\}$ OF i.i.d. RANDOM VARIABLES, AND LET $p(x)$ BE THEIR COMMON pdf.

WHAT IS THE STATISTICS OF X_{MAX}?

DEFINE THE CUMULATIVE DISTRIBUTION FUNCTION

$$Q_N(x) = \mathrm{Prob}\left[X_{MAX} \leq x\right]$$
$$= \mathrm{Prob}\left[X_1 \leq x, X_2 \leq x, \dots, X_N \leq x\right] = \left(\int^x p(x')dx'\right)^N \equiv \left(\mathcal{P}(x)\right)^N.$$

WHAT HAPPENS IF WE SEND $N \to \infty$? BEING

$$0 \leq \mathcal{P}(x) \leq 1$$

WE WOULD NAIVELY GET EITHER 0 OR 1, WHICH IS TRIVIAL.

DEFINE INSTEAD

$$z \equiv \frac{x - a_N}{b_N} \qquad\qquad x = b_N z + a_N$$

AND LET'S STUDY

$$\lim_{N\to\infty} Q_N\left(b_N z + a_N\right) = F(z).$$

CAN WE CHOOSE a_N, b_N SO THAT $F(z)$ IS FINITE?

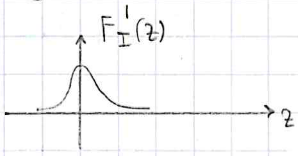
• FISHER-TIPPETT-GNEDENKO THEOREM

FOR i.i.d. RANDOM VARIABLES, THE SCALING FUNCTION $F(z)$ CAN ONLY BE ONE OF THREE DIFFERENT TYPES. LET

$$x^* \equiv \sup\left(x : \mathcal{P}(x) < 1\right)$$ UPPER ENDPOINT OF THE SUPPORT OF $p(x)$.

1) IF x^* IS FINITE OR INFINITE AND $p(x)$ FALLS OFF FASTER THAN ANY POWER FOR $x \to x^*$, THEN THE LIMITING DISTRIBUTION IS

$$F_I(z) = e^{-e^{-z}}$$ GUMBEL.



2) IF x^* IS INFINITE AND $p(x)$ FALLS OFF AS A POWER LAW,

FOR INSTANCE $p(x) \sim x^{-(\gamma+1)}$, THEN THE LIMITING DISTRIBUTION IS

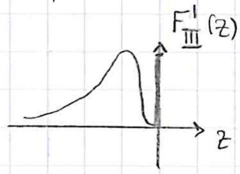
$$F_{II}(z) = \begin{cases} e^{-1/z^{\gamma}}, & z > 0 \\ 0, & \text{OTHERWISE.} \end{cases}$$

FRÉCHET

3) IF x^* IS FINITE AND $p(x) \sim (x^* - x)^{\gamma - 1}$, THEN THE LIMITING DISTRIBUTION IS

$$F_{III}(z) = \begin{cases} e^{-|z|^{\gamma}}, & z < 0 \\ 1 & z > 0. \end{cases}$$

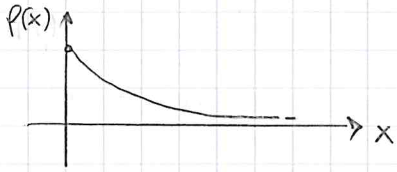
WEIBULL



NOTICE THESE 3 ARE ALL WELL POSED CUMULATIVE DISTRIBUTIONS.

- EXAMPLE: EXPONENTIAL pdf

$$p(x) = \mu e^{-\mu x}, \quad x \geq 0.$$



LET'S COMPUTE EXPLICITLY

$$Q_N(x) = \left[\int_0^x \mu e^{-\mu x'} dx' \right]^N = [1 - e^{-\mu x}]^N = e^{N \log(1 - e^{-\mu x})}.$$

AS $x \to \infty$,

$$Q_N(x) \simeq e^{-N e^{-\mu x}} = e^{-e^{-(\mu x - \log N)}}$$

NOTE: IN ORDER NOT TO GET A TRIVIAL RESULT, WE NEED TO SEND $x \to \infty$ IN SOME WAY AS WE TAKE $N \to \infty$.

WHICH SUGGESTS THAT WE MAY DEFINE

$$z \equiv \mu x - \log N$$

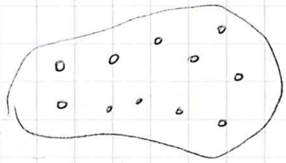
$$\begin{cases} a_N = \frac{1}{\mu} \log N \\ b_N = \frac{1}{\mu} \end{cases}$$

$$\lim_{N \to \infty} Q_N\left(\frac{z + \log N}{\mu} \right) = e^{-e^{-z}}.$$

NOTE: MORE IN THE FINAL Addendum.

- "Will a large complex system be stable"?

(Lord R. May, Nature 238, 413 (1972)).



SUPPOSE YOU HAVE AN ECOSYSTEM MADE OF MANY SPECIES, INITIALLY NON-INTERACTING AND STABLE: CALLING

$\rho_i \equiv$ POPULATION DENSITY OF THE i-TH SPECIES

$x_i = \rho_i - \rho_i^*$

WHERE x_i DECAYS TO ZERO IN TIME.

A SIMPLE MODEL WOULD BE

$$\frac{dx_i}{dt} = -x_i \, .$$

WHAT HAPPENS IF WE SWITCH ON THE INTERACTIONS? IMAGINE

$$\frac{dx_i}{dt} = -x_i + \alpha \sum_j A_{ij} x_j \qquad \alpha = \text{AVERAGE INTERACTION STRENGTH}$$

i.e. THE EVOLUTION EQUATIONS GET COUPLED. WE MAY TAKE A_{ij} TO BE A RANDOM MATRIX: WILL THE SYSTEM BE STABLE? OR BETTER, WHAT IS THE PROBABILITY OF IT BEING STABLE?

* LET $\hat{A}$ BE A GAUSSIAN SYMMETRIC MATRIX (GOE). WE REWRITE

$$\frac{d}{dt} \underline{x} = (\alpha \hat{A} - \mathbb{1}) \underline{x} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} .$$

WE DIAGONALIZE $\hat{A}$ BY AN ORTHOGONAL TRANSFORMATION,

$$\hat{A} = S \Lambda S^{-1}$$

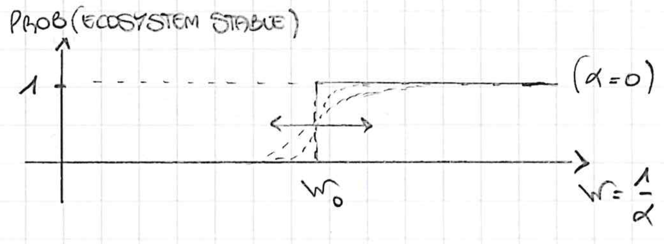
$$\frac{d}{dt} \underline{y} = (\alpha \Lambda - \mathbb{1}) \underline{y} \qquad \underline{y} = S^{-1} \underline{x}$$

AND NOW THE SYSTEM IS DECOUPLED.

THE CONDITION FOR STABILITY IS

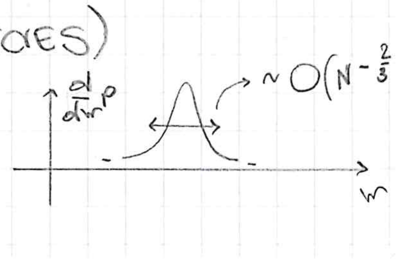
$$\underline{\alpha \lambda_i - 1 < 0 \quad \forall i} \quad \Rightarrow \quad \lambda_i < \frac{1}{\alpha} \equiv w \quad \forall i \, .$$

ALL THE EIGENVALUES MUST BE SMALLER THAN A CERTAIN THRESHOLD: THIS IS THEN A STATEMENT ABOUT THE MAXIMUM OF THE EIGENVALUES.



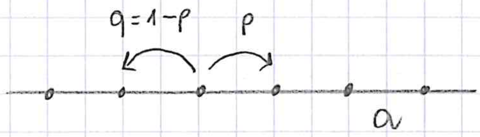
IN THE $N \to \infty$ LIMIT, THE PROBABILITY OF STABILITY UNDERGOES A SHARP PHASE TRANSITION, THAT IS (N = NUMBER OF SPECIES)

$$\lim_{N \to \infty} \text{PROB(ECOSYSTEM STABLE)} = \begin{cases} 1 & \alpha < \alpha_c \\ 0 & \alpha > \alpha_c \, . \end{cases}$$



CRASH COURSE ON LARGE DEVIATIONS

CONSIDER A RANDOM WALK IN 1d. LET



X_n = POSITION OF THE WALKER AT STEP n

$= X_{n-1} + \xi_n$ (MARKOV PROPERTY FOR R.W.)

WHERE

$$\mathcal{P}(\xi_n) = p\,\delta(\xi_n - 1) + q\,\delta(\xi_n + 1).$$

BY ITERATION WE EASILY OBTAIN

$$X_n = \sum_{k=1}^{n} \xi_n$$

WHENCE

$$\langle X_n \rangle = \langle \sum_{k=1}^{n} \xi_k \rangle = \sum_{k=1}^{n} \langle \xi_k \rangle = n(p-q)$$

$$\mathrm{Var}(X_n) = \langle X_n^2 \rangle - \langle X_n \rangle^2 = 4pqn.$$

THEN, BY CENTRAL LIMIT THEOREM (CLT),

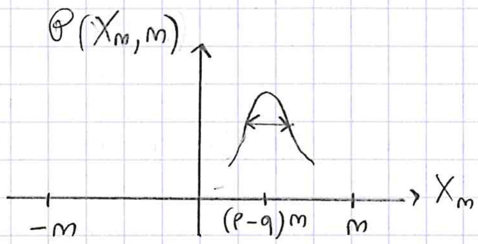
$$\mathcal{P}(X_n, n) \sim \frac{1}{\sqrt{2\pi 4pqn}} \exp\left\{-\frac{1}{8pqn}\left(X_n - (p-q)n\right)^2\right\}.$$

ONE MAY WRITE EQUIVALENTLY

$$X_n = (p-q)n + \sqrt{4pqn}\,\chi$$

χ RANDOM VARIABLE

$$p(\chi) = \frac{1}{\sqrt{2\pi}} e^{-\chi^2/2} \quad \text{AS } n \to \infty.$$



* NOTICE X_n IS REALLY BOUNDED, WHILE A GAUSSIAN IS DEFINED OVER THE WHOLE REAL AXIS. MOREOVER,

$$\mathrm{Prob}(X_n = n) = p^n = e^{n(\log p)} \quad \text{(EXACT RESULT)} \qquad (I)$$

WHILE BY CLT

$$\mathcal{P}(X_n = n) \sim \frac{1}{\sqrt{2\pi 4pqn}} \exp\left\{-\frac{1}{8pqn}\left(n - (p-q)n\right)^2\right\}$$

AND

$$\lim_{m\to\infty} \frac{\log P(X_m = m)}{m}$$

IS VERY DIFFERENT: THE RESULT IS ONLY ACCURATE UP UNTIL A SCALE $\sim \sqrt{4pqm}$.

THERE MUST BE A FUNCTION INTERPOLATING BETWEEN THE AVERAGE AND THE EXTREME: THIS IS CALLED THE LARGE DEVIATION (RATE) FUNCTION.

* IN THIS PROBLEM, WE CAN DEFINE

m_+ = # OF STEPS TO THE RIGHT

m_- = # OF STEPS TO THE LEFT

$$\Rightarrow \begin{cases} X_m = m_+ - m_- \\ m = m_+ + m_- \end{cases}$$

SO THAT THE RATE FUNCTION IS SIMPLY

$$P(X_m, m) = \binom{m}{m_+} p^{m_+} q^{m_-}$$

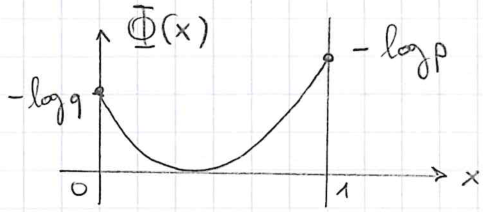
$$\overset{m\gg 1}{\simeq} e^{-m\,\Phi(x)} \qquad x = \frac{X_m}{m}$$

AND YOU CAN PROVE THAT

$$\Phi(x) = \frac{1+x}{2}\log\left(\frac{1+x}{2p}\right) + \frac{1-x}{2}\log\left(\frac{1-x}{2q}\right).$$

CHOOSING $X_m = m$ (EXTREME EVENT),

$$\lim_{x\to 1} \Phi(x) = -\log p$$



WHICH REPRODUCES THE EXACT RESULT (I).

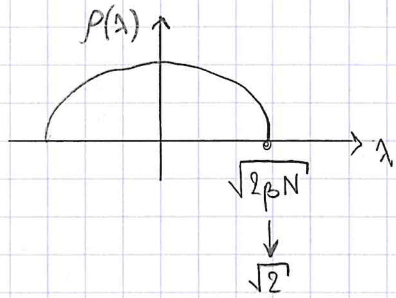
MOREOVER, EXPANDING $\Phi(x)$ AROUND ITS MINIMUM WE RECOVER

$$\Phi(x) \simeq \frac{1}{8pq}\left(x-(p-q)\right)^2$$

WHICH REPRODUCES THE CLT RESULT

$$P(X_m, m) \sim e^{-m\frac{1}{8pq}\left(\frac{X_m}{m}-(p-q)\right)^2} = e^{-\frac{1}{8pq}\left(X_m - m(p-q)\right)^2}.$$

LECTURE 5



GAUSSIAN RANDOM MATRICES' EIGENVALUES FOLLOW THE SEMICIRCLE LAW FOR $N \to \infty$.

WE MAY RESCALE THE EIGENVALUES AS

$$\lambda_i \to \frac{\lambda_i}{\sqrt{\beta N}}$$

SO THAT THE SEMICIRCLE EXTENDS FROM $-\sqrt{2}$ TO $\sqrt{2}$. INDEED,

$$\lim_{N\to\infty} \langle \lambda_{MAX}^{(N)} \rangle = \sqrt{2}.$$

BUT WHAT IS THE FULL DISTRIBUTION OF λ_{MAX}?

WE MAY REWRITE THE R.V. λ_{MAX} AS

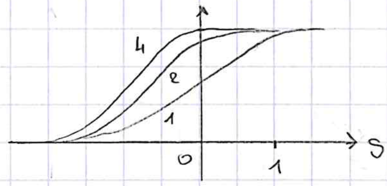
$$\lambda_{MAX} = \sqrt{2} + \frac{1}{\sqrt{2}} N^{-2/3} \chi_\beta$$

NOTE: THE SINGLE REALIZATION OF λ_{MAX} DOESN'T HAVE TO BE $\leq \sqrt{2}$.

$\beta = 1, 2, 4$.

IT HAS BEEN PROVEN (1994) THAT, DEPENDING ON β,

$$\lim_{N\to\infty} P[\chi_\beta \leq s] = \begin{cases} F_1(s) \\ F_2(s) \\ F_4(s) \end{cases}$$



WHICH ARE CALLED <u>TRACY-WIDOM</u> DISTRIBUTIONS. THESE ARE QUITE COMPLICATED; FOR INSTANCE,

$$F_2(s) = \exp\left\{ -\int_s^\infty (x-s)\, q^2(x)\, dx \right\}$$

WHERE $q(x)$ IS THE FUNCTION WHICH SATISFIES

$$q''(x) = 2q^3(x) + x q(x) \qquad \text{(PAINLEVÉ II EQUATION)}$$

WITH SOME APPROPRIATE BOUNDARY CONDITIONS*.

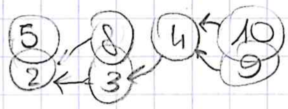
*NOTE: NAMELY, THAT $q(x)$ BE AN AIRY FUNCTION,
$q(x) \underset{x\to\infty}{\to} + Ai(x)$.

- <u>EXAMPLE</u>: Distribution of the length of the longest increasing subsequence of random permutations

OUT OF A SEQUENCE, EXTRACT AN INCREASING SUBSEQUENCE. THE LONGEST HERE HAS LENGTH 4.

5 2 8 3 4 10 9

SIMPLE ALGORITHM: READ THE SEQUENCE AND

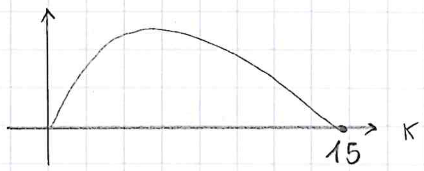


i) IF $X_{m+1} < X_m$, PUT IT ON TOP OF X_m.

ii) IF $X_{m+1} > X_m$, CREATE A NEW PILE WHERE EACH ELEMENT HAS AN ARROW POINTING TO THE TOP OF THE PREVIOUS PILE *

THEN, MIN NUMBER OF PILES = MAX LENGTH OF AN I.S..

* LET'S INTRODUCE RANDOMNESS. TAKE $1, 2, \dots, m$ AND CONSIDER ITS $m!$ PERMUTATIONS, GIVING TO EACH OF THEM EQUAL PROBABILITY. APPLY THE ALGORITHM ABOVE, WHICH GIVES YOU $m!$ NUMBERS (ONE FOR EACH PERMUTATION). HOW ARE THESE NUMBERS DISTRIBUTED?



ULAM (1961) CONJECTURED

$$C = \lim_{m \to \infty} \frac{1}{\sqrt{m}} \langle L_m \rangle$$

*NOTE: THE FOLLOWING X_m'S DON'T HAVE TO STAY ON THE NEW PILE.

WHERE IT WAS LATER FOUND THAT $C = 2$, AND FINALLY (1999)

$$\lim_{m \to \infty} P(\chi_m \leq x) = F_2(x) \qquad \chi_m = \frac{L_m - 2\sqrt{m}}{m^{1/6}}$$

(Baik, Deift, Johansson).

Shakespeare: "With caution judge of probability. Things deemed unlikely, e'en impossible, experience oft hath proven to be true".

- <u>LARGE DEVIATIONS OF THE LARGEST EIGENVALUE</u>

FOR GAUSSIAN MATRICES,

$$P(\lambda_1, \dots, \lambda_N) = \tilde{B}_N(\beta)\, e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i^2} \cdot \prod_{j<k} |\lambda_j - \lambda_k|^\beta .$$

APPLYING THE ABOVE RESCALING $\lambda_i \to \lambda_i / \sqrt{\beta N}$,

$$P(\lambda_1, \dots, \lambda_N) = B_N(\beta)\, e^{-\frac{\beta N}{2}\sum_{i=1}^{N} \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

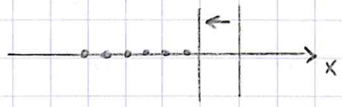
$$= B_N(\beta) \exp\left\{ -\beta \left[\frac{N}{2} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \right] \right\}$$

WHICH CAN BE INTERPRETED AS A "GAS" OF EIGENVALUES.

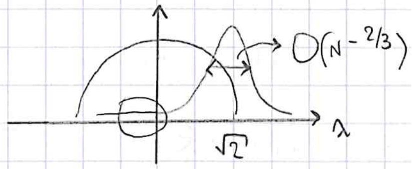
WE WANT TO COMPUTE

$$\mathbb{P}[\lambda_{max} \le x] = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} P(\lambda_1, \dots, \lambda_N)\, d\lambda_1 \dots d\lambda_N \equiv Z_N(x)$$

WHICH, IN STAT-MECH PARLANCE, IS SOME KIND OF PARTITION FUNCTION (CLASSICAL!) OVER A GIVEN REGION, AS IF WE WERE COMPRESSING A GAS WITH A BARRIER.



THE MOST LIKELY CONFIGURATION IS THEN FOUND BY MINIMIZING THE FREE ENERGY.



THIS WILL GIVE THE SHAPE OF THE FLUCTUATIONS OF THE LARGEST EIGENVALUE AROUND $\sqrt{2}$ (TRACY-WIDOM DISTRIBUTION), BUT LIMITED TO THE TWO TAILS. FOR INSTANCE, IT WILL ANSWER TO

$$\text{Prob}[\{\lambda_i\} < 0] = ?$$

- COULOMB GAS METHOD

GOAL:

$$Z_N(w) = \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} d\lambda_1 \dots d\lambda_N \exp\left\{-\beta N^2 \left(\frac{1}{2N} \sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln|\lambda_i - \lambda_j| \right)\right\}.$$

WHAT HAPPENS TO

$$\lim_{N \to \infty} \log Z_N(w) = ?$$

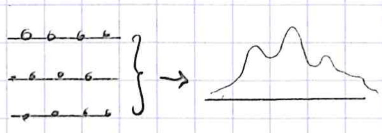
1) INTRODUCE A "COUNTING FUNCTION"

$$m(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - \lambda_i)$$

NOTE: WE ASSUME $m(x)$ TO BECOME, FOR $N \to \infty$, A SMOOTH FUNCTION OF x.

(IT'S THE NON-AVERAGE VERSION OF THE SPECTRAL DENSITY).

2) INSTEAD OF SUMMING (INTEGRATING) OVER THE MICROSTATES $\{\lambda_1, \dots, \lambda_N\}$, WE FIRST FIX A CERTAIN PROFILE $m(x)$ (NON-NEGATIVE, SMOOTH AND NORMALIZED) AND THEN WE SUM OVER ALL MICROSTATES WHICH ARE "COMPATIBLE" WITH THE $m(x)$ THAT WE FIXED (FINALLY, WE'LL SUM OVER ALL POSSIBLE $m(x)$).



3) BY YET ANOTHER DIRTY TRICK,

$$1 = \int \mathcal{D}[m(x)]\, \delta\left(m(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right)$$

WHICH IS THE FUNCTIONAL VERSION OF

$$1 = \int dx\, \delta(x) .$$

THIS WAY WE CAN REWRITE

$$Z_N(w) = \int \mathcal{D}[m(x)] \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} d\lambda_1 \ldots d\lambda_N \exp(\ldots) \cdot \delta\left(m(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)\right).$$

* NOW WE USE THE IDENTITIES

NOTE: $\int_{\mathbb{R}} dx\, m(x) = 1$.

$$\sum_{i=1}^{N} f(\lambda_i) = N\int dx\, f(x)\, m(x) \qquad m(x) \to \frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i)$$

$$\sum_{i,j=1}^{N} g(\lambda_i,\lambda_j) = N^2 \int dx\,dx'\, m(x)\,m(x')\, g(x,x') .$$

FOR EXAMPLE, IN THE HAMILTONIAN WE FIND

$$\frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 = \frac{1}{2N}\cdot N\int dx\, m(x)\, x^2$$

$$\frac{1}{2N^2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j| = ?$$

HERE THE $i=j$ CONTRIBUTION DIVERGES (CHARGED PARTICLES CAN'T SIT IN THE SAME PLACE). WHAT WE CAN DO IS TO ADD A CUTOFF

$$\frac{1}{2N^2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j| = \frac{1}{2N^2}\left[\sum_{i,j}\ln|\lambda_i-\lambda_j| - \sum_i \ln\Delta(\lambda_i)\right]$$

$$= \frac{1}{2N^2}\cdot N^2 \iint dx\,dx'\, m(x)m(x')\ln|x-x'| - (\text{CORRECTION}) .$$

INDEED, THE CUTOFF TERM WOULD PROVE SUBLEADING AND DISAPPEAR ANYWAY:

$$(\text{CORRECTION}) = \frac{1}{2N^2}\cdot N\int dx\, m(x) \ln\Delta(x)$$

$\Delta(x)$ = SHORT-DISTANCE CUTOFF.

★ SUMMARIZING,

$$Z_N(w) = \int \mathcal{D}[m(x)] \exp\left\{-\beta N^2\left(\frac{1}{2}\int dx\, m(x)\, x^2 - \frac{1}{2}\iint dx\, dx'\, m(x) m(x') \ln|x-x'|\right)\right\} \cdot$$

$$\cdot \int_{-\infty}^{w} d\lambda_1 \dots d\lambda_N \;\delta\left(m(x) - \frac{1}{N}\sum_{i=1}^{N} \delta(x-\lambda_i)\right)$$

WHERE THE SECOND LINE TERM IS BASICALLY AN ENTROPY (PROVE IT!),

$$I_N[m(x); w] \sim \exp\left[-N \int dx\, m(x) \ln m(x)\right]$$

NOTE: IT IS THE NUMBER OF MICROSTATES COMPATIBLE WITH $m(x)$. SEE FOCUS.

WHILE THE FIRST LINE TERM SCALES AS N^2: THE FREE ENERGY IS DOMINATED BY THE ENERGETIC COMPONENT, WHILE THE ENTROPIC COMPONENT IS SUBLEADING.

THIS IS TYPICAL OF LONG-RANGE SYSTEMS WITH PAIRWISE INTERACTIONS, AS THE NUMBER OF PAIRS IS $O(N^2)$.

THE REGIME WHERE THE ENERGY IS SUPEREXTENSIVE IS NOT THE ONE WE ARE USED TO CONSIDERING IN STAT-MECH.

OUR PARTITION FUNCTION THEN TAKES THE FORM

$$Z_N(w) \simeq \int \mathcal{D}[m(x)] \exp\left\{-\beta N^2 \mathcal{E}_w[m(x)] + O(N)\right\}$$

$$\mathcal{E}_w[m(x)] = \frac{1}{2}\int_{-\infty}^{w} dx\, m(x)\, x^2 - \frac{1}{2}\iint_{-\infty}^{w} dx\, dx'\, m(x) m(x') \ln|x-x'|.$$

★ NEXT, WE APPLY A "SADDLE-POINT" (LAPLACE) APPROXIMATION: THE MAIN CONTRIBUTION COMES FROM THE MINIMIZER

$$\left.\frac{\delta \mathcal{E}_w[m(x)]}{\delta m(x)}\right|_{m=m^*} = 0$$

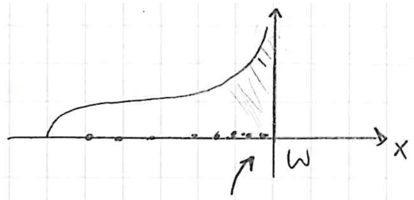
WHERE $m^*(x)$ WILL BE THE TYPICAL PARTICLE DENSITY.

WITHOUT THE WALL, THIS WOULD BE THE SEMICIRCLE (REMEMBER THE PARTICLES ARE SITTING INTO A HARMONIC WELL).

ADDING A BARRIER, WE EXPECT THE PARTICLES TO START

ACCUMULATING AGAINST THE WALL.

OUR TASK WILL BE THAT OF DESCRIBING THE RESULTING PROFILE.



★ PERFORMING THE FUNCTIONAL DERIVATION,

$$\frac{\delta}{\delta m(x)}\left[\frac{1}{2}\int_{-\infty}^{w} dy\, m(y)\, y^2\right] = \frac{1}{2}x^2$$

NOTE: I BELIEVE THEY SHOULD BOTH BE MULTIPLIED BY $\Theta(w-x)$, BUT IT WILL CANCEL OUT IN THE EQUATION.

$$\frac{\delta}{\delta m(x)}\left[\iint_{-\infty}^{w} dy\, dy'\, m(y) m(y') \ln|y-y'|\right] = 2\int_{-\infty}^{w} dy\, m(y)\ln|x-y|.$$

· <u>FOCUS</u>: ENTROPY TERM (Dean & Majumdar)

WE WANT TO COMPUTE THE JACOBIAN OF THE CHANGE $\{\lambda_i\}\to m(x)$, i.e.

$$I[m] \equiv \int_{-\infty}^{w}\prod_i d\lambda_i\, \delta\left(N m(x) - \sum_i \delta(x-\lambda_i)\right) \qquad (m(x) \text{ DENSITY FIELD})$$

$$= \int_{-\infty}^{w}\prod_i d\lambda_i \int \mathcal{D}[ig]\, e^{\int dx\, g(x)\left[N m(x) - \sum_i \delta(x-\lambda_i)\right]}$$

$$= \int \mathcal{D}[ig]\, e^{N\int dx\, g(x) m(x)} \underbrace{\int_{-\infty}^{w}\prod_i d\lambda_i\, e^{-\sum_i g(\lambda_i)}}_{\equiv A}$$

WHERE

$$A = \prod_i \int_{-\infty}^{w} d\lambda_i\, e^{-g(\lambda_i)} = \left(\int_{-\infty}^{w} d\lambda\, e^{-g(\lambda)}\right)^N.$$

THEN

$$I[m] = \int \mathcal{D}[ig]\, e^{N\left\{\int dx\, g(x) m(x) + \log\int_{-\infty}^{w} dx\, e^{-g(x)}\right\}}$$

$$\equiv \int \mathcal{D}[ig]\, e^{N F[g]}.$$

WE USE A SADDLE POINT BY REQUIRING

$$0 \equiv \frac{\delta F[g]}{\delta g(y)} = m(y) - \left(\int_{-\infty}^{w} dx\, e^{-g(x)}\right)^{-1} \cdot \int_{-\infty}^{w} dx\, e^{-g(x)}\, \delta(x-y).$$

CALLING

$$Z_W[g] = \int_{-\infty}^{W} dx\, e^{-g(x)}$$

WE HAVE

$$\frac{\delta F[g]}{\delta g(y)} = m(y) - \frac{1}{Z_W[g]} e^{-g(y)} \theta(W-y) \equiv 0$$

WHENCE THE SADDLE POINT (NOTE $m(x)$ IS CORRECTLY NORMALIZED)

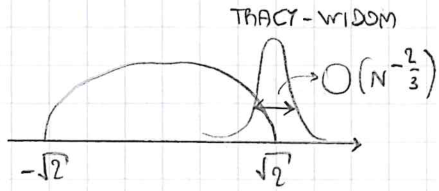
$$m(x) = \frac{1}{Z_W[g]} e^{-g(x)} \theta(W-x) \quad \leftrightarrow \quad g(x) = -\log\left(m(x) Z_W[g]\right) \theta(W-x).$$

SUBSTITUTING BACK INTO $I[m]$,

$$I[m] \simeq \exp\left\{N\left[-\int_{-\infty}^{W} dx\, m(x)\left(\log m(x) + \log Z_W[g]\right) + \log Z_W[g]\right]\right\}$$

$$= e^{-N\int_{-\infty}^{W} dx\, m(x) \log m(x)}.$$

LECTURE 6



COULOMB GAS TECHNIQUE

LAST TIME WE SET UP A SADDLE-POINT CALCULATION TO COMPUTE

$$P[\lambda_{max} \leq w] = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

WHERE WE EXPRESSED

$$Z_N(w) = \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} d\lambda_1 \ldots d\lambda_N \exp\left\{-\beta N^2 (\ldots)\right\}$$

$$\simeq \int_{\substack{m(x) \geq 0 \\ \int m(x) dx = 1}} \mathcal{D}[m(x)] \exp\left\{-\beta N^2 \mathcal{E}_w[m(x)] + O(N)\right\} \qquad (\mathrm{I})$$

AND NOTICED THAT THE ENERGY TERM IS SUPEREXTENSIVE. THIS IS

$$\mathcal{E}_w[m(x)] = \frac{1}{2}\int_{-\infty}^{w} dx\, m(x)\, x^2 - \frac{1}{2}\iint_{-\infty}^{w} dx\, dx'\, m(x) m(x') \ln|x - x'| .$$

THE SADDLE POINT IS GIVEN BY

$$\left.\frac{\delta \mathcal{E}_w[m(x)]}{\delta m(x)}\right|_{m = m^*} = 0$$

WHICH LEADS TO

$$\frac{1}{2}x^2 - \int_{-\infty}^{w} dx'\, m_w^*(x') \ln|x - x'| + C = 0 . \qquad (\mathrm{II})$$

HERE C IS A CONSTANT DUE TO THE NORMALIZATION OF $m(x)$. WE CAN ENFORCE THIS CONSTRAINT BY INSERTING INTO (I)

$$\delta\left(\int m(x)\, dx - 1\right) = 0$$

WHICH LEADS* TO THE ADDITIONAL FACTOR C IN (II).

WE OBSERVE (II) IS AN INTEGRAL EQUATION IN THE UNKNOWN $m_w^*(x)$ ("CARLEMAN EQUATION"), WHERE $m_w^*(x)$ ALSO DEPENDS PARAMETRICALLY ON THE POSITION w OF THE BARRIER.

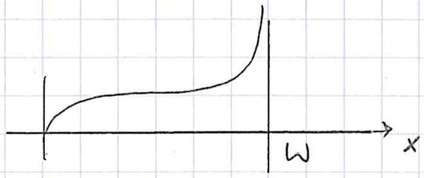
*NOTE: WE INTRODUCE A LAGRANGE MULTIPLIER, ADJUSTING THE SCALE SO AS TO HAVE THE CONSTRAINT AT LEADING ORDER
$\delta(\cdot) = \int d(\beta N^2 c)\, e^{-\beta N^2 C \left(\int dx\, m(x) - 1\right)}$
WHICH PRODUCES AN ADDITIONAL TERM INTO $\mathcal{E}_w[m(x)]$,
$+ C\left[\int dx\, m(x) - 1\right]$.
SETTING $x = 0$ IN (II) FINALLY GIVES
$C = \int_{-\infty}^{w} dx'\, m_w^*(x') \ln|x'|$.

· CLAIM: $m^*_\omega(x)$ CANNOT HAVE AN UNBOUNDED SUPPORT.

INDEED, SENDING $|x| \to \infty$ IN (II),

$LHS \sim |x|^2$

$RHS \sim \ln|x|$



AND THERE'S NO WAY THEY CAN BALANCE. WE THUS EXPECT SOME KIND OF CUTOFF.

1) DIFFERENTIATE (II) WRT x.

THIS HAS TO BE DONE WITH CARE ("WEAK DERIVATIVE"), BUT THE RESULT IS

NOTE: SEE FOCUS.

$$x - PV \int_{-\infty}^{\omega} dx' \, m^*_\omega(x') \frac{1}{x - x'} = 0 \qquad \text{(III)}$$

WHICH IS A SINGULAR INTEGRAL EQUATION (DUE TO THE PRESENCE OF THE CAUCHY SINGULAR VALUE).

WE HAVE A GENERAL TECHNIQUE TO SOLVE THIS KIND OF EQUATIONS (THE FIRST TERM x COMES FROM OUR PARTICULAR CHOICE OF $V(x)$, BUT OTHER MODELS GIVE DIFFERENT TERMS).

· TRICOMI'S FORMULA

$$g(\lambda) = PV \int_{a_1}^{a_2} dx' \, \frac{\rho(x')}{x - x'}$$

$$\Downarrow$$

$$\rho(\lambda) = \frac{1}{\pi \left[(a_2 - \lambda)(\lambda - a_1)\right]^{1/2}} \left\{ C_0 - PV \int_{a_1}^{a_2} \frac{dt}{\pi} \frac{\left[(a_2 - t)(t - a_1)\right]^{\frac{1}{2}}}{\lambda - t} g(t) \right\}$$

WHERE

$$C_0 = \int_{a_1}^{a_2} \rho(\lambda) d\lambda .$$

*NOTE: AND $\rho(\lambda) \equiv m^*_\omega(\lambda)$.

IN OUR CASE*, $g(\lambda) \equiv \lambda$ AND $C_0 \equiv 1$. AFTER A LOT OF ALGEBRA,

$$m^*_\omega(\lambda) = \frac{1}{8\pi \left[(a_2 - \lambda)(\lambda - a_1)\right]^{1/2}} \left[8 + (a_2 - a_1)^2 + 4(a_2 + a_1)\lambda - 8\lambda^2 \right]$$

WHICH IS THE GENERAL SOLUTION OF (II) WITH EDGES a_1, a_2.

WE PLUG THIS BACK INTO THE ENERGY FUNCTIONAL $\mathcal{E}_W[m^*_W(x)]$, TRUNCATING THE SUPPORT AS

$$\int_{-\infty}^{W} dx \to \int_{a_1}^{W} dx .$$

THIS GIVES

$$\mathcal{E}_W[m^*_W(x)] \equiv f(a_1, a_2; W)$$

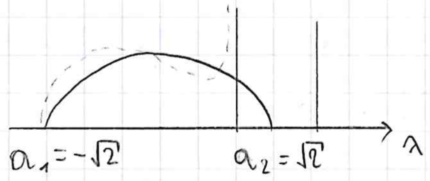
NOTE: FROM THE PHYSICAL POINT OF VIEW, WE COULD HAVE SET $a_2 = W$ FROM THE START (BUT a_1, a_2 IS FORMALLY PLEASANT).

WHICH CAN BE MINIMIZED WRT a_1, a_2, WHICH ARE THE ENDPOINTS OF THE SUPPORT.

FOLLOWING THIS PROCEDURE LEADS TO:

i) IF $W > a_2$, THEN $a_1 = -a_2 = -\sqrt{2}$ AND

$$m^*_W(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}.$$



ii) IF $W < (a_2 = \sqrt{2})$,

$$m^*_W(\lambda) = \frac{\sqrt{\lambda + L(W)}}{2\pi\sqrt{W-\lambda}}\left[W + L(W) - 2\lambda\right] \qquad L(W) = \frac{1}{3}\left[2\sqrt{W^2+6} - W\right]$$

WHICH INDEED PRESENTS A SQUARE ROOT DIVERGENCE AT THE POSITION OF THE WALL.

PLUGGING THIS BACK INTO $\mathcal{E}_W[m^*_W(\lambda)]$ GIVES THE FREE ENERGY FOR OUR PROBLEM.

- <u>FREE PROBABILITY</u>

LET A, B BE $N \times N$ HERMITIAN MATRICES (NON RANDOM).

WE WANT TO FIND THE EIGENVALUES OF $A+B$: THIS REQUIRES KNOWLEDGE OF THE EIGENVALUES OF A, B, BUT ALSO OF THE RELATIVE POSITION OF THEIR EIGENSPACES (NOT GOOD!).

DEFINE INSTEAD AN OPERATION (NON-STANDARD ADDITION OF MATRICES) WHICH DEPENDS <u>ONLY</u> ON THE EIGENVALUES OF INDIVIDUAL MATRICES.

WE CALL IT

NOTE: THIS WAY THE TRANSFORMATION PRESERVES THE EIGENVALUES OF A, B.

$$A \boxplus B = U^{+}AU + V^{+}BV$$

WHERE U, V ARE RANDOM UNITARY MATRICES (DRAWN UNIFORMLY FROM THE UNITARY GROUP).
NOTICE WE ARE RANDOMIZING THE EIGENVECTORS OF A, B:
WE HAVE THUS WASHED AWAY THE INFORMATION ABOUT THE EIGENSPACES. $A \boxplus B$ IS A RANDOM MATRIX (EVEN IF A, B WERE DETERMINISTIC/FIXED).

- FREENESS: GENERALIZATION OF STATISTICAL INDEPENDENCE FOR NON-COMMUTATIVE OBJECTS. IT REQUIRES 3 INGREDIENTS:
i) $N \to \infty$
ii) A, B ARE STATISTICALLY INDEPENDENT
iii) EIGENVECTORS ARE IN GENERIC POSITIONS
(i.e. THEY SHOULD NOT PLAY A ROLE). A WAY TO ACHIEVE THIS IS BY THE RANDOMIZATION ABOVE.

- CLASSICAL (COMMUTATIVE) PROBABILITY

$X_1, \dots, X_L$ INDEPENDENT R.V. DRAWN FROM $P_{X_i}(x)$
$S = X_1 + X_2 + \dots + X_L$.

HERE WE HAVE THE CHARACTERISTIC FUNCTION (GENERATING F. OF MOMENTS)

$$\varphi_X(t) = \langle e^{itx} \rangle = \int_{-\infty}^{+\infty} dx\, P_X(x)\, e^{itx}$$

AND

$$\varphi_S(t) = \varphi_{X_1}(t)\varphi_{X_2}(t) \dots \varphi_{X_L}(t)$$

SO THAT DEFINING

$$g_{X_1}(t) = \log \varphi_{X_1}(t)$$

WE HAVE THE GENERATING FUNCTION OF CUMULANTS

$$g_S(t) = g_{X_1}(t) + \dots + g_{X_L}(t).$$

★ CAN WE DEFINE SOMETHING SIMILAR? WE ALREADY DID: IT IS THE RESOLVENT, OR GREEN'S FUNCTION,

$$G_A(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z-A} \right\rangle = \int d\lambda \, \frac{\rho_A(\lambda)}{z-\lambda} .$$

REF: A. Zee, "Law of addition in random matrix theory".
"For the sake of convenience, we may, with due respect to Green, somewhat forcefully define a Blue's function ..."

$$G_A(B_A(z)) = B_A(G_A(z)) = z$$ <u>BLUE'S FUNCTION</u> B_A

i.e. IT'S THE FUNCTIONAL INVERSE OF THE GREEN'S FUNCTION.
IT SATISFIES* THE <u>FREE ADDITION RULE</u>

$$B_{H_1+\dots+H_L}(z) = B_{H_1}(z) + B_{H_2}(z) + \dots + B_{H_L}(z) + \frac{1-L}{z}$$

FOR FREE HERMITIAN MATRICES $\{H_1, \dots, H_L\}$. IN ORDER TO GET RID OF THE LAST TERM, DEFINE THE <u>R-FUNCTION</u>

$$R_A(z) \equiv B_A(z) - \frac{1}{z}$$

*NOTE: WE WILL NOT PROVE THAT G_A, R_A SATISFY THE FREE ADDITION RULE.

WHICH IS STRICTLY ADDITIVE (LIKE THE CUMULANT G.F.!),

$$R_{H_1+\dots+H_L}(z) = R_{H_1}(z) + \dots + R_{H_L}(z) .$$

- <u>THEOREM</u> (SUFFICIENT CONDITION FOR FREENESS)

LET N×N RANDOM MATRICES A_N, B_N s.t.

i) A_N, B_N HAVE AN ASYMPTOTIC EIGENVALUE DENSITY FOR $N \to \infty$
ii) A_N, B_N ARE INDEPENDENT
iii) B_N IS A UNITARY INVARIANT ENSEMBLE.

THEN A_N AND B_N ARE ASYMPTOTICALLY FREE.
IN OTHER TERMS,

$$A+B \iff A \boxplus B$$

IF B IS ROTATIONALLY INVARIANT.

1) THE R TRANSFORM IS THE GENERATING FUNCTION OF FREE CUMULANTS:

$$R(z) = \sum_{m=1}^{\infty} K_m z^{m-1} \qquad K_m \text{ FREE CUMULANTS.}$$

IN CLASSICAL PROBABILITY, FOR INSTANCE,

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

WHOSE CUMULANTS ARE ALL ZERO EXCEPT FOR $K_2 = 1$.

SIMILARLY, CONSIDER HERE THE CASE

$$R(z) = z$$

AND LET'S TRY TO COMPUTE THE CORRESPONDING GREEN'S FUNCTION:

$$B_A(G_A(z)) = z$$

$$R_A(G_A(z)) + \frac{1}{G_A(z)} = z \quad \Rightarrow \quad G_A(z) = \frac{1}{z - R_A(G_A(z))}$$

WHICH LOOKS A LOT LIKE THE I^{ST} DYSON-SCHWINGER EQUATION IN QFT (WITH $R_A(G_A(z)) \simeq \Sigma(z)$, THE SELF-ENERGY). HENCE

$$R(z) = z \quad \Rightarrow \quad G_A(z) = \frac{1}{z - G_A(z)}$$

WHICH IS OUR FAMILIAR

$$G_A(z) \cdot z - G_A^2(z) - 1 = 0$$

AND

$$G_A(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right) \quad \overset{\text{PLEMELJ SOKHOTSKI}}{\Rightarrow} \quad \rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}$$

WHICH IS THE SEMICIRCLE LAW.

SUMMING RANDOM MATRICES WITH A SEMICIRCLE LAW, YOU STILL GET A SEMICIRCLE LAW (AS FOR THE STABILITY OF GAUSSIAN DISTRIBUTIONS UNDER THE SUM).

* CONSIDER NOW $A_1 \dots A_L$ TAKEN FROM A ROTATIONAL INVARIANT ENSEMBLE, EACH WITH ITS INDIVIDUAL SPECTRAL DENSITY $\rho_{A_i}(\lambda)$.

TAKING

$$S = A_1 + A_2 + \ldots + A_L$$

HOW DO WE COMPUTE $\rho_S(\lambda)$ (AS $N \to \infty$)? FOR EACH A_i,

$$\rho_{A_i}(\lambda) \to G_{A_i}(z) \to B_{A_i}(z) \to R_{A_i}(z)$$

AND THEN FORM

$$R_S(z) = R_{A_1}(z) + \ldots + R_{A_L}(z)$$

WHICH CAN BE INVERTED AS

$$R_S(z) \to B_S(z) \to G_S(z) \to \rho_S(\lambda)$$

(THE LAST PASSAGE BY SOKOHOTSKI-PLEMELJ FORMULA).

THIS GIVES THE EXACT SPECTRAL DENSITY.

- FOCUS: PROOF OF (III) FROM VIVO'S NOTES
- DEF: WEAK DERIVATIVE

LET u BE A FUNCTION IN $\mathcal{L}^1([a,b])$. WE SAY THAT $v \in \mathcal{L}^1([a,b])$ IS A WEAK DERIVATIVE OF u IF

$$\int_a^b dx\, u(x)\varphi'(x) = -\int_a^b dx\, v(x)\varphi(x)$$

FOR ALL INFINITELY DIFFERENTIABLE FUNCTIONS φ WITH $\varphi(a) = \varphi(b) = 0$.

THE NOTION OF WEAK DERIVATIVE EXTENDS THE STANDARD (STRONG) DERIVATIVE TO FUNCTIONS THAT ARE NOT DIFFERENTIABLE, BUT INTEGRABLE. SETTING

$$u(x) \equiv \int dx'\, m_w^*(x') \ln|x - x'|$$

WE CAN WRITE

$$\int dx\, \varphi'(x) \left\{ \int dx'\, m_w^*(x') \ln|x - x'| \right\}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int dx\, \varphi'(x) \left\{ \int dx'\, m_w^*(x') \ln\left[(x - x')^2 + \varepsilon^2\right] \right\}$$

$$= -\frac{1}{2} \lim_{\varepsilon \to 0} \int dx\, \varphi(x) \left\{ \int dx' \frac{2(x - x') m_w^*(x')}{(x - x')^2 + \varepsilon^2} \right\} = -\int dx\, \varphi(x)\, PV \int dx' \frac{m_w^*(x')}{x - x'} .$$

LECTURE 7

· EDWARDS-JONES FORMULA

PARTICULARLY USEFUL IN THE A, B CLASS (INDEPENDENT ENTRIES, NO ROTATIONAL INVARIANCE) WHERE WE DON'T HAVE OTHER TOOLS.

GOAL: CONNECT THE JOINT pdf OF THE ENTRIES $\overbrace{P(x_{11},\dots,x_{NN})}^{\text{INPUT}}$ WITH THE AVERAGE SPECTRAL DENSITY $\underbrace{\rho_N(\lambda)}_{\text{OUTPUT}}$. NORMALLY, ONE USES

$$\underbrace{P(x_{11},\dots,x_{NN})}_{O(N^2)} \xrightarrow{①} P(\underbrace{\lambda_1,\dots,\lambda_N}_{O(N)}) \xrightarrow{②} \rho_N(\lambda).$$

BUT WE KNOW THE INTEGRATION OVER EIGENVECTORS (STEP 1) CANNOT ALWAYS BE COMPUTED IN CLOSED FORM, SO WE NEED A WAY TO CIRCUMVENT IT. NOTE: SEE LECTURE 3.

THE E-J FORMULA READS

NOTE: Log IS THE COMPLEX LOGARITHM.

$$\rho_N(\lambda) = -\frac{2}{\pi N}\lim_{\varepsilon\to 0^+}\mathrm{Im}\frac{\partial}{\partial\lambda}\langle \mathrm{Log}\, Z(\lambda)\rangle$$

$$Z(\lambda) = \int_{\mathbb{R}^N} d\underline{y}\, \exp\left\{-\frac{1}{2}\underline{y}^T(\lambda_\varepsilon \mathbb{1} - X)\underline{y}\right\}$$

WHERE

$$\lambda_\varepsilon = \lambda - i\varepsilon \qquad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

$$\langle\cdot\rangle = \int dx_{11}\dots dx_{NN}\, P(x_{11},\dots,x_{NN})(\cdot)$$ (AVERAGE OVER THE "DISORDER").

· REMARK: THE FORMULA IS IN PRINCIPLE VALID FOR FINITE N, BUT IN PRACTICE IT'S USED IN THE LIMIT $N\to\infty$, WHERE SEVERAL SIMPLIFICATIONS TAKE PLACE.

· PROOF

$$\rho_N(\lambda) = \left\langle \frac{1}{N}\sum_{i=1}^N \delta(\lambda-\lambda_i)\right\rangle$$

$$\frac{1}{x\pm i\varepsilon} \xrightarrow{\varepsilon\to 0^+} \mathrm{Pr}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$ (SOKHOTSKI - PLEMELJ FORMULA).

THEN

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon\to 0^+} \operatorname{Im} \left\langle \sum_{i=1}^{N} \frac{1}{\lambda - i\varepsilon - \lambda_i} \right\rangle$$

$$= -\frac{1}{\pi N} \lim_{\varepsilon\to 0^+} \operatorname{Im} \left\langle \sum_{i=1}^{N} \frac{1}{\lambda_i + i\varepsilon - \lambda} \right\rangle .$$

- COMPLEX LOGARITHM: THE $\log$ OF A COMPLEX NUMBER z IS w s.t. $e^w = z$.

PROBLEM: THE COMPLEX EXPONENTIAL IS NOT INJECTIVE, BECAUSE $e^{w+2\pi i} = e^w \quad \forall w$.

SOLUTION: RESTRICT THE DOMAIN OF THE EXP FUNCTION TO A REGION WHICH DOES NOT CONTAIN ANY TWO NUMBERS DIFFERING BY AN INTEGER MULTIPLE OF $2\pi i$. FOR ANY COMPLEX NUMBER $z = x + iy$

THE "PRINCIPAL VALUE" $\operatorname{Log} z$ IS DEFINED AS THE LOGARITHM WHOSE IMAGINARY PART LIES IN THE INTERVAL $(-\pi, \pi]$.

NOT ALL THE FAMILIAR PROPERTIES OF THE REAL $\log$ CARRY OVER TO THE COMPLEX Log. FOR INSTANCE

i) $\operatorname{Log} e^z$ MAY NOT BE JUST EQUAL TO z

ii) $\operatorname{Log}(z_1 z_2)$ MAY BE DIFFERENT FROM $\operatorname{Log} z_1 + \operatorname{Log} z_2$.

- EXAMPLE

$$\operatorname{Log}((-1)\, i) = -\frac{\pi i}{2}$$

$$\operatorname{Log}(-1) + \operatorname{Log}(i) = \frac{3\pi i}{2}.$$

★ BACK TO THE PROOF, WE CAN STILL WRITE

$$\sum_{i=1}^{N} \frac{1}{\lambda_i + i\varepsilon - \lambda} = -\frac{\partial}{\partial\lambda} \sum_{i=1}^{N} \operatorname{Log}(\lambda_i + i\varepsilon - \lambda).$$

WE WANT TO DEVISE SOMETHING SIMILAR TO THE USUAL

"$\operatorname{Tr} \log = \log \operatorname{Det}$".

NOTE: INCIDENTALLY, THIS IS BY FAR THE BEST WAY TO MEMORIZE THIS IDENTITY.

YOU CAN CHECK BY EXERCISE THAT ?

$$\mathcal{Z}(\lambda) = \int_{\mathbb{R}^N} d\underline{y}\, \exp\left\{ -\frac{i}{2} \underline{y}^T (\lambda_\varepsilon \mathbb{1} - X) \underline{y} \right\}$$

$$= (2\pi)^{N/2} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \mathrm{Log}(\lambda_i + i\varepsilon - \lambda) + \frac{iN\pi}{4} \right\} \qquad \text{(I)}$$

(MULTIDIMENSIONAL FRESNEL INTEGRAL). NOTICE

$$\lambda_\varepsilon \mathbb{1} - X$$

IS A COMPLEX SYMMETRIC MATRIX (BUT NOT HERMITIAN).

THE SECOND CHUNK OF THIS FORMULA IS EXACTLY WHAT WE NEED.

WITH SOME CARE, FROM (I) WE CAN GET

$$\sum_{i=1}^{N} \mathrm{Log}(\lambda_i + i\varepsilon - \lambda) = -2\,\mathrm{Log}\, \mathcal{Z}(\lambda) + (\ldots)$$

WHERE THE OTHER TERMS ARE KILLED BY $\frac{\partial}{\partial \lambda}$.

PUTTING EVERYTHING TOGETHER,

$$\rho_N(\lambda) = -\frac{2}{\pi N} \lim_{\varepsilon \to 0^+} \mathrm{Im} \frac{\partial}{\partial \lambda} \langle \mathrm{Log}\, \mathcal{Z}(\lambda) \rangle . \qquad \therefore$$

* EXPLICITLY, THIS READS

$$\langle \mathrm{Log}\, \mathcal{Z}(\lambda) \rangle = \int dx_{11} \ldots dx_{NN}\, P(x_{11}, \ldots, x_{NN})\, \mathrm{Log} \overbrace{\int_{\mathbb{R}^N} d\underline{y}\, e^{-\frac{i}{2} \underline{y}^T (\lambda_\varepsilon \mathbb{1} - X) \underline{y}}}^{\mathcal{Z}(\lambda)}$$

WHICH SEEMS HOPELESS: YOU CAN'T JUST COMPUTE $\int d\underline{y} (\ldots)$, BECAUSE YOU WOULD END UP RUNNING E-J FORMULA BACKWARDS.

THERE ARE, HOWEVER, TWO POSSIBLE STRATEGIES:

i) "ANNEALED" DISORDER. LITERALLY, "TO HEAT (METAL OR GLASS) AND ALLOW IT TO COOL SLOWLY, IN ORDER TO REMOVE INTERNAL STRESSES AND TOUGHEN IT."

ii) "QUENCHED" DISORDER. LITERALLY, "MADE LESS SEVERE OR INTENSE", "SUBDUED", "OVERCOME".

WE CAN INTERPRET $\mathcal{Z}(\lambda)$ AS A "CANONICAL" PARTITION FUNCTION CORRESPONDING TO A "GIBBS - BOLTZMANN" DISTRIBUTION

$$P(y_1, \dots, y_N) = \frac{1}{\mathcal{Z}(\lambda)} e^{-\mathcal{H}(\underline{y}; X, \lambda)}.$$

NOTE: THIS IS ONLY FORMAL, AS THE ENERGY FUNCTION $\mathcal{H}$ IS COMPLEX.

THIS WAY, $\langle \operatorname{Log} \mathcal{Z}(\lambda) \rangle$ IS THE AVERAGE OF A "FREE ENERGY" OVER THE DISORDER DEGREES OF FREEDOM $x_{11} \dots x_{NN}$.
NOTICE $\int d\underline{y}$ IS INSTEAD AN AVG OVER BOLTZMANN DISTRIBUTION.
THE TWO "LEVELS" OF RANDOMNESS ACT ON DIFFERENT TIME SCALES:

i) FIRST, THE DYNAMICAL VARIABLES $\{\underline{y}\}$ THERMALIZE FOR A FIXED INSTANCE OF THE MATRIX X (DISORDER). THIS IS CALLED A QUENCHED AVERAGE.

ii) THEN, THE FREE ENERGY IS AVERAGED OVER THE DISORDER (DIFFERENT REALIZATIONS OF X).

THIS IS THE <u>QUENCHED ROUTE</u>.

NOTE: THIS IS THE CORRECT WAY TO PROCEED, BUT IT HAS THE DRAWBACK THAT THE Log IS IN THE WAY.

- <u>ANNEALED STRATEGY</u>

THE ASSOCIATED STAT-MECH MODEL IS DESCRIBED IN TERMS OF THE JOINT SET OF VARIABLES $\{X, \underline{y}\}$:

$$\mathcal{Z}^{(ANN)}(\lambda) = \int dX \, d\underline{y} \, (\dots).$$

NOTE: X AND $\underline{y}$ FLUCTUATE AND THERMALIZE TOGETHER.

BUT THIS WOULD MEAN PULLING THE Log OUT OF THE INTEGRAL!
A WAY TO DESCRIBE THIS OPERATION IS

$$\langle \operatorname{Log} \mathcal{Z}(\lambda) \rangle \to \operatorname{Log} \langle \mathcal{Z}(\lambda) \rangle \equiv \operatorname{Log} \mathcal{Z}^{(ANN)}(\lambda).$$

NOTE: THE NAME $\mathcal{Z}^{(ANN)}$ IS MORE CORRECT.

- <u>EXAMPLE</u>: ANNEALED CALCULATION FOR GOE

$$P[X] = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi/N}} e^{-\frac{N}{2} x_{ii}^2} \prod_{i<j} \frac{1}{\sqrt{\pi/N}} e^{-N x_{ij}^2}.$$

WE RESCALED THE VARIANCE BY $1/N$ IN ORDER TO ENSURE A

GOOD LIMIT FOR $N \to \infty$ (THIS WAY THE EIGENVALUES ARE IN FACT RESCALED BY $\sqrt{N}$).

WE HAVE TO COMPUTE

$$Z^{(ANN)}(\lambda) = \int_{\mathbb{R}^N} d\underline{y} \int \prod_{i \leq j} dx_{ij}\, P[X]\, e^{-\frac{i}{2}\underline{y}^T(\lambda_\varepsilon \mathbb{1} - X)\underline{y}}$$

$$\propto \int_{\mathbb{R}^N} d\underline{y}\; e^{-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^N y_i^2} \cdot \underbrace{\langle e^{\frac{i}{2}\sum_{i=1}^N x_{ii} y_i^2} \rangle}_{\text{DIAGONAL ENTRIES}} \underbrace{\langle e^{i \sum_{i<j} x_{ij} y_i y_j} \rangle}_{\text{OFF-DIAGONAL}} .$$

USING

$$e^z \simeq 1 + z + \frac{z^2}{2} \dots$$

NOTE: $\langle x_{ij}^2 \rangle = \begin{cases} \frac{1}{N}, & i=j \\ \frac{1}{2N}, & i\neq j \end{cases}$ BY CONSTRUCTION.

$$\langle x_{ij} \rangle = 0 \qquad\qquad \langle x_{ij}^2 \rangle = \frac{1}{N(2-\delta_{ij})}$$

WE HAVE

$$\langle e^{\frac{i}{2}\sum_{i=1}^N x_{ii} y_i^2} \rangle = \langle \prod_{i=1}^N e^{\frac{i}{2} x_{ii} y_i^2} \rangle = \prod_{i=1}^N \langle 1 + \frac{i}{2} x_{ii} y_i^2 - \frac{1}{8} x_{ii}^2 y_i^4 + \dots \rangle$$

$$= \prod_{i=1}^N \left(1 - \frac{1}{8N} y_i^4 + \dots\right) \underset{N \gg 1}{\simeq} \prod_{i=1}^N e^{-\frac{1}{8N} y_i^4}$$

AND SIMILARLY

$$\langle \quad \rangle_{\text{OFF DIAG.}} = \prod_{i<j} e^{-\frac{1}{4N} y_i^2 y_j^2} .$$

THUS WE FIND

$$Z^{(ANN)}(\lambda) \propto \int_{\mathbb{R}^N} e^{-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^N y_i^2}\; e^{-\frac{1}{8N}\left(\sum_i y_i^2\right)^2}$$

WHERE WE USED

$$\sum_{ij} y_i^2 y_j^2 = \left(\sum_i y_i^2\right)^2 .$$

$\ast$ RECALL NOW THE GAUSSIAN IDENTITY (<u>HUBBARD-STRATONOVICH</u> TRANSF.)

$$\int_{-\infty}^{+\infty} dq\; e^{-\alpha q^2 + i\gamma q} \propto e^{-\frac{\gamma^2}{4\alpha}}$$

NOTE: EXACT IF THE GAUSSIAN IS NORMALIZED, $\sqrt{\frac{\pi}{\alpha}}$.

$$\gamma \equiv \sum_{i=1}^N y_i^2 \qquad\qquad \alpha = 2N .$$

THIS ALLOWS US TO REWRITE

$$\mathcal{Z}^{(ANN)}(\lambda) \propto \int_{-\infty}^{+\infty} dq\, e^{-2Nq^2} \underbrace{\int d\underline{y}\, e^{-\frac{i}{2}\lambda_\varepsilon \sum_i y_i^2 + iq\sum_i y_i^2}}_{\equiv I_N}$$

WHERE

$$I_N = \left\{ \int_{-\infty}^{+\infty} dy\, e^{-\frac{i}{2}\lambda_\varepsilon y^2 + iqy^2} \right\}^N \equiv e^{N \operatorname{Log}(\dots)}$$

YOU CAN CHECK BY EXERCISE* THAT

$$\mathcal{Z}^{(ANN)}(\lambda) \propto \int_{-\infty}^{+\infty} dq\, \exp\Big\{-N \underbrace{\Big(2q^2 - \frac{1}{2}\operatorname{Log}\frac{2\pi}{\varepsilon + i(\lambda - 2q)}\Big)}_{\varphi_\lambda(q)}\Big\}$$

$$\simeq e^{-N\varphi_\lambda(q^*)}$$

*NOTE: $\sqrt[N]{I_N} = \left[\frac{2\pi}{i(\lambda_\varepsilon - 2q)}\right]^{\frac{1}{2}}$.

WHERE WE USED A SADDLE-POINT

$$\varphi'_\lambda(q^*) = 0 \qquad \Rightarrow \qquad 4q^* + \frac{1}{2q^* - \lambda_\varepsilon} = 0$$

WHICH GIVES

$$q^* = \frac{1}{4}\left(\lambda_\varepsilon \pm \sqrt{\lambda_\varepsilon^2 - 2}\right)$$

(BIRTH OF A SEMICIRCLE).

★ THERE REMAINS TO APPLY E-J FORMULA (ANNEALED APPROXIMATION):

$$\rho_{N\to\infty}(\lambda) = -\frac{2}{\pi N} \lim_{\varepsilon\to 0^+} \operatorname{Im} \frac{\partial}{\partial\lambda} \operatorname{Log} \mathcal{Z}^{(ANN)}(\lambda)$$

$$= \frac{2}{\pi} \lim_{\varepsilon\to 0^+} \operatorname{Im} \frac{\partial}{\partial\lambda} \varphi_\lambda(q^*).$$

BUT

$$\frac{\partial}{\partial\lambda}\varphi_\lambda(q^*) = q^{*\prime} \underbrace{\frac{\partial}{\partial q}\varphi_\lambda(q)\Big|_{q=q^*}}_{=0} + \frac{\partial}{\partial\lambda}\varphi_\lambda(q)\Big|_{q=q^*}$$

WHENCE

$$\frac{\partial}{\partial\lambda}\left\{\frac{1}{2}\operatorname{Log}(\varepsilon + i(\lambda - 2q))\right\}\Big|_{q=q^*} = \frac{1}{2}\frac{i}{\varepsilon + i(\lambda - 2q)}\Big|_{q=q^*}$$

$$= \frac{1}{\lambda_\varepsilon \mp [\lambda_\varepsilon^2 - 2]^{1/2}}.$$

PUTTING EVERYTHING TOGETHER,

$$\rho_{N\to\infty}(\lambda) = \frac{2}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im} \frac{1}{\lambda_\varepsilon \mp \sqrt{\lambda_\varepsilon^2 - 2}}$$

$$= \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im} \left\{ \lambda_\varepsilon \pm \sqrt{\lambda_\varepsilon^2 - 2} \right\}$$

$$= \begin{cases} \frac{1}{\pi}\sqrt{2-\lambda^2}, & |\lambda| < 2 \\ 0 & \text{OTHERWISE.} \end{cases}$$

WHERE WE USED THE PROPERTIES* OF COMPLEX ROOTS.

THIS SHOWS THAT THIS RUDE TRICK SEEMS TO BE WORKING, AT LEAST FOR OUR FULLY CONNECTED MODEL.

*PROPERTY USED:

$$\sqrt{a+ib} = p+iq$$

WITH

$$p = \frac{1}{\sqrt{2}}\left[\left[a^2+b^2\right]^{1/2}+a\right]^{\frac{1}{2}} \qquad q = \frac{\mathrm{sgn}(b)}{\sqrt{2}}\left[\left[a^2+b^2\right]^{\frac{1}{2}}-a\right]^{\frac{1}{2}}.$$

LECTURE 8

EDWARD - JONES FORMULA

$$\rho_N(\lambda) = -\frac{2}{\pi N}\lim_{\varepsilon\to 0^+} \mathrm{Im}\,\frac{\partial}{\partial\lambda}\left\langle \operatorname{Log}\int_{\mathbb{R}^N} dY\, e^{-\frac{i}{2}\underline{Y}^T(\lambda_\varepsilon \mathbb{1}-X)\underline{Y}}\right\rangle$$

$$\langle\cdot\rangle = \int dx_{11}\ldots dx_{NN}\, P(x_{11}\ldots x_{NN})\,(\cdot) \qquad \cdot\ \lambda_\varepsilon = \lambda - i\varepsilon.$$

LAST TIME WE APPLIED IT TO THE GOE USING THE ANNEALED APPROXIMATION; BUT THERE'S A LESS DIRTY WAY TO DO IT, REMAINING IN A QUENCHED FASHION.
REMEMBER WHAT WE ARE AFTER IS

$$\int dx_{11}\ldots dx_{NN}\, P(x_{11}\ldots x_{NN})\operatorname{Log}\int_{\mathbb{R}^N} dY\,(\ldots).$$

* FOR THE GOE, RECALL

$$P(x_{11},\ldots,x_{NN}) = \prod_{i=1}^{N}\frac{e^{-Nx_{ii}^2/2}}{\sqrt{2\pi/N}}\prod_{i<j}\frac{e^{-Nx_{ij}^2}}{\sqrt{\pi/N}}.$$

· REPLICA IDENTITY (TRICK)

$$\langle \operatorname{Log} \mathcal{Z}(\lambda)\rangle = \lim_{m\to 0}\frac{1}{m}\operatorname{Log}\langle \mathcal{Z}^m(\lambda)\rangle.$$

PROOF:

$$\mathcal{Z}^m(\lambda) = e^{m\operatorname{Log}\mathcal{Z}(\lambda)} = 1 + m\log\mathcal{Z}(\lambda) + O(m^2)$$

$$\langle \mathcal{Z}^m(\lambda)\rangle = 1 + m\langle \log\mathcal{Z}(\lambda)\rangle + O(m^2)$$

$$\operatorname{Log}\langle\mathcal{Z}^m(\lambda)\rangle = \operatorname{Log}\left\{1+m\langle\operatorname{Log}\mathcal{Z}(\lambda)\rangle + O(m^2)\right\} = m\langle\operatorname{Log}\mathcal{Z}(\lambda)\rangle + O(m^2).$$

* IF m IS AN INTEGER, WE ARE BASICALLY REPLICATING $\mathcal{Z}(\lambda)$ m TIMES.
BUT NOTICE

$$\left[\int dx\,\varphi(x)\right]^2 = \int dx\,dY\,\varphi(x)\varphi(Y)$$

AND HERE IS WHERE THE DIRT COMES: AT THE END OF THE CALCULATION, WE WILL NEED A CONTINUATION TO ZERO, WHICH IS NOT INTEGER.

★ LET'S THEN COMPUTE

$$\langle Z^m(\lambda)\rangle = \int\left(\prod_{i\le j} dx_{ij}\right)\prod_{i=1}^{N}\frac{e^{-\frac{N}{2}x_{ii}^2}}{\sqrt{2\pi/N}}\prod_{i<j}\frac{e^{-N x_{ij}^2}}{\sqrt{\pi/N}}\cdot$$

$$\cdot\int_{\mathbb{R}^{N\cdot m}}\left(\prod_{a=1}^{m} d\underline{y}_a\right)\exp\left\{-\frac{i}{2}\sum_{i,j}\sum_{a=1}^{m} y_{ia}\left(\lambda_\varepsilon\delta_{ij}-x_{ij}\right)y_{ja}\right\}.$$

THIS WAY, THE Log HAS DISAPPEARED AND THIS ALLOWS US TO SWAP THE ORDER OF INTEGRATIONS, AND AVERAGE OVER THE DISORDER FIRST:

$$\langle Z^m(\lambda)\rangle = \int_{\mathbb{R}^{N\cdot m}}\left(\prod_{a=1}^{m} d\underline{y}_a\right)e^{-\frac{i}{2}\lambda_\varepsilon\sum_{i=1}^{N}\sum_{a=1}^{m}y_{ia}^2}\cdot\int\prod_{i=1}^{N}\frac{dx_{ii}}{\sqrt{2\pi/N}}\,e^{-\frac{N}{2}\sum_i x_{ii}^2+\frac{i}{2}\sum_i x_{ii}\sum_a y_{ia}^2}$$

$$\cdot\int\prod_{i<j}\frac{dx_{ij}}{\sqrt{\pi/N}}\,e^{-N\sum_{i<j}x_{ij}^2+i\sum_{i<j}\sum_{a=1}^{m}y_{ia}x_{ij}y_{ja}}.$$

WE CAN PERFORM THE GAUSSIAN INTEGRALS USING

$$\int_{-\infty}^{+\infty}dq\,e^{-\alpha q^2+i\gamma q}\propto e^{-\frac{\gamma^2}{4\alpha}}$$

REPEATEDLY, WITH

$$\alpha=\frac{N}{2}\ \text{OR}\ N \qquad\qquad \gamma=\frac{1}{2}\sum_a y_{ia}^2\ \text{OR}\ \sum_a y_{ia}y_{ja}.$$

THIS LEADS TO

$$\langle Z^m(\lambda)\rangle\propto\int_{\mathbb{R}^{N\cdot m}}\left(\prod_{a=1}^{m}d\underline{y}_a\right)e^{-i\frac{\lambda_\varepsilon}{2}\sum_{i=1}^{N}\sum_{a=1}^{m}y_{ia}^2\underbrace{-\frac{1}{8N}\sum_{i=1}^{N}\left(\sum_a y_{ia}^2\right)^2-\frac{1}{4N}\sum_{i<j}\left(\sum_a y_{ia}y_{ja}\right)^2}_{A}}. \qquad (\mathrm{I})$$

TERMS $\sim(\cdot)^2$ COUPLE DIFFERENT SITES (i,j). FIRST, NOTICE THAT

$$A=-\frac{1}{8N}\sum_{i,j}\left(\sum_a y_{ia}y_{ja}\right)^2.$$

NOW INTRODUCE THE NORMALIZED DENSITY $\mu(\vec{y})$, WHERE

$\vec{y}=(y_1,\dots,y_m)$ $\qquad$ ($\underline{y}=(y_1,\dots,y_N)$ INSTEAD)

DEFINED AS

$$\mu(\vec{y}) \equiv \frac{1}{N}\sum_{i=1}^{N}\prod_{a=1}^{m}\delta(y_a - y_{ia}).$$

NOTE: THE KEY POINT DOWN HERE IS THAT

$$\mu(\vec{w}) = \frac{1}{N}\sum_i \prod_a \delta(w_a - y_{ia})$$

THIS ALLOWS US TO REWRITE (PROVE IT BY EXERCISE)

$$A = -\frac{N}{8}\int d\vec{y}\, d\vec{w}\, \mu(\vec{y})\mu(\vec{w})\left(\sum_a y_a w_a\right)^2$$

WHERE

$$d\vec{y} = \prod_{a=1}^{m} dy_a .$$

NOTE:

$$A = -\frac{1}{8N}\sum_{i,j}\int d\vec{y}\, d\vec{w} \prod_{a=1}^{m}\delta(y_a - y_{ia})\delta(w_a - w_{ja})\cdot\left(\sum_c y_c w_c\right)^2$$

$$= -\frac{1}{8N}\sum_{i,j}\left(\sum_c y_{ic} w_{jc}\right)^2.$$

THIS PROCEDURE IS LONGER THAN HUBBARD - STRATONOVICH, BUT THE LATTER IS SPECIFIC FOR GAUSSIAN ENSEMBLES.

★ LET'S INTRODUCE THE REPRESENTATION OF THE IDENTITY

$$1 = \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\, \exp\left\{-i\int d\vec{y}\, \hat{\mu}(\vec{y})\left[N\mu(\vec{y}) - \sum_{i=1}^{N}\prod_{a=1}^{m}\delta(y_a - y_{ia})\right]\right\}$$

i.e. THE FUNCTIONAL ANALOGUE OF

$$\delta(x) = \int dk\, e^{ikx}.$$

NOTE: NO, IT'S THE ANALOGUE OF

$$1 = \int dx \frac{dk}{2\pi} e^{-ikx} = \int dx\, \delta(x).$$

USING IT INTO (I) WE GET

$$\langle \mathcal{Z}^m(\lambda)\rangle \propto \int \mathcal{D}\mu\,\mathcal{D}\hat{\mu}\, \exp\left\{-iN\int d\vec{y}\,\mu(\vec{y})\hat{\mu}(\vec{y}) - \frac{N}{8}\int d\vec{y}\,d\vec{w}\,\mu(\vec{y})\mu(\vec{w})\left(\sum_a y_a w_a\right)^2\right\}$$

$$\cdot \int_{\mathbb{R}^{Nm}}\left(\prod_a d\underline{y}_a\right)\exp\left\{-i\frac{\lambda_\varepsilon}{2}\sum_{i=1}^{N}\sum_{a=1}^{m} y_{ia}^2 + i\sum_{i=1}^{N}\int d\vec{y}\,\hat{\mu}(\vec{y})\prod_a \delta(y_a - y_{ia})\right\}.$$

BUT THE LAST PIECE IS REALLY N COPIES OF A "SINGLE" INTEGRAL:

$$\int_{\mathbb{R}^{Nm}}\left(\prod_a d\underline{y}_a\right)(\dots) = \left\{\int_{\mathbb{R}^m} d\vec{y}_1\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{m} y_{1a}^2 + i\int d\vec{y}\,\hat{\mu}(\vec{y})\prod_a\delta(y_a - y_{1a})\right]\right\}^N$$

$$= \left\{\int_{\mathbb{R}^m} d\vec{y}_1\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{m} y_{1a}^2 + i\hat{\mu}(\vec{y}_1)\right]\right\}^N$$

$$= \left\{\int_{\mathbb{R}^m} d\vec{y}\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{m} y_a^2 + i\hat{\mu}(\vec{y})\right]\right\}^N = e^{N\,\mathrm{Log}\left(\int_{\mathbb{R}^m} d\vec{y}\dots\right)}.$$

* PUTTING EVERYTHING TOGETHER, WE FOUND

$$\langle Z^m(\lambda)\rangle \propto \int \mathcal{D}\mu\,\mathcal{D}\hat{\mu}\; e^{N S_m[\mu,\hat{\mu};\lambda]}$$ ACTION

WHERE

$$S_m[\mu,\hat{\mu};\lambda] = -i\int d\vec{y}\,\mu(\vec{y})\hat{\mu}(\vec{y}) - \frac{1}{8}\int d\vec{y}\,d\vec{w}\,\mu(\vec{y})\mu(\vec{w})\left(\sum_{a=1}^{m} y_a w_a\right)^2$$

$$+ \operatorname{Log}\left\{\int_{\mathbb{B}^m} d\vec{y}\,\exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{m} y_a^2 + i\hat{\mu}(\vec{y})\right]\right\}.$$

THIS SUGGESTS WE CAN USE A SADDLE-POINT FOR LARGE N.

BUT RECALL WE HAD TO TAKE THE LIMITS

$$\lim_{N\to\infty}\lim_{m\to 0}(\ldots)$$

SO WE ARE ACTUALLY GOING TO EXCHANGE THE ORDER OF THE TWO LIMITS (CLOSE OUR EYES AND HOPE FOR THE BEST). HENCE

$$0 \equiv \left.\frac{\delta S_m}{\delta\mu}\right|_{\mu^*,\hat{\mu}^*} \qquad 0 \equiv \left.\frac{\delta S_m}{\delta\hat{\mu}}\right|_{\mu^*,\hat{\mu}^*}.$$

THE FIRST CONDITION GIVES

$$-i\hat{\mu}^*(\vec{y}) = \frac{1}{4}\int d\vec{w}\,\mu^*(\vec{w})\Big(\underbrace{\sum_{a=1}^{m} y_a w_a}_{=\vec{y}\cdot\vec{w}}\Big)^2$$

AND THE SECOND

$$+i\mu^*(\vec{y}) = \frac{\exp\left\{-i\frac{\lambda_\varepsilon}{2}\sum_a y_a^2 + i\hat{\mu}^*(\vec{y})\right\}\cdot(+i)}{\int_{\mathbb{B}^m} d\vec{y}\,'\exp\left\{-i\frac{\lambda_\varepsilon}{2}\sum_a {y'_a}^2 + i\hat{\mu}^*(\vec{y}\,')\right\}}$$

WHICH IS A SYSTEM OF TWO COUPLED INTEGRAL EQUATIONS.

PLUGGING THE SECOND INTO THE FIRST YIELDS

$$-i\hat{\mu}^*(\vec{y}) = \frac{\frac{1}{4}\int d\vec{w}\,\exp\left\{-i\frac{\lambda_\varepsilon}{2}\sum_a w_a^2 + i\hat{\mu}^*(\vec{w})\right\}(\vec{y}\cdot\vec{w})^2}{\int d\vec{w}\,\exp\left\{-i\frac{\lambda_\varepsilon}{2}\sum_a w_a^2 + i\hat{\mu}^*(\vec{w})\right\}} \qquad (\mathrm{II})$$

WHICH IS A SINGLE INTEGRAL EQUATION (CONTAINING m-DIMENSIONAL INTEGRALS) FOR THE FUNCTION $\hat{\mu}^*(\vec{y})$.

- <u>ASSUMPTION</u> ("REPLICA-SYMMETRIC HIGH-TEMPERATURE ANSATZ"):

$$\hat{\mu}^*(\vec{y}) = \hat{\mu}^*(|\vec{y}|).$$

INDEED, REPLICAS WERE INTRODUCED AS A MATHEMATICAL TOOL AND THERE'S NO REASON TO EXPECT ONE COPY TO BEHAVE DIFFERENTLY FROM THE OTHERS. HOWEVER, THIS IS NOT ALWAYS THE CASE (RSB).

THIS ALLOWS US TO STEP TO m-DIMENSIONAL SPHERICAL COORDINATES IN (II):

$$\begin{cases} y_1 = y\cos\phi_1 \\ y_2 = y\sin\phi_1\cos\phi_2 \\ y_3 = y\sin\phi_1\ldots\cos\phi_3 \\ \vdots \\ y_m = y\sin\phi_1\ldots\sin\phi_{m-2}\sin\phi_{m-1} \end{cases}$$

NOTE: AND
$y_{m-1} = y\sin\phi_1\ldots\sin\phi_{m-2}\cos\phi_{m-1}$.

WHERE THE VOLUME ELEMENT IS

$$r^{m-1}\sin^{m-2}\phi_1\sin^{m-3}\phi_2\ldots\sin\phi_{m-2}.$$

MOST INTEGRALS OVER THE ANGULAR VARIABLES CANCEL OUT BETWEEN NUMERATOR AND DENOMINATOR, SO THAT

$$-i\hat{\mu}^*(y) = \frac{\frac{y^2}{4}\int_0^\infty dw\, w^{m-1}\exp\left\{-i\frac{\lambda_\varepsilon}{2}w^2 + i\hat{\mu}^*(w)\right\}w^2\int_0^\pi d\phi(\sin\phi)^{m-2}\cos^2\phi}{\int_0^\infty dw\, w^{m-1}\exp\left\{-i\frac{\lambda_\varepsilon}{2}w^2 + i\hat{\mu}^*(w)\right\}\int_0^\pi d\phi\,(\sin\phi)^{m-2}}. \quad \text{(III)}$$

NOTICE THIS EXPRESSION DOESN'T KNOW THAT m IS AN INTEGER ANYMORE, AND THIS IS THANKS TO THE ANSATZ.

NOTICE ALSO $\hat{\mu}^*(y) \propto y^2$ FOR THE GAUSSIAN ENSEMBLE.

* THE ONES INVOLVED IN (III) ARE KNOWN INTEGRALS:

$$\int_0^{\pi} d\phi \,(\sin\phi)^{m-2} \cos^2\phi = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{2}(m-1)\right)}{\Gamma\left(1+\frac{m}{2}\right)}$$

$$\int_0^{\pi} d\phi \,(\sin\phi)^{m-2} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}(m-1)\right)}{\Gamma(m/2)} .$$

CALLING

$$G(w) \equiv e^{-\frac{i}{2}\lambda_\varepsilon w^2 + i\hat{\mu}^*(w)}$$

WE CAN REWRITE (III) AS

$$i\hat{\mu}^*(y) = \underbrace{\frac{\Gamma(m/2)\cdot m}{2\,\Gamma\left(1+\frac{m}{2}\right)}}_{b} \cdot \frac{y^2}{4} \cdot \frac{\int_0^\infty dw\, G(w)\, w^{m+1}}{\int_0^\infty dw\, G'(w)\, w^m}$$

WHERE WE INTEGRATED BY PARTS THE DENOMINATOR

$$-\int_0^\infty dw\, G(w)\, w^{m-1}$$

(WE WILL HAVE TO SEND $m \to 0$) AND THE PREFACTOR IS S.T.

$$b \xrightarrow{m\to 0} 1 .$$

IN THE LIMIT $m \to 0$, THEN,

$$\underline{i\hat{\mu}^*(y) = C(\lambda)\, y^2}$$

WHERE

$$C(\lambda) = \frac{1}{4} \frac{\int_0^\infty dw\; w\, G(w)}{\int_0^\infty dw\; G'(w)} = \frac{1}{4} \frac{\int_0^\infty dw\; w\, e^{-\frac{i}{2}\lambda_\varepsilon w^2 + C(\lambda) w^2}}{\int_0^\infty dw\, \exp\left\{-i\frac{\lambda_\varepsilon}{2} w^2 + C(\lambda) w^2\right\}\cdot 2w\left[-\frac{i}{2}\lambda_\varepsilon + C(\lambda)\right]}$$

WHICH SELF-CONSISTENTLY DETERMINES THE CONSTANT $C(\lambda)$.

YOU CAN PROVE THAT THE SOLUTION IS*

$$C(\lambda) = \frac{1}{4}\left\{ i\lambda_\varepsilon \pm \sqrt{2-\lambda_\varepsilon^2} \right\}$$

WHICH IS, AGAIN, THE BIRTH OF A SEMICIRCLE.

*NOTE: ACTUALLY THE INTEGRALS AT NUM/DEN SIMPLIFY AND ONE GETS

$$C^2(\lambda) - \frac{i}{2}\lambda_\varepsilon C(\lambda) - \frac{1}{8} = 0.$$

LECTURE 9

- QUENCHED CALCULATION FOR GOE (CODA)

LAST TIME WE FOUND

$$i\hat{\mu}^*(y) = C(\lambda)\, y^2 \qquad C(\lambda) = \frac{1}{4}\left\{ i\lambda_\varepsilon \pm \sqrt{2-\lambda_\varepsilon^2} \right\}.$$

$\lambda_\varepsilon = \lambda - i\varepsilon$

BY E-J FORMULA,

$$\rho_N(\lambda) = -\frac{2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im} \frac{\partial}{\partial\lambda} \left\{ \lim_{m\to 0} \frac{1}{m} \mathrm{Log} \langle Z^m(\lambda)\rangle \right\}$$

$$\langle Z^m \rangle \simeq \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\; e^{N S_m[\mu,\hat{\mu},\lambda]} \underset{N\gg 1}{\simeq} e^{N S_m[\mu^*,\hat{\mu}^*;\lambda]}.$$

THEN

$$\rho_N(\lambda) = -\frac{2}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im} \lim_{m\to 0} \frac{1}{m} \frac{\partial}{\partial\lambda} S_m[\mu^*,\hat{\mu}^*;\lambda]$$

WHERE WE RECALL

$$S_m[\mu,\hat{\mu};\lambda] = -i\int d\vec{y}\, \mu(\vec{y})\hat{\mu}(\vec{y}) - \frac{1}{8}\int d\vec{y}\, d\vec{w}\, \mu(\vec{y})\mu(\vec{w})(\vec{y}\cdot\vec{w})^2$$

$$+ \mathrm{Log}\left\{ \int_{\mathbb{R}^m} d\vec{y}\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a y_a^2 + i\hat{\mu}(\vec{y})\right] \right\}.$$

ONLY THE 3^{RD} TERM COUNTS:

$$\rho_{N\to\infty}(\lambda) = -\frac{2}{\pi} \lim_{\varepsilon\to 0} \mathrm{Im} \lim_{m\to 0} \frac{1}{m} \frac{-\frac{i}{2}\int_0^\infty dy\, y^{m+1} \exp\left[-\frac{i}{2}\lambda_\varepsilon y^2 + C(\lambda) y^2\right]}{\int_0^\infty dy\, y^{m-1} \exp\left[-\frac{i}{2}\lambda_\varepsilon y^2 + C(\lambda)y^2\right]}$$

$$= \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Re} \frac{1}{-2C(\lambda) + i\lambda_\varepsilon}$$

NOTE: USE $\Gamma(m+1) = m\Gamma(m)$ AND $\int_0^\infty x^k e^{-a x^2} dx = \Gamma\left(\frac{k+1}{2}\right)\Big/\left(2a^{\frac{k+1}{2}}\right)$.

AS YOU CAN EASILY VERIFY, SINCE THE TWO INTEGRALS CAN BE COMPUTED EXACTLY. WE NOW PLUG IN $C(\lambda)$, WHICH WE REWRITE AS

$$C(\lambda) = P_\varepsilon(\lambda) + i Q_\varepsilon(\lambda)$$

WITH

$$P_\varepsilon(\lambda) = \frac{1}{\sqrt{2}}\left\{ 2-\lambda^2+\varepsilon^2 + \left[(2-\lambda^2+\varepsilon^2)^2 + (2\varepsilon\lambda)^2\right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

AND

$$Q_\varepsilon(\lambda) = \frac{\mathrm{sgn}(2\varepsilon\lambda)}{\sqrt{2}} \left\{ \left[(2-\lambda^2+\varepsilon^2)^2 + (2\varepsilon\lambda)^2 \right]^{\frac{1}{2}} - (2-\lambda^2+\varepsilon^2) \right\}^{\frac{1}{2}} .$$

HENCE

$$\mathrm{Re}\, \frac{1}{-2C(\lambda)+i\lambda_\varepsilon} = \frac{-2P_\varepsilon(\lambda)}{4P_\varepsilon^2(\lambda)+(\lambda-2Q_\varepsilon(\lambda))^2} .$$

IN THE LIMIT $\varepsilon \to 0^+$ AND FOR $-\sqrt{2} < \lambda < \sqrt{2}$, P_ε AND Q_ε CONVERGE TO

$$P_0(\lambda) = \pm \frac{\sqrt{2-\lambda^2}}{4} \qquad\qquad Q_0(\lambda) = \frac{\lambda}{4}$$

FROM WHICH WE FINALLY GET THE SEMICIRCLE LAW

$$\rho_{N\to\infty}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2} \qquad \text{(FOR } |\lambda| < \sqrt{2}\text{, AND 0 OTHERWISE).}$$

- BRAY - RODGERS EQUATION

(SPECTRAL DENSITY OF SPARSE RANDOM MATRICES)

$$\rho(x) = -\frac{2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im}\, \frac{\partial}{\partial x} \langle \mathrm{Log}\, Z(x) \rangle$$

$$Z(x) = \int_{\mathbb{R}^N} d\underline{y}\ \exp\left\{ -\frac{i}{2}\underline{y}^T (x_\varepsilon \mathbb{1} - H)\underline{y} \right\}.$$

NOTICE IF WE HAD A REAL NUMBER IN PLACE OF $\frac{i}{2}$, SINCE H IS RANDOM, THE CONVERGENCE OF $Z(x)$ WOULDN'T BE GUARANTEED.

WE TAKE

$$H_{ij} = c_{ij} K_{ij} \qquad\qquad c_{ij} \text{ CONNECTIVITY MATRIX}$$

$$\mathcal{P}(\{c_{ij}\}) = \prod_{i<j} p(c_{ij})\,\delta_{c_{ij},c_{ji}} \qquad\qquad p(c_{ij}) = \left(1-\frac{c}{N}\right)\delta_{c_{ij},0} + \frac{c}{N}\delta_{c_{ij},1}$$

(ERDÖS - RENYI GRAPH).

THE DISTRIBUTION OF THE WEIGTHS $\{K_{ij}\}$ IS UNSPECIFIED AT THIS LEVEL.

NOTE: IT MAKES SENSE TO SEPARATE THE TOPOLOGY OF THE GRAPH, $c_{ij} = \{0,1\}$, FROM THE WEIGHTS $K_{ij} \in \mathbb{R}$ ON THE LINKS.

WE WILL USE THE REPLICA TRICK, WHERE

$$\langle Z^m(x)\rangle = \langle \int \prod_{i<j} dc_{ij}\, p(c_{ij}) \int_{\mathbb{R}^{Nm}} \prod_{a=1}^{m} d\underline{y}_a \exp\left\{-\frac{i}{2}\sum_{i,j}\sum_{a=1}^{m} y_{ia}\left(x_\varepsilon \delta_{ij} - H_{ij}\right) y_{ja}\right\}\rangle_{\{K\}}$$

$$= \int_{\mathbb{R}^{Nm}} \prod_{a=1}^{m} d\underline{y}_a \exp\left\{-\frac{i}{2} x_\varepsilon \sum_{i=1}^{N}\sum_{a=1}^{m} y_{ia}^2\right\} \cdot \int \prod_{i<j} dc_{ij}\left[\left(1-\frac{c}{N}\right)\delta_{c_{ij},0} + \frac{c}{N}\delta_{c_{ij},1}\right]$$

$$\cdot \langle \exp\left\{ i \sum_{i<j}\sum_{a=1}^{m} y_{ia} c_{ij} K_{ij} y_{ja}\right\}\rangle_{\{K\}}.$$

THE AVERAGE OVER THE CONNECTIVITY c_{ij} IS STRAIGHTFORWARD:

$$\int \prod_{i<j} dc_{ij}\,(\dots) = \prod_{i<j} \langle \left(1-\frac{c}{N}\right) + \frac{c}{N}\exp\left\{ i \sum_{a=1}^{m} y_{ia} K_{ij} y_{ja}\right\}\rangle_K$$

$$= \prod_{i<j} \langle 1 + \frac{c}{N}\left[\exp(\dots) - 1\right]\rangle_K$$

$$\simeq \exp\left\{\frac{c}{2N}\sum_{i,j}\left(\langle e^{iK\sum_a y_{ia}y_{ja}}\rangle_K - 1\right)\right\}$$

↑ SINGLE K

NOTE: WE USED THE FACT THAT $c_{ii}=0$, WHENCE

$$\prod_{i<j} = \frac{1}{2}\prod_{i\neq j} \equiv \frac{1}{2}\prod_{i,j}$$

AND THE PROPERTY

$$\int dK_{11}\dots dK_{NN}\, P(K_{11})\dots P(K_{NN}) \prod_{i<j}\phi_{ij}(K_{ij}) = \prod_{i<j}\int dK\, P(K)\,\phi_{ij}(K).$$

WITH

$$\langle f\rangle_K = \int P(K) f(K)\, dK$$

(PROVE THE LAST PASSAGE!). WE ARE LEFT WITH

$$\langle Z^m(x)\rangle \simeq \int_{\mathbb{R}^{Nm}} \prod_{a=1}^{m} d\underline{y}_a\, e^{-\frac{i}{2}x_\varepsilon \sum_{i=1}^{N}\sum_{a=1}^{m} y_{ia}^2}\, e^{\frac{c}{2N}\sum_{i,j}\left[\langle \exp\left(iK\sum_a y_{ia}y_{ja}\right)\rangle_K - 1\right]}.$$

TO MAKE PROGRESS, WE INTRODUCE AGAIN

$$\mu(\vec{y}) = \frac{1}{N}\sum_i \prod_a \delta(y_a - y_{ia}) \qquad \vec{y} = (y_1, y_2, \dots, y_m)$$

AND THE REPRESENTATION IN TERMS OF THE CONJUGATED FIELD $\hat{\mu}$

$$1 = \int \mathcal{D}\mu\,\mathcal{D}\hat{\mu}\, \exp\left\{-i\int d\vec{y}\,\hat{\mu}(\vec{y})\left(N\mu(\vec{y}) - \sum_i \prod_a \delta(y_a - y_{ia})\right)\right\}.$$

THIS LEADS TO

$$\langle Z(x)^m \rangle = \int \mathcal{D}\mu \mathcal{D}\hat{\mu} \exp\left\{ -iN \int d\vec{y}\, \hat{\mu}(\vec{y})\mu(\vec{y}) + \frac{CN}{2} \int d\vec{y}\, d\vec{y}\,'\, \mu(\vec{y})\mu(\vec{y}\,') \right.$$

$$\left. \cdot \left(\langle e^{ik \sum_a y_a y_a'} \rangle_K - 1 \right) \right\} \cdot \int_{\mathbb{R}^{Nm}} \prod_{a=1}^{m} d\underline{y}_a \exp\left\{ -\frac{i}{2} x_\varepsilon \sum_{i=1}^{N} \sum_{a=1}^{m} y_{ia}^2 \right\} \cdot$$

$$\cdot \exp\left\{ i \sum_i \int d\vec{y} \prod_a \delta(y_a - y_{ia})\, \hat{\mu}(\vec{y}) \right\}$$

WHERE THE INTEGRATIONS IN $d\underline{y}_a$ ARE ANALOGOUS TO THOSE OF LAST TIME AND GIVE

$$\left\{ \int d\vec{y} \exp\left[-\frac{i}{2} x_\varepsilon \sum_a y_a^2 + i\hat{\mu}(\vec{y}) \right] \right\}^N = e^{N \log(\dots)} .$$

SIMILARLY TO WHAT WE DID WITH THE GOE, HERE WE FIND

$$\langle Z(x)^m \rangle = \int \mathcal{D}\mu \mathcal{D}\hat{\mu} \exp\left\{ N S_m(\mu, \hat{\mu}; x) \right\}$$

WHERE

$$S_m(\mu, \hat{\mu}; x) = \frac{c}{2} \int d\vec{y}\, d\vec{y}\,'\, \mu(\vec{y})\mu(\vec{y}\,') \left(\langle e^{ik \sum_a y_a y_a'} \rangle_K - 1 \right)$$

$$- i \int d\vec{y}\, \hat{\mu}(\vec{y})\mu(\vec{y}) + \log \int d\vec{y} \exp\left\{ -\frac{i}{2} x_\varepsilon \sum_a y_a^2 + i\hat{\mu}(\vec{y}) \right\} .$$

THE SADDLE - POINT EQUATIONS GIVE

$$g(\vec{y}) = \frac{\int d\vec{y}\,' \exp\left\{ -\frac{i}{2} x_\varepsilon \sum_a y_a'^2 + c g(\vec{y}\,') \right\} \cdot f(\vec{y} \cdot \vec{y}\,')}{\int d\vec{y}\,' \exp\left\{ -\frac{i}{2} x_\varepsilon \sum_a y_a'^2 + c g(\vec{y}\,') \right\}}$$

WHERE WE DEFINED

$i\hat{\mu}^*(\vec{y}) =: c\, g(\vec{y})$ $\qquad\qquad$ $f(z) := \langle e^{ikz} \rangle_K - 1 .$

THIS IS CALLED <u>BRAY - RODGERS INTEGRAL EQUATION</u> (EVEN THOUGH THEY ONLY DERIVED IT IN A SPECIFIC CASE).

WE STEP TO SPHERICAL m - DIMENSIONAL COORDINATES AS WE

ASSUME REPLICA SYMMETRY: THIS GIVES

$$g(q) = \frac{\int_0^\infty dr\, r^{m-1} \exp\left\{-\frac{i}{2} x_\varepsilon r^2 + c g(r)\right\} \int_0^\pi d\phi\, (\sin\phi)^{m-2} f(qr\cos\phi)}{\int_0^\infty dr\, r^{m-1} \exp\{\dots\} \int_0^\pi d\phi\, (\sin\phi)^{m-2}}.$$

SPECIALIZING TO B-B ORIGINAL CHOICE

$$\varphi(H_{ij}) = \left(1-\frac{c}{N}\right)\delta_{H_{ij},0} + \frac{c}{2N}\left[\delta_{H_{ij},1} + \delta_{H_{ij},-1}\right]$$

(SPARSE MATRIX, WITH SYMMETRICALLY DISTRIBUTED NONZERO ENTRIES), WE SEE THIS CORRESPONDS TO

$$\varphi(\{K_{ij}\}) = \frac{1}{2}\delta_{K_{ij},1} + \frac{1}{2}\delta_{K_{ij},-1}$$

AND WE FIND

$$f(z) = \langle e^{ikz}\rangle_k - 1 = -1 + \frac{1}{2}\sum_{k=\pm 1} e^{ikz} = -1 + \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$= \cos z - 1.$$

THEN

$$f(qr\cos\phi) = -1 + \cos(qr\cos\phi)$$

AND USING

$$\int_0^\pi d\phi\, (\sin\phi)^{m-2}\left\{\cos(qr\cos\phi) - 1\right\}$$

$$= \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{m-1}{2}\right) \cdot \left\{\left(\frac{2}{qr}\right)^{\frac{m}{2}}\left[m I_{\frac{m}{2}}(qr) - qr I_{\frac{m}{2}+1}(qr)\right] - \frac{2}{\Gamma(m/2)}\right\}$$

WHERE I_k DENOTES THE K-TH ORDER MODIFIED BESSEL FUNCTION OF THE FIRST KIND, WE GET

$$g(q) = -\frac{m}{2}\Gamma\left(\frac{m}{2}\right)\frac{\int_0^\infty dr\, r^{m-1} G(r)\left\{\left(\frac{2}{qr}\right)^{\frac{m}{2}}\left[m I_{\frac{m}{2}}(qr) + qr I_{\frac{m}{2}+1}(qr)\right) - \frac{2}{\Gamma(\frac{m}{2})}\right\}}{\int_0^\infty dr\, r^m G'(r)}$$

WHERE WE INTEGRATED BY PARTS IN THE DENOMINATOR AND WE DEFINED

$$G(r) = e^{-\frac{1}{2}\chi_\varepsilon r^2 + c g(r)} .$$

SINCE

$$m \Gamma\left(\frac{m}{2}\right) \xrightarrow[m\to 0]{} 2$$

$$\int_0^\infty dr\, r^m G'(r) \xrightarrow[m\to 0]{} \int_0^\infty dr\, G'(r) = G(\infty) - G(0) = -1$$

$$r^{m-1}\left[\ldots\right] \longrightarrow y\, I_1(ry)$$

WE GET IN THE $m \to 0$ LIMIT

$$g(y) = y \int_0^\infty dy\, I_1(ry)\, e^{-\frac{1}{2}\chi_\varepsilon r^2 + c g(r)}$$

WHICH IS AN INTEGRAL EQUATION FOR THE AUXILIARY FUNCTION $g(x)$ (TO BE SOLVED NUMERICALLY: THE PROBLEM CANNOT BE CRACKED COMPLETELY AS IN THE GAUSSIAN CASE).

ADDENDUM TO LECTURE 4

- 3 PROPERTIES (MAYBE NOT SO WIDELY KNOWN)

1) a_N, b_N CAN BE FOUND AS FOLLOWS:

- GUMBEL: $a_N = P^{-1}\left(1-\frac{1}{N}\right)$, $\qquad b_N = P^{-1}\left(1-\frac{1}{Ne}\right) - a_N$

- FRÉCHET: $a_N = 0$, $\qquad b_N = P^{-1}\left(1-\frac{1}{N}\right)$

- WEIBULL: $a_N = x^*$, $\qquad b_N = x^* - P^{-1}\left(1-\frac{1}{N}\right)$

WHERE P IS THE CDF OF EACH INDIVIDUAL RANDOM VARIABLE.

2) HOW TO PREDICT THE BASIN OF ATTRACTION?

$$\lim_{\varepsilon \to 0} \frac{P^{-1}(1-\varepsilon) - P^{-1}(1-2\varepsilon)}{P^{-1}(1-2\varepsilon) - P^{-1}(1-4\varepsilon)} = 2^c$$

WHERE $c=0$, $c>0$, $c<0$ IF G, F, W.

3) THE CONSTANTS $\{a_N, b_N\}$ ARE NOT UNIQUE, BUT CAN BE REPLACED BY $\{a_N', b_N'\}$ PROVIDED THAT

$$\lim_{N\to\infty} \frac{b_N'}{b_N} = 1, \qquad \lim_{N\to\infty} \frac{a_N - a_N'}{b_N} = 0.$$

KPZ STORY (SATYA MAJUMDAR)

· PHASE 1: GROWTH MODELS

(EXPERIMENTS ON LIQUID CRYSTAL GROWTH).

GROWING CLUSTERS ARE UBIQUITOUS!

* EDEN (1961): A STOCHASTIC GROWTH MODEL FOR TUMOURS ON A 2d LATTICE.

AT EACH STEP, CHOOSE 1 NEIGHBOUR AND INFECT IT (ONLY 1, NOT 1 PER ALREADY INFECTED SITE). AFTER MANY STEPS, THE RESULTING SURFACE IS ROUGH AND IT GETS ROUGHER AND ROUGHER.

AVERAGE RADIUS (HEIGHT):

$\langle h(t) \rangle \sim t$ AT LATE TIMES $t = \sqrt{N}$.

HEIGHT FLUCTUATIONS (WIDTH):

$$w = \left[\langle (h - \langle h \rangle)^2 \rangle \right]^{\frac{1}{2}} \sim \underset{\uparrow \text{ POWER LAW}}{t^{1/3}} \quad \text{AS } t \to \infty.$$

* LIQUID CRYSTAL EXPERIMENTS (2011):

$\langle h(t) \rangle \sim t$ AT LATE TIMES t

$w \sim t^{1/3}$ AS $t \to \infty$.

SAME ASYMPTOTIC GROWTH LAW $t^{1/3}$: UNIVERSAL!

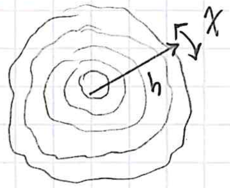
THE SAME HAPPENS, FOR INSTANCE, FOR SLOW COMBUSTION OF PAPER.

* SEVERAL DISCRETE GROWTH MODELS IN (1+1) DIMENSIONS SHARE THE COMMON ASYMPTOTIC GROWTH LAW FOR THE WIDTH $w \sim t^{1/3}$: EDEN MODEL, SOLID-ON-SOLID MODEL, BALLISTIC DEPOSITION MODELS, POLYNUCLEAR GROWTH MODEL, DIRECTED POLYMER IN A RANDOM MEDIUM, ASYMMETRIC EXCLUSION PROCESSES...

THEY ALSO DISPLAY

$h(t) = vt + t^{1/3}\chi$ $h(t) \sim$ DETERMINISTIC + FLUCTUATING.

· KPZ (1986): EFFECTIVE MODEL TO DESCRIBE THE UNIVERSAL GROWTH PROCESS,



$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \sqrt{D}\, \eta(x,t)$$

WHERE η IS A GAUSSIAN NOISE WITH

$$\langle \eta(x,t) \rangle = 0 \qquad \langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t').$$

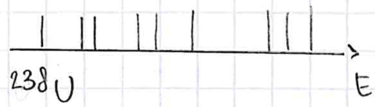
NOTICE $\partial_x^2 h$ TRIES TO SMOOTHEN THE SURFACE, OPPOSING THE NOISE. THE NONLINEAR TERM $(\partial_x h)^2$ DRIVES THE SYSTEM OUT OF EQUILIBRIUM AND IT BECOMES DOMINANT AT LATE TIMES $t \gg t^*$: IT IS RESPONSIBLE FOR THE TEMPORAL GROWTH OF THE WIDTH $w \sim t^{1/3}$. HERE THE CHARACTERISTIC TIME t^* IS THE TIME AT WHICH $(\partial_x h)^2$ BECOMES DOMINANT.

→ ASYMPTOTIC GROWTH OF WIDTH IN VARIOUS DISCRETE MODELS BELONGS TO THE <u>KPZ UNIVERSALITY CLASS</u> (SURFACE WIDTH $w \sim t^{1/3}$).

· DOES THE UNIVERSALITY IN VARIOUS GROWTH MODELS BELONGING TO THE KPZ CLASS HOLD BEYOND THE SECOND MOMENT?
UP TO THE EARLY 00'S, THERE WHERE ONLY NUMERICAL HINTS...

RANDOM MATRIX THEORY & TRACY-WIDOM DISTRIBUTION

RMT WAS STARTED IN 1928 BY WISHART AND TAKEN UP IN NUCLEAR PHYSICS BY WIGNER AND DYSON.



IN ORDER TO EXPLAIN THE SPECTRA OF HEAVY NUCLEI, WIGNER THOUGHT OF REPLACING THE COMPLEX HAMILTONIAN H BY RANDOM MATRICES WITH THE SAME SYMMETRY (THIS WORKS PRECISELY BECAUSE OF UNIVERSALITY). NOW RMT HAS MANY APPLICATIONS IN PHYSICS, MATHS, STATISTICS, INFORMATION, BIOLOGY, ECONOMICS & FINANCE...

⋆ GAUSSIAN RANDOM MATRICES

CONSIDER A $N\times N$ GAUSSIAN RANDOM MATRIX $J \equiv (J_{ij})$,

$$P[J] \propto e^{-\beta \frac{N}{2} \sum_{ij}^{N} |J_{ij}|^2} \propto e^{-\beta \frac{N}{2} \mathrm{Tr}(J^\dagger J)} \qquad \text{(jpdf OF ENTRIES)}$$

WHICH IS INVARIANT UNDER ROTATION. WE DISTINGUISH

(i) GOE : REAL SYMMETRIC

(ii) GUE : COMPLEX HERMITIAN

(iii) GSE : COMPLEX QUATERNIONIC.

THE N REAL EIGENVALUES $\{\lambda_1, \dots, \lambda_N\}$ ARE STRONGLY CORRELATED AND WE WANT TO GET THEIR SPECTRAL STATISTICS. INDEED, THE JOINT DISTRIBUTION OF THE EIGENVALUES TURNS OUT TO BE (WIGNER '51)

$$P(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

WHERE $\beta = 1, 2, 4$ IS THE DYSON INDEX FOR THE 3 CASES ABOVE. THE LAST TERM (THE JACOBIAN) IS RESPONSIBLE FOR THE REPULSION (STRONG CORRELATION).

COULOMB GAS INTERPRETATION (DYSON '62):

$$P(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} e^{-\frac{\beta}{2}\left\{ N \sum_{i=1}^{N} \lambda_i^2 - \sum_{j\neq k} \log|\lambda_j - \lambda_k| \right\}} \sim e^{-\beta E(\{\lambda_i\})}$$

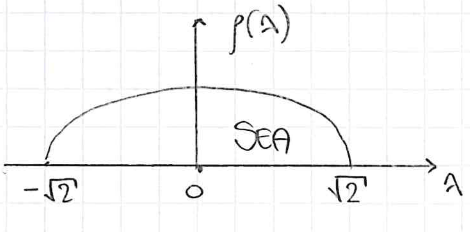
WHICH IS THE BOLTZMANN WEIGHT OF A GAS OF N PAIRWISE REPELLING CHARGES (log REPULSION) IN AN EXTERNAL HARMONIC POTENTIAL $V(\lambda) = \lambda^2$.

SPECTRAL DENSITY: WIGNER'S SEMICIRCLE LAW

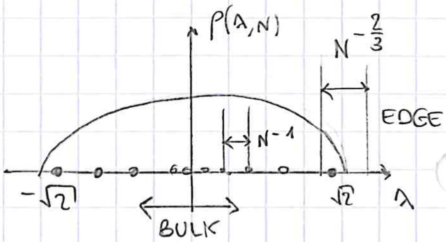
THE AVG DENSITY OF EIGENVALUES, NORMALIZED TO UNITY, IS

$$\rho(\lambda, N) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle$$

$$\xrightarrow[N\to\infty]{} \rho(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



i.e. $\rho(\lambda, N) \to \rho(\lambda)$ WITH EDGES AT $\pm\sqrt{2}$.



THERE ARE TWO ESSENTIAL LENGTHSCALES IN THIS PROBLEM: THE INTERPARTICLE DISTANCE NEAR THE BULK AND NEAR THE EDGES.

TO ESTIMATE THE BULK INTERPARTICLE DISTANCE, NOTICE

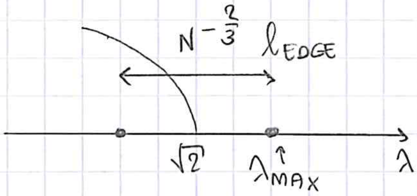
$$\int_0^{l_{BULK}} \rho(\lambda, N)\, d\lambda \cong \frac{1}{N} \qquad \Rightarrow \qquad l_{BULK} \sim N^{-1}$$

WHILE FOR THE EDGES

$$\int_{\sqrt{2}-l_{EDGE}}^{\sqrt{2}} \rho(\lambda, N)\, d\lambda \cong \frac{1}{N} \qquad \Rightarrow \qquad l_{EDGE} \sim N^{-\frac{2}{3}}.$$

WE CONCLUDE THAT "PARTICLES" ARE SPARSER NEAR THE EDGE:

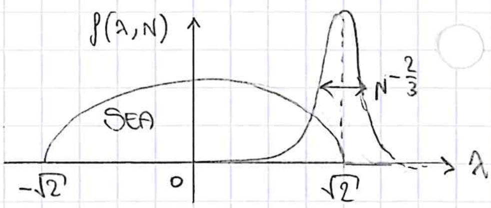
$l_{EDGE} \gg l_{BULK}$.



IT THEN MAKES SENSE TO STUDY λ_{MAX}, THE TOP EIGENVALUE OF A RANDOM MATRIX:

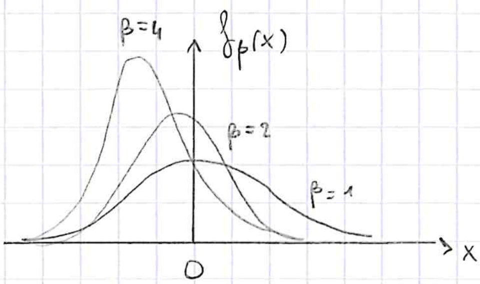
$\langle \lambda_{MAX} \rangle = \sqrt{2}$ AVERAGE

$|\lambda_{MAX} - \sqrt{2}| \sim l_{EDGE} \sim N^{-2/3}$ TYPICAL FLUCTUATIONS.



FOR LARGE N, IT TURNS OUT TYPICAL FLUCTUATIONS ARE DISTRIBUTED VIA TRACY-WIDOM ('94, T & W)

$$\lambda_{MAX} \to \sqrt{2} + \frac{1}{\sqrt{2}} N^{-\frac{2}{3}} \chi_\beta$$



WHERE

$\text{Prob}(\chi_\beta = x) = f_\beta(x) \to$ PAINLEVÈ II

WITH NON-GAUSSIAN ASYMPTOTICS:

$$f_\beta(x) \sim \begin{cases} \exp\left(-\frac{\beta}{24}|x|^3\right) & x \to -\infty \\ \exp\left(-\frac{2\beta}{3} x^{3/2}\right) & x \to +\infty. \end{cases}$$

* SO WHAT?

IN 1999, BDJ FULLY SOLVED THE OUTSTANDING ULAM PROBLEM, i.e. THAT OF THE LONGEST INCREASING SUBSEQUENCE (LIS) PROPOSED IN 1961.

TAKE A SEQUENCE OF m INTEGERS AND CONSIDER ANY ONE OF THE $m!$ PERMUTATIONS; FOR EACH OF THEM, CONSTRUCT ALL POSSIBLE INCREASING SUBSEQUENCES AND FIND OUT THE LONGEST ONES.

LET L_m BE THE LENGTH OF THE LIS: IT WILL FLUCTUATE FROM ONE PERMUTATION TO ANOTHER. GIVEN THAT EACH PERMUTATION OCCURS WITH EQUAL PROBABILITY $\frac{1}{m!}$, THEN L_m IS A RANDOM VARIABLE.

ULAM PROBLEM: WHAT IS THE STATISTICS OF L_m?

* LARGE m RESULTS

AVERAGE LENGTH OF LIS:

$\langle L_m \rangle \simeq 2\sqrt{m}$ FOR LARGE m.

TYPICAL FLUCTUATIONS AROUND THE AVERAGE ARE $\sim m^{1/6}$, SO

$$L_m = 2\sqrt{m} + m^{1/6} \chi_2 \qquad \text{ULAM PROBLEM}$$

WHERE χ_2 IS A m-INDEPENDENT RANDOM VARIABLE.

AFTER A TOUR DE FORCE CALCULATION, BDJ PROVED THAT

$$\mathrm{Prob}(\chi_2 = x) = f_2(x) \qquad \text{TRACY-WIDOM GUE.}$$

· PHASE 2: KPZ STORY

INITIATED BY BDJ AND FOSTERED BY THE MAPPINGS

DISCRETE KPZ GROWTH MODELS $\longleftrightarrow$ ULAM PROBLEM.

THEN FOR KPZ GROWTH MODELS (IN CURVED GEOMETRY) IT WAS FOUND

$$h(t) = v t + t^{1/3} \chi_2 \qquad \text{FOR LARGE } t$$

UNDER THE EXACT MAPPING

$h(t) \leftrightarrow L_{m=t^2}$.

IT WAS THEN FOUND THAT THE EXACT DISTRIBUTION $P(H,t)$ OF THE CENTERED HEIGHT FLUCTUATION

$H = h - \langle h \rangle$

IN SEVERAL DISCRETE GROWTH MODELS COINCIDES WITH THE TRACY-WIDOM DISTRIBUTION IN RMT.

· UNIVERSALITY BEYOND THE SECOND MOMENT

TO SUMMARIZE, THE TYPICAL HEIGHT FLUCTUATION IN DISCRETE GROWTH MODELS (CURVED GEOMETRY)

$H = h(x,t) - \langle h(x,t) \rangle$

HAS A LIMITING DISTRIBUTION AT LATE TIMES t

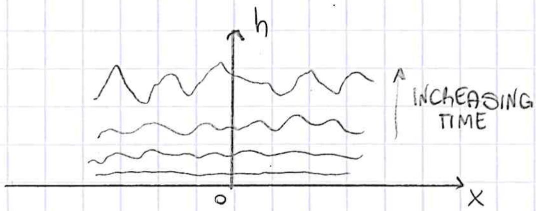
$$P(H,t) \longrightarrow \frac{1}{ct^{1/3}} f_2\left(\frac{H}{ct^{1/3}}\right)$$

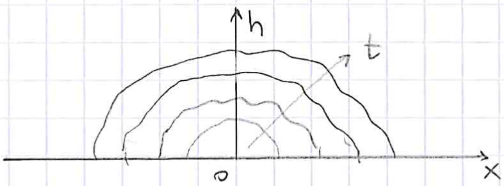
WHERE c IS NON-UNIVERSAL (MODEL DEPENDENT), BUT $f_2(x)$ IS UNIVERSAL AND IT'S THE TRACY-WIDOM (GUE) DISTRIBUTION IN RANDOM MATRIX THEORY.

DISCRETE KPZ GROWTH MODELS $\rightarrow$ ULAM PROBLEM $\xrightarrow{BDJ}$ RMT.

GROWTH IN FLAT VS CURVED GEOMETRY

LET'S START FROM FLAT INITIAL CONDITIONS (OR DROPLET). THE HEIGHT $h(0,t)$ IS A RANDOM VARIABLE. AT LATE TIMES,





$$h(0,t) \rightarrow at + bt^{1/3} \chi_\beta$$

WHERE AS USUAL χ_β IS A RANDOM VARIABLE COMING FROM A TRACY WIDOM DISTRIBUTION PARAMETRIZED BY β, WITH $\beta = 1$ FOR GOE AND $\beta = 2$ FOR GUE.

IT TURNS OUT

$\chi_1 \rightarrow$ FLAT GEOMETRY

$\chi_2 \rightarrow$ CURVED (DROPLET) GEOMETRY.

WHY? NOBODY KNOWS YET (2017): IT JUST EMERGES FROM THE CALCULATIONS.

★ SINCE THE WORK OF BDJ, TRACY-WIDOM IS UBIQUITOUS: DIRECTED POLYMER, RANDOM PERMUTATION, GROWTH MODELS (KPZ EQUATION), SEQUENCE ALIGNMENT, LARGE N GAUGE THEORY, LIQUID CRYSTALS, SPIN GLASSES...

IT IS ALSO OBSERVED IN MANY EXPERIMENTS.

· PHASE 3: WHAT ABOUT THE HEIGHT DISTRIBUTION IN THE KPZ EQUATION ITSELF?

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \sqrt{D}\, \eta(x,t).$$

IT TOOK A WHILE TO FIGURE OUT THE CORRECT REGULARIZATION SCHEME; A REGULARITY STRUCTURE WAS FOUND BY Hairer (2014). THE COMBINED EFFORT OF PHYSICISTS AND MATHEMATICIANS LED TO A BREAKTHROUGH FROM 2010:

$$h_{KPZ}(0,t) \rightarrow at + bt^{1/3} \chi_\beta \qquad \text{AS } t \rightarrow \infty$$

χ_β FROM TRACY-WIDOM.