Lecture notes on

STATISTICAL FIELD THEORY

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STATISTICAL FIELD THEORY

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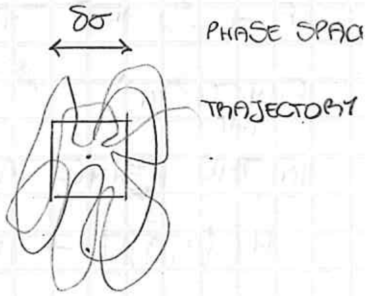
$\sigma = \{\sigma_1 \dots \sigma_N\}$ PHASESPACE, N LARGE $(\sim 10^{23})$

THE SET OF σ'S DESCRIBES THE <u>CONFIGURATION</u> ($\neq$ <u>STATE</u>, i.e. BUNCH OF CONFIGURATIONS WITH A GIVEN PROPERTY).



LOOKING AT A REGION IN PHASESPACE,

$$\lim_{T\to\infty} \frac{t(\sigma)}{\underset{\underset{\text{TOTAL TIME OF OBSERVATION}}{\uparrow}}{T}} \equiv \delta w(\sigma) \equiv \text{PROBABILITY} \propto \underset{\text{VOLUME}}{\delta\sigma} \cdot \underset{\text{PROBABILITY DENSITY}}{\rho(\sigma)}$$

NORMALIZATION CONDITION (i.e. THE SYSTEM HAS TO BE SOMEWHERE):

$$\int D\sigma \; \rho(\sigma) = 1$$

TAKE A FUNCTIONAL $f(\sigma)$: WE WANT ITS <u>AVERAGE</u>.

i) TIME AVG:

$$\langle f \rangle_{exp} = \lim_{T\to\infty} \frac{1}{T} \int^{T} dt \; f(t)$$

THIS IS WHAT YOU DO IN EXPERIMENTS (HOPEFULLY) AND SIMULATIONS

ii) PHASE AVG:

$$\langle f \rangle_{\rho} = \int D\sigma \; \underset{\underset{\text{STATIC DISTRIBUTION}}{\uparrow}}{\rho(\sigma)} \; f(\sigma)$$

THE AIM OF STATISTICAL PHYSICS IS TO WRITE

$$\langle f \rangle_{exp} \simeq \langle f \rangle_{\rho}$$

AS CLOSE AS POSSIBLE TO AN EQUALITY.

<u>HOPE</u>: $\rho(\sigma)$ IS STATIONARY.

IN SOME CASES YOU CAN ACTUALLY PROVE IT, e.g. LIOUVILLE THEOREM, WHICH UNFORTUNATELY ONLY HOLDS IF YOU USE $\rho(\{q,p\})$.

IN BIOLOGY – WHERE YOU ALL WILL END UP IF YOU WANT TO STUDY

STATISTICAL PHYSICS - YOU DON'T USE $\{p, q\}$, AS YOU DO FOR HAMILTONIAN SYSTEMS.

IN GENERAL, IF ρ DOES NOT DEPEND ON t, $\underline{P}$ AND $\underline{L}$ ARE INTEGRALS* OF MOTION:

$$\rho = \rho(E, \underline{P}, \underline{L}) \quad \xrightarrow[\text{REFERENCE FRAME}]{\text{TRANSFORM.}} \quad \rho = \rho(E), \quad H(\sigma) = E$$

IF H IS NOT THE HAMILTONIAN, WE SEARCH FOR A SIMILAR "COST FUNCTION".

IN THE ISING MODEL, IT'S HARD TO STEP FROM

$H(\{p, q\}) \to H_{ISING}(\sigma)$ (MICROSCOPICAL → MACROSCOPICAL)

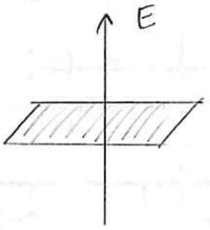
SO EVEN IN THAT CASE WE ACTUALLY guessed H.

*NOTE: HOW CAN I STEP TO A FRAME WHERE $\underline{L} = 0$ WITHOUT PROBLEMS WITH RELATIVITY? PROBABLY RELATIVITY, BEING "LONG RANGE", IS PROBLEMATIC.

- MICROCANONICAL ENSEMBLE

THE SYSTEM IS ISOLATED → FIXED ENERGY, $H(\sigma) = E$, $\forall \sigma$ THAT YOU VISIT.

WHAT IS $\rho(\sigma)$?



CRITERIUM: "RESPECT YOUR IGNORANCE".

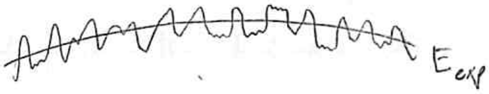
$$\rho(\sigma) = \begin{cases} 0 & H(\sigma) \neq E_{exp} \\ \text{CONST.} & H(\sigma) = E_{exp} \end{cases}$$

$\rho(\sigma) \sim \delta(H(\sigma) - E_{exp})$ MAXIMUM IGNORANCE

- CANONICAL ENSEMBLE

NOT ISOLATED (HEAT BATH). IF $\rho(\sigma)$ IS STATIONARY, WE CAN COMPUTE

$$\langle H \rangle_{exp} = \lim_{T \to \infty} \frac{1}{T} \int^{T} dt \, \underset{\substack{\uparrow \\ \text{FLUCTUATES}}}{H(\sigma(t))} = E_{exp}$$



MAXIMUM ENTROPY PRINCIPLE

$\rho \neq \delta(H(\sigma) - E_{exp})$

$\rho(\sigma)$ s.t.

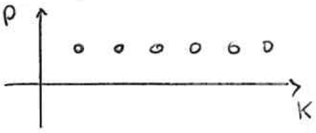
1) $\int D\sigma \, \rho(\sigma) = 1$ NORMALIZATION

2) $\int D\sigma \, \rho(\sigma) H(\sigma) = E_{exp} = \lim_{T \to \infty} \frac{1}{T} \int^{T} dt \, H(\sigma(t))$

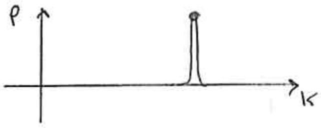
SHANNON ENTROPY:

$$S[P] = -\int \mathcal{D}\sigma \; P(\sigma) \ln P(\sigma)$$

LARGE S ↔ LOW INFO CONTENT



$$S = -\sum_{k} \frac{1}{k} \ln \frac{1}{k} = \ln k$$



$$S = 0$$

BEST PROOF:

E.T. JAYNES, PHYS. REV. 106, 620 (1957) I
108, 171 (1957) II

* LET'S NOW MAXIMIZE $S[P]$ WITH CONSTRAINTS (1) AND (2):

$$\mathcal{L}[P] = -\int \mathcal{D}\sigma \, P(\sigma) \ln P(\sigma) - \beta\left(\int \mathcal{D}\sigma \, P(\sigma) H(\sigma) - E_{exp}\right) - \lambda\left(\int \mathcal{D}\sigma \, P(\sigma) - 1\right)$$

$$\frac{\delta \mathcal{L}[P]}{\delta P} = 0 = -\ln P - 1 - \beta H(\sigma) - \lambda$$

$$P(\sigma) = e^{-(1+\lambda)} e^{-\beta H(\sigma)} \equiv \frac{1}{Z} e^{-\beta H(\sigma)}$$

IMPOSING NORMALIZATION,

$$Z = \int \mathcal{D}\sigma \, e^{-\beta H(\sigma)}$$

NOTE: Z STANDS FOR "ZUSTANDSSUMME", THE GERMAN WORD FOR "SUM OVER STATES"

WHICH IS THE <u>PARTITION FUNCTION</u>.

IMPOSING $E_{exp} = \langle H(\sigma) \rangle_P$,

$$E_{exp} = \frac{1}{Z(\beta)} \int \mathcal{D}\sigma \, e^{-\beta H(\sigma)} H(\sigma) \qquad (I)$$

WHICH FIXES β GIVEN E_{exp} $\left(\beta = \frac{1}{T}\right)$.
SO β IS THE CORRECT LAGRANGE MULTIPLIER.

WE FOUND

$$P(\sigma; E_{exp}) = \frac{1}{Z(\beta(E_{exp}))} e^{-\beta(E_{exp}) H(\sigma)}$$

ALTERNATIVELY, (I) FIXES E_{exp}, GIVEN β. THIS WAY WE OBTAIN THE GIBBS-BOLTZMANN DISTRIBUTION

$$P(\sigma; \beta) = \frac{1}{Z(\beta)} e^{-\beta H(\sigma)}$$

LET'S NOW EVALUATE THE FUNCTIONAL $S[P]$ IN $P = P(\sigma; \beta)$:
WE HAD A FUNCTIONAL AND NOW WE HAVE A NUMBER, i.e. ITS MAXIMUM,

$$S(\beta) = S[P]\Big|_{P = P(\sigma;\beta)}$$

★ TO SUM UP, WE SAW TWO ENTROPIES (SO FAR):

$$S_0[P] = -\int D\sigma\, P(\sigma) \ln P(\sigma)$$

$$S_1(\beta) = S_0[P = P_\beta(\sigma)]$$

BUT WHERE ARE THE CONFIGURATIONS?

- FREE ENERGY (HELMOTZ)

LET'S COMPUTE

$$Z = \int D\sigma\, e^{-\beta H(\sigma)} = \int D\sigma \underbrace{\int dE\, \delta(E - H(\sigma))}_{=1} e^{-\beta H(\sigma)}$$

→ MOST IMPORTANT TRICK IN THEORETICAL PHYSICS: MULTIPLY BY 1.

$$= \int dE\, e^{-\beta E} \int D\sigma\, \delta(E - H(\sigma))$$

DEFINE

$$\Omega(E) = \int D\sigma\, \delta(E - H(\sigma))$$

NOTA: È LA $\omega(E)$ DI VULPIANI.

NOW THIS COUNTS THE CONFIGURATIONS: IT'S THE VOLUME OF A MANIFOLD.

DEFINE A NEW ENTROPY (GEOMETRICAL)

$$S_2(E) = \ln \Omega(E) = \ln \int \mathcal{D}\sigma\, \delta(E - H(\sigma))$$

WHICH IS A FUNCTION OF ENERGY (NOT β!).

IN THESE TERMS,

$$\mathcal{Z} = \int dE\, e^{-\beta(E - TS_2(E))} \qquad T = 1/\beta$$

ASSUME $H(\sigma)$ IS AN EXTENSIVE QUANTITY, SO THAT

$$\frac{E}{N} \sim O(1)$$

AND WE CAN USE A SADDLE-POINT FOR BIG N:

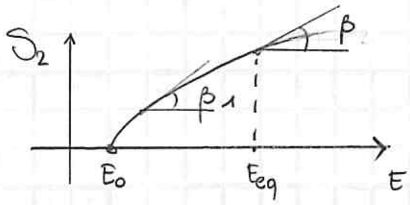
$$\mathcal{Z} = \int dE\, e^{-\beta N\left(\frac{E}{N} - T\frac{S_2(E)}{N}\right)} \simeq e^{-\beta \min_E (E - TS(E))} \equiv e^{-\beta(\underset{\substack{\uparrow \\ \text{EQUILIBRIUM ENERGY}}}{E_{eq}} - TS(E_{eq}))} \qquad (\mathrm{II})$$

DEFINE THE <u>FREE ENERGY</u>

$$F(E) = E - TS_2(E)$$

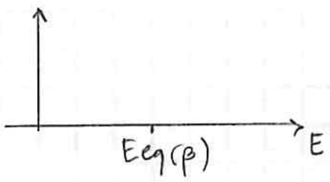
$$\left.\frac{dF}{dE}\right|_{E_{eq}} = 0 \qquad \Rightarrow \qquad \underline{\left.\frac{dS_2(E)}{dE}\right|_{E_{eq}} = \beta} \qquad (\mathrm{III})$$

THE SYSTEM MUST HAVE A GROUND STATE AND $S''_2(E) < 0$.



$S''_2(E) < 0$

$\beta\downarrow,\ T\uparrow,\ E_{eq}\uparrow$



· <u>HOMEWORK</u>

WHAT HAPPENS IF $S''_2(E) > 0$?

* NOTICE E_{eq} DEPENDS ON β VIA (III).

HENCE

$$F(E = E_{eq}(\beta)) = F(\beta) = E_{eq} - TS_2(E_{eq}(\beta)) \qquad (\mathrm{IV})$$

IS THE HELMOTZ FREE ENERGY. FROM (II),

$$F(\beta) = -\frac{1}{\beta} \ln \mathcal{Z}(\beta) \qquad (\mathrm{V})$$

- EXERCISE

PROVE $E_{exp} = E_{eq}$, WHERE

$$E_{exp} = \langle H_{exp} \rangle = \langle H \rangle_P = \int \mathcal{D}\sigma\, P(\sigma)\, H(\sigma)$$

$$E_{eq}: \quad \beta = \frac{dS_2}{dE}\bigg|_{E_{eq}}$$

PROOF:

$$E_{exp} = -\frac{d}{d\beta} \ln Z(\beta) \overset{(\mathrm{V})}{\overset{\downarrow}{=}} \frac{d}{d\beta}\left(\beta F(\beta)\right) \overset{(\mathrm{IV})}{\overset{\downarrow}{=}} \frac{d}{d\beta}\left(\beta E_{eq}(\beta) - S(E_{eq})\right)$$

$$= E_{eq}(\beta) + \beta \frac{dE_{eq}}{d\beta} - \frac{dS}{dE}\bigg|_{E_{eq}} \frac{dE}{d\beta} \overset{(\mathrm{III})}{\overset{\downarrow}{=}} E_{eq}(\beta)$$

SO WE CAN SIMPLY TALK ABOUT the ENERGY.

★ SUMMARY OF ENTROPIES

$$S_0[P] = -\int \mathcal{D}\sigma\, P(\sigma) \ln P(\sigma)$$

$\downarrow$

$$S_1(\beta) = S_0[P = P(\sigma;\beta)]$$

$$S_2(E) = \ln \int \mathcal{D}\sigma\, \delta(E - H(\sigma))$$

$\downarrow$

$$S_3(\beta) = S_2(E = E_{eq}(\beta)) \qquad \frac{dS}{dE}\bigg|_{E_{eq}} = \beta$$

IN BOTH CASES WE HAD AN OPTIMIZATION PRINCIPLE AT WORK.

BUT ARE THEY EQUAL, i.e.

$$S_1(\beta) \overset{?}{=} S_3(\beta)$$

PROOF

$$S_0[P] = -\langle \ln P(\sigma) \rangle$$

$$S_1(\beta) = -\langle \ln P_\beta(\sigma) \rangle = -\langle \ln \frac{e^{-\beta H(\sigma)}}{Z} \rangle = \ln Z + \beta \langle H(\sigma) \rangle = \ln Z + \beta E_{eq}$$

RECALL

$$Z = e^{-\beta F(\beta)}$$

SO

$$S_1(\beta) = -\beta F + \beta E_{eq} = -\beta\left(E_{eq} - TS_2(E_{eq}(\beta))\right) + \beta E_{eq} = S_2(E_{eq}(\beta)) = S_3(\beta)$$

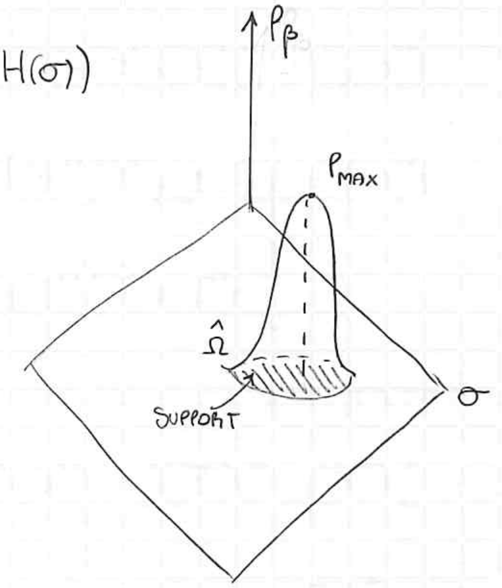
$\therefore$

WE FOUND

$$S_1(\beta) = S_3(\beta)$$

$$-\int \mathcal{D}\sigma\, P_\beta(\sigma) \ln P_\beta(\sigma) = \ln \int \mathcal{D}\sigma\, \delta\left(E_{eq}(\beta) - H(\sigma)\right)$$

$$\underline{-\langle \ln P_\beta \rangle = \ln \Omega(E_{eq}(\beta))}$$



PICTORIALLY,

$$-\langle \ln P_\beta \rangle = -\int \mathcal{D}\sigma\, P_\beta(\sigma) \ln P_\beta(\sigma)$$

$$= -\int_{\hat{\Omega}} \mathcal{D}\sigma\, P_\beta(\sigma) \ln P_\beta(\sigma) \underset{\substack{\uparrow \\ \text{BASE} \\ \times \text{ ALTEZZA}}}{\simeq} - P_{MAX} \cdot \ln(P_{MAX}) \cdot \hat{\Omega}$$

NOTE: TO MAKE THIS EXACT IT'S ENOUGH TO SUBSTITUTE P_{MAX} WITH AN INTEGRAL AVERAGE, OR TO CHOOSE $\hat{\Omega}$ WITH THE SAME CRITERIUM.

BUT

$$1 = \int \mathcal{D}\sigma\, P_\beta(\sigma) \simeq P_{MAX} \cdot \hat{\Omega}$$

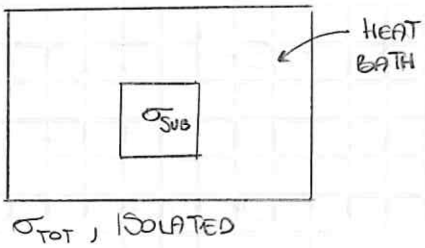
SO

$$-\langle \ln P_\beta \rangle \simeq -\ln P_{MAX} \simeq \ln \hat{\Omega}$$

$$\underset{\substack{\uparrow \\ \text{\# OF CONF.} \\ \text{ON WHICH THE} \\ \text{DISTRIBUTION IS PEAKED}}}{\ln \hat{\Omega}(\beta)} \simeq \underset{\substack{\uparrow \\ \text{\# OF CONF. WITH A GIVEN ENERGY } E_{eq}(\beta)}}{\ln \Omega(\beta)}$$

THERE IS ACTUALLY ONLY ONE ENTROPY.

• STANDARD DERIVATION OF CANONICAL ENSEMBLE (À LA LANDAU)



$H_{TOT}(\sigma_{TOT})$ → WHOLE ISOLATED SYSTEM

$H_{ext}(\sigma_{ext})$ → EXTERNAL SYSTEM (HEAT BATH)

$H(\sigma)$ → SUB-SYSTEM ("THE SYSTEM")

$$H_{TOT} = \underbrace{H(\sigma) + H_{ext}(\sigma_{ext})}_{\sim O(L^3)} + O(L^2)$$

IF THE INTERACTIONS ARE SHORT-RANGED, IN COMPUTING INTENSIVE QUANTITIES WE CAN NEGLECT

$$\frac{O(L^2)}{L^3}$$

THEN

$$P(\sigma_{TOT}) = \delta(H_{TOT}(\sigma_{TOT}) - E_{TOT})$$

$$P(\sigma) = \int D\sigma_{ext}\, \delta(H(\sigma) + H_{ext}(\sigma_{ext}) - E_{TOT})$$

$$= \int dE_{ext}\, \delta(H(\sigma) + E_{ext} - E_{TOT}) \int D\sigma_{ext}\, \delta(E_{ext} - H_{ext}(\sigma_{ext}))$$

RECOGNIZE

$$\int D\sigma_{ext}\, \delta(E_{ext} - H_{ext}(\sigma_{ext})) = e^{S_2^{ext}(E_{ext})}$$

SO THAT

$$\underline{P(\sigma) = e^{S_2^{ext}(E_{TOT} - H(\sigma))}}$$

BUT BY CONSTRUCTION

$$H(\sigma) << E_{TOT}$$

SO WE CAN EXPAND

$$P(\sigma) = e^{S_2^{ext}(E_{TOT})} e^{-H(\sigma) \frac{dS_2^{ext}}{dE}(E_{TOT})} = \frac{1}{Z} e^{-\beta H(\sigma)}$$

- HOMEWORK

BY USING CARNOT THEOREM,

$$\oint \frac{\delta Q}{T_{TH.}} \leq 0$$

PROVE THAT, IN A MECHANICALLY ISOLATED SYSTEM,

$$F_{TH.} = E - T_{TH.} S_{TH.}(E)$$

i.e. PROVE THE COMPATIBILITY BETWEEN STATISTICAL MECHANICS AND THERMODYNAMICS.

· SMALL SUMMARY OF LAST LECTURE

26.02.2019

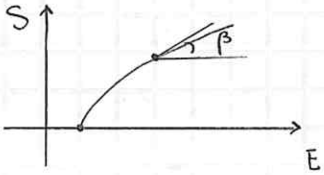
$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta)$$

$$F(\beta) = \min_E \{E - TS(E)\} \quad \Rightarrow \quad \frac{dS}{dE} = \beta$$

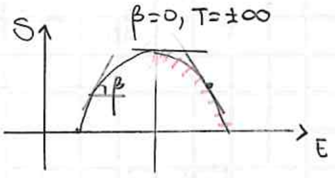
$F\downarrow$: $E\downarrow$ $S\uparrow$

T TUNES THIS BALANCE: IT'S A TRADE-OFF BETWEEN MINIMIZING E AND MAXIMIZING S, THERE'S A COMPETITION BETWEEN THE TWO.

IN SOME CASES, THIS TRADE-OFF GIVES RISE TO PHASE TRANSITIONS.



WHAT HAPPENS IF S(E) LOOKS LIKE THIS? THIS HAPPENS FOR INSTANCE IN THE P-SPIN SPHERICAL MODEL.



THE RIGHT BRANCH IS THERMODYNAMICALLY UNACCESSIBLE (T<0).

· GIBBS VS HELMOTZ FREE ENERGY

$$Z = e^{-\beta F(\beta)} = \int \mathcal{D}\sigma \, e^{-\beta H(\sigma)}$$

CONSIDER A SYSTEM DESCRIBED BY

$$H_{TOT} = H(\sigma) - h \sum_i \sigma_i$$

$$H_{TOT} = H(\sigma) - h \int d^d x \, \sigma(x)$$

h IS AN EXTERNAL FIELD AND THE SECOND TERM (BECAUSE OF THE MINUS) FAVOURS THE ALIGNMENT BETWEEN h AND σ_i.

NOW F DEPENDS ON h AS WELL:

$$e^{-\beta F(\beta, h)} = \int \mathcal{D}\sigma \, e^{-\beta H(\sigma) + \beta h \sum_i \sigma_i}$$

USING

$$1 = \int dm \, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

WHERE m IS THE MAGNETIZATION, $\overset{\substack{\text{ORDER PARAMETER} \\ \downarrow}}{m} \sim O(1)$, $M = \sum_i \sigma_i = mN$.

THEN

$$e^{-\beta F(\beta, h)} = \int dm \, e^{\beta h N m} \int D\sigma \, e^{-\beta H(\sigma)} \, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

DEFINE

$$\underline{e^{-\beta \mathcal{G}(\beta, m)} \equiv \int D\sigma \, e^{-\beta H(\sigma)} \, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)}$$

WHERE $\mathcal{G}$ IS GIBBS' FREE ENERGY. ITS RELATION WITH F IS

$$\underline{e^{-\beta F(\beta, h)} = \int dm \, e^{-\beta \mathcal{G}(\beta, m) + N\beta h m}} \qquad (\mathrm{I})$$

CALL

$$f(\beta, h) \equiv \frac{F(\beta, h)}{N}$$

$$g(\beta, h) \equiv \frac{\mathcal{G}(\beta, m)}{N}$$

SO AS TO REWRITE (I) AS

$$e^{-\beta N f(\beta, h)} = \int dm \, e^{-\beta N [g(\beta, m) - hm]}$$

FOR $N \to \infty$ WE CAN USE THE SADDLE-POINT:

$$\left.\frac{\partial g}{\partial m}\right|_{m_{eq}} = h \qquad \rightarrow \quad m_{eq} = m(\beta, h)$$

$$f(\beta, h) = g(\beta, m_{eq}(\beta, h)) - h \cdot m_{eq}(\beta, h)$$

WHICH IS A LEGENDRE TRANSFORM:

NOTE: h_{eq} IS UNUSUAL, AS h IS TUNED EXPERIMENTALLY.

$$g(\beta, m) = f(\beta, \underset{\sim}{h_{eq}}(\beta, m)) + \underset{\sim}{h_{eq}}(\beta, m) \cdot m$$

$$h(\beta, m) \text{ s.t. } -\left.\frac{\partial f}{\partial h}\right|_{h_{eq}(\beta, m)} = m$$

NOTE: LET $a(x)$ BE A CONVEX FUNCTION. THEN

$$p = \frac{da(x)}{dx}$$

IMPLICITLY DEFINES A FUNCTION $x(p)$. WE CAN DEFINE

$$\tilde{a}(p) = xp - a(x) \equiv x(p) \cdot p - a(x(p))$$

WHICH IS THE LEGENDRE TRANSFORM OF $a(x)$.

* LET'S COMPUTE THE PROBABILITY DISTRIBUTION OF m:

$$P(m;\beta,h) = \int D\sigma \, P(\sigma;\beta,h)\, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

$$= \frac{1}{\mathcal{Z}} \int D\sigma \, e^{-\beta H(\sigma) + \beta h \sum_i \sigma_i}\, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

$$= \frac{1}{\mathcal{Z}}\, e^{\beta h N m} \int D\sigma \, e^{-\beta H(\sigma)}\, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

$$= \frac{1}{\mathcal{Z}(\beta,h)}\, e^{\beta h N m}\, e^{-\beta N g(m,\beta)}$$

WE FOUND

$$P(m;\beta,h) = \frac{1}{\mathcal{Z}(\beta,h)}\, e^{-\beta N\left[g(m,\beta) - hm\right]}$$

AND THIS IS WHY g IS SO IMPORTANT.

DEFINE THE GENERALIZED FREE ENERGY

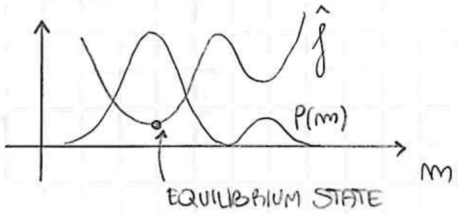
$$\hat{f}(m,h,\beta) = g(m,\beta) - hm$$

$$P(m) \sim e^{-\beta N \hat{f}(m,h,\beta)}$$

*NOTE: YES, THE SADDLE POINT OF g IS THE MINIMUM OF $\hat{f}$.

$\hat{f}$ IS NOT f (IT'S NOT CALCULATED IN ITS MINIMUM*). IF

$$h = 0 \rightarrow \hat{f}(m,\beta) \equiv g(m,\beta)$$



COMPUTING $P(h)$ IS NOT INTERESTING (WE SET IT EXPERIMENTALLY).

RECALLING

$$\mathcal{Z}(\beta,h) = e^{-\beta F(\beta,h)}$$

$$P(m) = \frac{e^{-\beta N[g(\beta,m) - hm]}}{e^{-\beta N f(\beta,h)}} = e^{-\beta N[g(\beta,m) - hm - f(\beta,h)]}$$

BUT

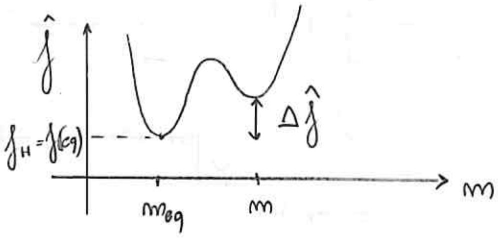
$$f(\beta, h) = g(\beta, m_{eq}(h)) - h\, m_{eq}(h)$$

$$\frac{\partial g}{\partial m}\bigg|_{m_{eq}} = h$$

SO

$$P(m) = e^{-\beta N\left[g(\beta, m) - g(\beta, m_{eq}) - h(m - m_{eq})\right]}$$

$$= e^{-\beta N\left[\hat{f}(\beta, m, h) - \hat{f}(\beta, m_{eq}, h)\right]} = e^{-\beta N \Delta \hat{f}}$$



AND WE SEE SUCH METASTABLE CONFIGURATIONS ARE HIGHLY SUPPRESSED IN THE $N \to \infty$ LIMIT:

$$P(m) = e^{-\beta N \Delta \hat{f}} \xrightarrow[N \to \infty]{} 0$$

- EXERCISE

$$m_{exp} = \langle \frac{1}{N} \sum_i \sigma_i \rangle$$

PROVE THAT

$$m_{exp} = m_{eq} .$$

PROOF

$$m_{exp} = \frac{1}{Z(\beta, h)} \int D\sigma \frac{1}{N}\left(\sum_i \sigma_i\right) e^{-\beta H + \beta h \sum_i \sigma_i} = -\frac{\partial}{\partial h} f(\beta, h)$$

$$= -\frac{\partial}{\partial h}\left[g(\beta, m_{eq}(\beta, h)) - h\, m_{eq}(\beta, h)\right]$$

$$= -\underbrace{\frac{\partial g}{\partial m_{eq}}}_{=h} \frac{\partial m_{eq}}{\partial h} + m_{eq} + h \frac{\partial m_{eq}}{\partial h} = m_{eq}$$

- SUSCEPTIBILITY : χ

$$H_{TOT} = H(\sigma) - h \sum_i \sigma_i$$

$$\frac{1}{N} \sum_i \langle \sigma_i \rangle = m_{eq}(h) = \langle \sigma_k \rangle = \langle \sigma_j \rangle$$

(TRUE IF THE SYSTEM IS HOMOGENEOUS). LET'S DEFINE

$$\chi(h) = \frac{\partial m(h)}{\partial h}$$

HOW DOES THE AVG $m(h)$ CHANGE BY CHANGING h?
IT'S A STATIC RESPONSE.

WE JUST LEARNED THAT

$$m(h) = - \frac{\partial f}{\partial h}(\beta, h)$$

HENCE

$$\chi = - \frac{\partial^2 f}{\partial h^2}(\beta, h)$$

NOTE (IMPORTANT): WE'RE USING THE LEGENDRE TRANSFORM, SO THIS g IS THE REAL GIBBS FREE ENERGY g_G AND NOT OUR $g(m, \beta)$.

USING LEGENDRE TRANSFORM,

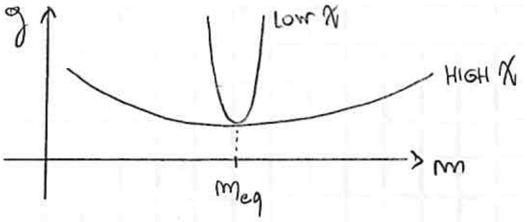
$$g(m) = f(h) + hm \qquad h(m) : \frac{\partial f(h)}{\partial h} = -m$$

$$\frac{\partial g}{\partial m} = \frac{\partial f}{\partial h}\frac{\partial h}{\partial m} + h + m\frac{\partial h}{\partial m} = h(m)$$

$$\frac{\partial^2 g}{\partial m^2} = \frac{\partial h}{\partial m}(m) = \left(\frac{\partial m(h)}{\partial h}\right)^{-1} = \chi^{-1}$$

HENCE

$$\chi = \left[\frac{\partial^2 g}{\partial m^2}\right]^{-1}\Bigg|_{eq}$$



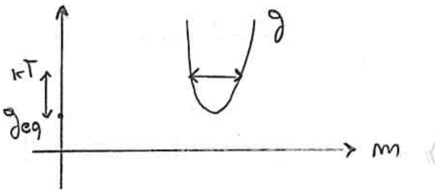
i.e. χ IS THE INVERSE OF THE CURVATURE OF g AT EQUILIBRIUM:

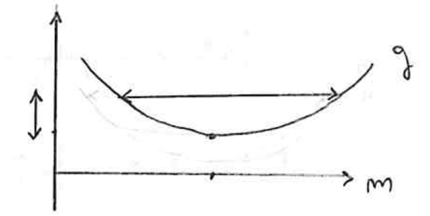
HIGH CURVATURE ↔ LOW χ

LOW CURVATURE ↔ HIGH χ

NOTICE

$$[k_B T] = \text{ENERGY}$$





TEMPERATURE FLUCTUATIONS GIVE RISE TO δg AND THIS CONTROLS δm.

IN PARTICULAR,

$$g''(m_{eq}) \to 0 \qquad \to \qquad \chi \to \infty$$

(MARGINAL STATE, FLAT DIRECTION).

χ IS IN FACT CONNECTED TO THE SPONTANEOUS FLUCTUATIONS OF THE SYSTEM...

· SUSCEPTIBILITY VS SPONTANEOUS FLUCTUATIONS

$$\chi = \frac{\partial m(h)}{\partial h}\bigg|_{h=0} = \frac{\partial}{\partial h}\frac{1}{N}\sum_i \langle\sigma_i\rangle\bigg|_{h=0}$$

$$= \frac{1}{N}\sum_i \frac{\partial}{\partial h}\frac{1}{Z(h)}\int D\sigma\, \sigma_i\, e^{-\beta H + \beta h \sum_j \sigma_j}\bigg|_{h=0}$$

$$= \frac{1}{N}\sum_i \left\{ \frac{1}{Z(h)}\int D\sigma\, \beta \sum_j \sigma_i\sigma_j\, e^{-\beta H} - \frac{1}{Z^2}\int D\sigma\, e^{-\beta H}\beta\sum_j \sigma_j \int D\sigma\, \sigma_i\, e^{-\beta H}\right.$$

HENCE

$$\chi = \frac{\beta}{N}\sum_{ij}\left\{\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle\right\} = \frac{\beta}{N}\sum_{ij}\langle\delta\sigma_i\,\delta\sigma_j\rangle$$

WHERE $\delta\sigma_i$ IS THE FLUCTUATION,

$$\delta\sigma_i = \sigma_i - \langle\sigma_i\rangle$$

INTRODUCING

$$m_\sigma = \frac{1}{N}\sum_i \sigma_i = \frac{1}{N}M_\sigma$$

NOTE: WE'RE CALLING
$m(h) = \langle m_\sigma\rangle$
WHERE m_σ IS THE OLD "m".

$$\chi = \beta N\left\{\langle \frac{1}{N}\sum_i \sigma_i \frac{1}{N}\sum_j \sigma_j\rangle - \frac{1}{N}\sum_i\langle\sigma_i\rangle\frac{1}{N}\sum_j\langle\sigma_j\rangle\right\}$$

$$= \beta N\left\{\langle m_\sigma^2\rangle - \langle m_\sigma\rangle^2\right\} = \frac{\beta}{N}\left\{\langle M_\sigma^2\rangle - \langle M_\sigma\rangle^2\right\}$$

SO χ IS ACTUALLY RELATED TO THE FLUCTUATIONS OF THE INTENSIVE ORDER PARAMETER.

* NOTICE

$$\sum_{ij}^{N} \sim N^2 \qquad\qquad \frac{1}{N}\sum_{ij}^{N} \sim N \to \infty$$

SO IS χ ALWAYS DIVERGENT?

ACTUALLY NOT ALL THE NUMBERS IN THE SUM ARE EQUALLY BIG: VARIABLES WHICH ARE FARTHER APART DON'T INTERACT SO MUCH.

· <u>CONNECTED CORRELATION FUNCTION</u>

$$G_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle\langle \sigma_j \rangle = \langle (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) \rangle = \langle \delta\sigma_i \cdot \delta\sigma_j \rangle$$

IF $|i-j| \to \infty$,

$$G_{ij} \to \langle \delta\sigma_i \rangle\langle \delta\sigma_j \rangle = 0 \qquad \text{(BY DEFINITION)}$$

WHILE

$$G_{ij}^{m.c.} = \langle \sigma_i \sigma_j \rangle \to \langle \sigma_i \rangle\langle \sigma_j \rangle = \langle \sigma \rangle^2 \neq 0$$

G_{ij} GROWS IF $\delta\sigma_i$, $\delta\sigma_j$ ARE IN THE SAME DIRECTION, i.e. THE TWO VARIABLES HAVE THE SAME FLUCTUATIONS W.R.T. THE MEAN.

WE CAN WRITE

$$\chi = \beta \frac{1}{N} \sum_{ij} G_{ij}$$

LET $r_{ij} = |r_i - r_j|$,

$$1 = \int dr\, \delta(r - r_{ij})$$

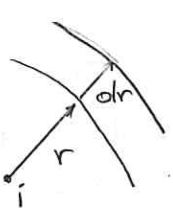
NOTE: I THINK HE MEANS $G_{ij} = G(\underline{r}_i, \underline{r}_j) = G(r_{ij})$.

SINCE $G_{ij} = G_{ij}(\underline{r}_i, \underline{r}_j) \overset{\substack{\text{ISOTROPIC}\\ \downarrow}}{=} G_{ij}(|\underline{r}_i - \underline{r}_j|) = G_{ij}(r_{ij})$,

$$\chi = \frac{\beta}{N} \sum_{ij} \int dr\, \delta(r - r_{ij})\, G_{ij} = \frac{\beta}{N} \int dr\, G(r) \sum_{ij} \delta(r - r_{ij})$$

LET'S ESTIMATE

$$\sum_{ij} \delta(r-r_{ij})\,dr$$



FOR EACH CENTRE i,

$$\sum_{j} \delta(r-r_{ij})\,dr = 4\pi r^2\,dr\,\rho$$

SO

$$\chi = \frac{\beta}{N}\int dr\, 4\pi r^2 \rho N\, G(r) = \beta\rho\int d^3r\, G(r)$$

IN GENERAL,

$$\underline{\chi = \beta\rho\int^{V} d^d r\, G(r)}$$

G(r) CONNECTED CORRELATION FUNCTION

AS LONG AS V IS FINITE, WE HAVE NO PROBLEMS. COMPARE WITH

$$\chi = \frac{\beta}{N}\sum_{ij}^{N}\langle\delta\sigma_i\,\delta\sigma_j\rangle$$

ITS CONVERGENCE DEPENDS ON G(r): IT MUST BE INTEGRABLE.
IF G(r) DOESN'T DECAY FAST ENOUGH (e.g. POWER LAW), THEN

$$\chi \to \infty$$

CORRELATION LENGTH

IN GENERAL,

$$G(r) = \frac{1}{r^\alpha}\, f\left(\frac{r}{\xi}\right)$$

MORALLY, BUT IT MIGHT BE SOMETHING ELSE

$$f\left(\frac{r}{\xi}\right) \overset{\downarrow}{\sim} e^{-r/\xi}$$

ξ SETS THE LENGTH SCALE AND IT'S CALLED CORRELATION LENGTH. THE SCALE-FREE PART IS $1/r^\alpha$

(POWER LAWS ARE THE ONLY SCALE-FREE FUNCTIONS). THEN

$$\chi = \beta\int d^d r\, \frac{f(r/\xi)}{r^\alpha} \underset{\substack{\uparrow \\ x=r/\xi}}{=} \beta\,\xi^{d-\alpha}\int d^d x\, f(x)/x^\alpha \sim \xi^{d-\alpha}$$

- HOMEWORK

$$\chi \sim \left(\frac{\partial^2 g}{\partial m^2}\bigg|_{eq} \right)^{-1}$$

WHAT HAPPENS WHEN $g'' = 0$, $N < \infty$? DOES χ DIVERGE?

Ⓐ YES, $\chi = \infty$ IF $g'' = 0$, $N < \infty$.

Ⓑ NO, $\chi < \infty$ IF $N < \infty$, BECAUSE g'' CANNOT BE $\emptyset$ IF $N < \infty$.

Ⓒ YES, g'' CAN BE $\emptyset$ AT $N < \infty$, BUT THERE IS SOMETHING ELSE GOING ON SO THAT $\chi < \infty$.

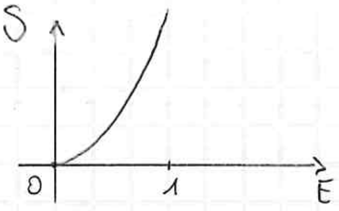
TIP:

WHEN YOU'RE STRIVING TO UNDERSTAND SOMETHING DIFFICULT, PRODUCE A SIMPLE EXAMPLE AND TRY TO SOLVE THAT FIRST.

- LESSON 01/03/2019

TOY MODEL FOR MELTING (NO VIBRATIONS)

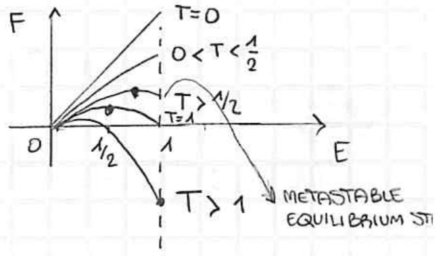
$S''(E) > 0$?



SIMPLIFIED VERSION:

$S(E) = E^2$

$F(E) = E - TS(E) = E - TE^2$



AT $T = 0$,

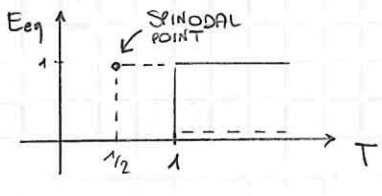
$F(E)_{MIN} = F(0) = 0 \quad \Rightarrow \quad Z = \int dE\, e^{-\beta F_{MIN}(E)}$

AT $T \neq 0$,

$E_{MAX} \equiv \hat{E} = \frac{1}{2T}$

NOTE: THE ENERGY HAS AN UPPER BOUND FOR INSTANCE IN THE P-SPIN MOD

BUT IT'S REALLY A MAXIMUM!



WITH

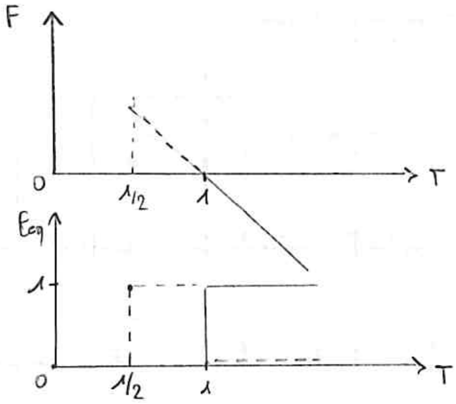
$\hat{E} = E_{MAX} \downarrow$ AS $T \uparrow$

AT $T = 1$ WE GET ANOTHER MINIMUM IN $E = 1$ (COEXISTENCE).

FOR $T > 1$, THE NEW ONE IS THE ONLY REAL MINIMUM.

DRAW $F(E_{eq}(T))$.

$$F(E) = E - TE^2$$



- RECAP OF LAST LESSON

$$\chi = \frac{\partial\langle\sigma_i\rangle}{\partial h} = \frac{\partial}{\partial h}\frac{1}{N}\sum_i\langle\sigma_i\rangle$$

$$= \frac{\beta}{N}\sum_{ij}\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$$

AND WE DEFINED THE CONNECTED CORRELATION FUNCTION

$$G_{ij} = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle = \langle(\sigma_i - \langle\sigma_i\rangle)(\sigma_j - \langle\sigma_j\rangle)\rangle$$

CORRELATION ≠ INTERACTION (DIRECT TRANSFER OF INFORMATION).

HENCE

$$\chi = \beta\rho\int^{r} d^d r\, G(r) \underset{\uparrow}{\sim} \xi^{d-\alpha}$$

$$G(r) \sim \frac{1}{r^\alpha} e^{-r/\xi}$$

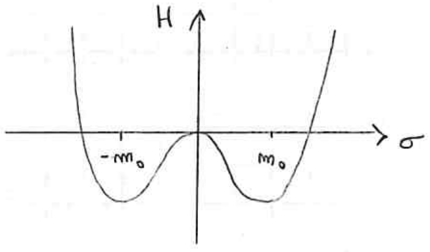
$$\begin{tikzcd}
-\frac{\partial^2 f}{\partial h^2} \arrow[dr] & \frac{\partial m}{\partial h} \arrow[d] & \left(\frac{\partial^2 g}{\partial m^2}\right)^{-1} \arrow[dl] \\
\beta N\langle\delta m^2\rangle \arrow[r] & \chi & \frac{\beta}{N}\sum_{ij}\langle\delta\sigma_i\delta\sigma_j\rangle \arrow[l] \\
\frac{\beta}{N}\langle\delta M^2\rangle \arrow[ur] & \beta\rho\int d^d r\, G(r) \arrow[u] &
\end{tikzcd}$$

- PLAUSIBILITY OF g'' CHANGING SIGN AT $N<\infty$

$$e^{-\beta N g(m,\beta)} \equiv \int D\sigma\, e^{-\beta H(\sigma)}\,\delta\left(m - \frac{1}{N}\sum_i\sigma_i\right)$$

$$\underset{\boxed{T\to 0}}{\simeq} e^{-\beta E_0}\delta(m - m_0) + e^{-\beta E_0}\delta(m + m_0)$$

$$Z(\beta) = e^{-\beta E_0} + e^{-\beta E_0}$$

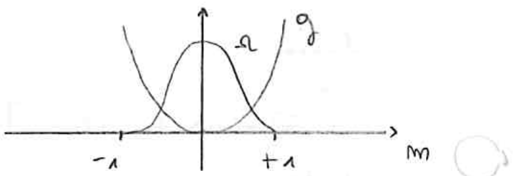


NOTE: RECALL
$P(m) = \frac{1}{2}e^{-\beta G(m,\beta)}$ IF $h=0$.

SO g'' MUST HAVE 2 MINIMA FOR VERY LOW T. IF INSTEAD $T\to\infty$ $(\beta\to 0)$,

$$e^{-\beta N g(m,0)} = \int D\sigma\,\delta\left(m - \frac{1}{N}\sum_i\sigma_i\right) \equiv \Omega$$ (NUMBER OF CONF. WITH MAGNETIZATION m)

SO g CHANGES ITS CONCAVITY!



NOTE: AT $T=0$, $g(m)$ LOOKS LIKE $H(\sigma)$, SO WE EXPECT A MAXIMUM AT $m=0$.

* WHAT'S THE RUB?

$$f(h) = -\frac{1}{\beta N}\ln\int\partial\sigma\, e^{-\beta H + \beta h\sum_i \sigma_i} = -\frac{1}{\beta N}\ln\int dm\, e^{-\beta N(g(m) - hm)}$$

$$m \overset{\circledast}{=} m_{eq} = -\frac{\partial f}{\partial h} = \frac{1}{\beta N}\frac{\int dm\,(-\beta N)\,e^{-\beta N(g(m)-hm)}(-m)}{\int dm\; e^{-\beta N(g(m)-mh)}} = \frac{\int dm\cdot e^{-\beta N(g-hm)}\,m}{\underbrace{\int dm\; e^{-\beta N(g-hm)}}_{A}}$$

$$\chi = \frac{\partial m_{eq}}{\partial h} = \beta N\left\{\frac{1}{A}\int dm\, m^2 e^{-\beta(\;)} - \left(\frac{1}{A}\int dm\; m\, e^{-\beta(\;)}\right)^2\right\} = \beta N\langle \delta m^2\rangle$$

AS LONG AS $g(m)$ GOES TO INFINITY FAST ENOUGH, χ DOESN'T SEEM TO DIVERGE... LET'S EXPAND

⊛ NOTE: THAT $\langle m\rangle = m_{eq}$ IS NOT TRUE AT THIS STAGE. TAKE m_{eq} AS THE NAME WE GIVE TO THE POINT AROUND WHICH WE EXPAND.

$$\hat{f}(m,h) = g(m) - mh$$

$$\hat{f}(m,h) = \hat{f}(m_{eq},h) + (0) + \frac{1}{2}\delta m^2\,\frac{\partial^2\hat{f}}{\partial m^2}(m_{eq})$$

$$= \hat{f}(m_{eq},h) + \frac{\partial^2 g}{\partial m^2}(m_{eq})\cdot\frac{1}{2}\delta m^2$$

HENCE

$$\chi = \beta N\,\frac{e^{-\beta N\hat{f}_{eq}}}{e^{-\beta N\hat{f}_{eq}}}\int d(\delta m)\,\delta m^2 e^{-\beta\frac{N}{2}g''(eq)\delta m^2}\cdot\left(\int d(\delta m)\, e^{-\beta\frac{N}{2}g''(eq)\delta m^2}\right)^{-1}$$

$$\underset{\substack{\uparrow\\ \text{IF } g''(eq)\neq 0}}{=} \beta N\,\frac{1}{\beta N\, g''(eq)} = \frac{1}{g''(eq)}$$

NOTE: IF YOU'RE NOT SURE ABOUT THE GAUSSIAN INTEGRAL, USE DIMENSIONAL ANALYS

$$\beta N g''(eq)\delta m^2 \sim 1 \rightarrow \langle\delta m^2\rangle \sim \frac{1}{\beta N g''(eq}$$

IF $g''(eq) = 0$, JUST KEEP ON EXPANDING UP TO THE FIRST NONZERO TERM:

$$\hat{f}(m,h) = \hat{f}(eq) + (\delta m)^m\,\frac{1}{m!}\frac{\partial^m g}{\partial m^m}\Big|_{eq} \qquad m > 2$$

$$\chi = \beta N\int d(\delta m)\,\delta m^2\, e^{-\beta N(\delta m)^m\cdot b_m}\left(\int d(\delta m)\, e^{-\beta N(\delta m)^m b_m}\right)^{-1} \sim \beta N\,\frac{1}{(\beta N)^{2/m}} \simeq N^{\frac{m-2}{m}}$$

$$\beta N(\delta m)^m \sim 1 \rightarrow (\delta m)^2 \sim \frac{1}{(\beta N)^{2/m}}$$

WE FOUND

$$\chi \sim N^{\frac{m-2}{m}} \xrightarrow[N\to\infty]{} \infty$$

$\uparrow$ ONLY IF $N\to\infty$

WHAT WE KNOW FOR SURE IS

$$e^{-\beta N f(h)} = \int dm\, e^{-\beta N (g(m) - hm)} \qquad (I)$$

$$e^{-\beta N g(m)} = \int D\sigma\, e^{-\beta H(\sigma)}\, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right) \qquad (II)$$

THE POINT IS $f(h)$ IS THE L.T. OF $g(m)$ ONLY IF $\frac{\partial^2 g}{\partial m^2} \neq 0$.

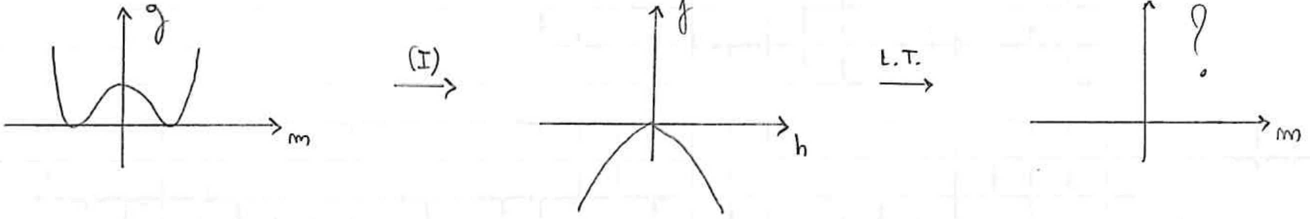
REMEMBER

$$\frac{\partial^2 f}{\partial h^2} = -\chi = -(\text{SUM OF POSITIVE TERMS}) < 0$$

SO ITS LEGENDRE TRANSFORM ALWAYS EXISTS:

$$g_G(m) = f(h(m)) + h(m)\cdot m \qquad h(m): \frac{\partial f}{\partial h}(h(m)) = -m$$

AND IT'S CONVEX (BUT IT MAY NOT COINCIDE WITH $g(m_{eq})$).



- HOMEWORK

PROVE THAT $g''(m)$ CAN CHANGE SIGN IN THE $N<\infty$ FULLY CONNECTED ISING MODEL, WHOSE HAMILTONIAN IS

$$H(\sigma) = -\frac{J}{2N}\sum_{ij} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

COMPUTE

$$g(m) = -\frac{1}{\beta N} \ln \int D\sigma\, e^{-\beta H(\sigma)}\, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

$$g'(m)$$

$$g''(m)\Big|_{m=0} \quad \text{AND PROVE THAT} \rightarrow \begin{cases} >0 & \text{FOR LARGE } T \\ <0 & \text{AT SMALL } T \end{cases}$$

AT FINITE N (NO SADDLE POINT IN N).

TIP: WRITE THE FOURIER REPRESENTATION OF $\delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$.

· LESSON 05/03/2019

· SMALL RECAP

$$\chi = \frac{\partial \langle\sigma\rangle}{\partial h} = \frac{\beta}{N} \sum_{ij} \{ \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \}$$

$$= \beta N \{ \langle m^2\rangle - \langle m\rangle^2 \} = O(1)$$

(NORMAL CASES, NO PHASE TRANSITIONS)

$$\langle \delta m^2 \rangle = O\left(\frac{1}{N}\right)$$

$$\frac{\sqrt{\langle \delta m^2 \rangle}}{\langle m \rangle} \sim O\left(\frac{1}{\sqrt{N}}\right) \xrightarrow[N\to\infty]{} 0$$

SO IT DOESN'T FLUCTUATE "MUCH" IN THE THERMODYNAMIC LIMIT.

SIMILARLY

$$\chi = \frac{\beta}{N} \{ \langle M^2\rangle - \langle M\rangle^2 \}$$

$$\langle \delta M^2 \rangle \sim O(N)$$

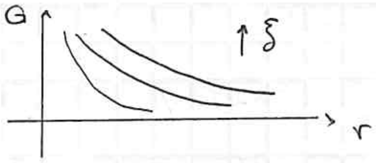
$$\frac{\sqrt{\langle \delta M^2 \rangle}}{\langle M \rangle} \sim O\left(\frac{1}{\sqrt{N}}\right)$$

RECALL

$$\chi = \beta\rho \int d^d r \, G(r)$$

$$G(r) = \frac{e^{-r/\xi}}{r^\alpha}$$

$$\chi \sim \xi^{d-\alpha}$$



$$\xi \to \infty \quad \leftrightarrow \quad \chi \to \infty$$

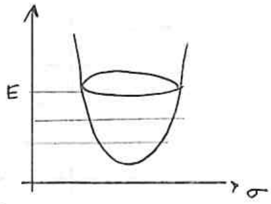
BUT WHEN DOES

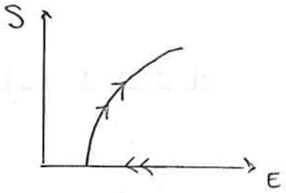
$$\chi = \left(\frac{\partial^2 g}{\partial m^2} \right)^{-1} \xrightarrow[N\to\infty]{} \infty \quad ?$$

PHASE TRANSITIONS

COMPETITION BETWEEN ENERGY AND ENTROPY

$$F = E - TS(E)$$



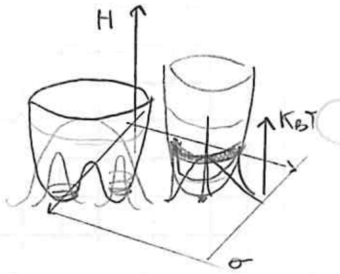


IN GENERAL, S(E) GROWS WITH 'E'; WE WOULD LIKE TO GO UP IN ENTROPY AND DOWN IN ENERGY.
'T' IS WHAT TUNES THIS COMPETITION.

CONSIDER A "GAUSSIAN" SYSTEM LIKE

$$H = \sum_{ij} J_{ij} \sigma_i \sigma_j \qquad |J| > 0$$

$$P(\sigma) = e^{-\beta H(\sigma)}$$



THINGS ARE EASY WITH A SINGLE GROUND STATE.
IF THERE ARE 2 GROUND STATES, EVEN AT FINITE N THERE EXISTS A TEMPERATURE S.t. WE NOTICE A QUALITATIVE CHANGE IN THE DISTRIBUTION, BECAUSE THE PRESENCE OF 2 MINIMA BECOMES IRRELEVANT.
YOU NEED DEGENERACY TO SEE PHASE TRANSITIONS.

ISING MODEL

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

THE BRAKETS STAND FOR "NEAREST NEIGHBOURS" ON A LATTICE.
J IS THE STRENGTH OF THE INTERACTION.
IT'S A MODEL FOR "IMITATION", i.e. FOR ELECTROMAGNETISM.
THERE ARE CLEARLY 2 GROUND STATES: ↑ OR ↓ $\forall i$. THEY ARE IN VERY FAR REGIONS OF PHASE SPACE WRT ONE ANOTHER.
NOTICE THERE'S NO MICROSCOPIC TERM LIKE

$$\sum_i \frac{p_i^2}{2m}$$

IN THE "HAMILTONIAN": IT'S JUST A COST FUNCTION.

IT'S BEEN ONLY SOLVED IN 1 OR 2D, BECAUSE OF THE PRESENCE OF $\sum_{\langle ij \rangle}$

SO WE PROMOTE THIS SUM TO ALL PAIRS AND GET

$$H = -\frac{J}{2N} \sum_{ij} \sigma_i \sigma_j \qquad \infty - \text{RANGE}$$

WHICH IS THE <u>FULLY CONNECTED (MEAN FIELD)</u> ISING MODEL. WE'VE ACTUALLY GIVEN UP SPACE (TOPOLOGY): IT'S JUST A BAG OF SPINS. HENCE, THERE IS NO CORRELATION FUNCTION, NOR CORRELATION LENGTH.

- <u>P-SPIN MODEL</u>

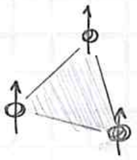
$$H = -\frac{J}{p!\, N^{p-1}} \sum_{i_1 \dots i_p}^{N} \sigma_{i_1} \dots \sigma_{i_p}$$

NOTE: THE FACTOR N^{1-p} MAKES H EXTENSIVE, $H \sim O(N)$.

IT'S THE GENERALIZATION TO P-BODY INTERACTIONS.

FOR $p=3$,

$$H = -\frac{J}{6N^2} \sum_{ijk} \sigma_i \sigma_j \sigma_k$$



WHICH IS NOT MERELY THE SUM OF 2-BODY INTERACTIONS (THE 3 OF US HAVE TO AGREE).

IN THE PICTURE, $p=2$ VS $p=3$.

IN $p=3$ YOU EXPECT THE ENTROPY HAS A BIGGER ROLE, BECAUSE THERE ARE MANY MORE CONFIGURATIONS COMPATIBLE WITH DISORDER.

+ + · 2 − − ORDER, FERRO	+ − 2 − + DISORDER, PARA	$p=2$
+ + + ORDER	+ − + − + + + + − ⋮ DISORDER	$p=3$

TO SOLVE IT, USE THE TRICK

$$H = -\frac{J}{p!\, N^{p-1}} \left(\sum_i \sigma_i \right)^p$$

$$f(h) = -\frac{1}{\beta N} \ln \int D\sigma\, e^{-\beta H + \beta h \sum_k \sigma_k} = -\frac{1}{\beta N} \ln \int D\sigma\, e^{\frac{\beta J}{p!\, N^{p-1}} \left(\sum_i \sigma_i\right)^p + \beta h \sum_i \sigma_i}$$

AS USUAL, WE INTRODUCE

$$1 = \int dm \, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

AND REWRITE

$$f(h) = -\frac{1}{\beta N} \ln \int dm \, e^{\beta N h m + \frac{\beta J N^p}{p! \, N^{p-1}} m^p} \int D\sigma \, \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

SINCE THE PREFACTOR WON'T COUNT IN THE N→∞, WE EXPRESS

$$\delta\left(m - \frac{1}{N}\sum_i \sigma_i\right) = \frac{1}{\beta N}\, \delta\left(\beta N m - \beta \sum_i \sigma_i\right) = \frac{1}{\beta N} \int_{-i\infty}^{+i\infty} dx \, e^{\beta N m x - \beta x \sum_i \sigma_i}$$

AND, NEGLECTING A CONSTANT TERM OF $O(\ln N/N)$,

$$f(h) = -\frac{1}{\beta N} \ln \int dm \int_{-i\infty}^{+i\infty} dx \, e^{-\beta N\left(-m^p \frac{J}{p!} - hm - xm\right)} \int D\sigma \, e^{-\beta x \sum_i \sigma_i}$$

BUT

$$\int D\sigma \, e^{-\beta x \sum_i \sigma_i} = \left(\prod_i \sum_{\sigma_i = \pm 1}\right) e^{-\beta x \sum_i \sigma_i} = \left(\sum_{\sigma = \pm 1} e^{-\beta x \sigma}\right)^N = 2^N \cosh(\beta x)^N$$

SO IT FACTORIZED IN THE PRODUCT OF N PARTITION FUNCTIONS OF THE SINGLE PARTICLE. HENCE

$$f(h) = -\frac{1}{\beta N} \ln \int dm \int_{-i\infty}^{+i\infty} dx \, e^{-\beta N\left\{-\frac{J m^p}{p!} - xm - \frac{1}{\beta}\ln\cosh(\beta x) - \frac{1}{\beta}\ln 2\right\}} \, e^{\beta N h m}$$

$$\underset{\substack{\uparrow \\ \text{def}}}{\equiv} -\frac{1}{\beta N} \ln \int dm \, e^{-\beta N g(m)} \, e^{+\beta h m N}$$

RECOGNIZE

$$e^{-\beta N g(m)} = \int_{-i\infty}^{+i\infty} dx \, e^{-\beta N\left\{-\frac{J m^p}{p!} - xm - \frac{1}{\beta}\ln\cosh(\beta x) - \frac{1}{\beta}\ln 2\right\}}$$

$$H = -J\left(\sum_i \sigma_i\right)^p \sim -J m^p$$ (ENERGY)

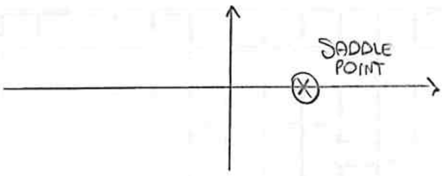
$$-T \ln 2 \to \frac{1}{N} \ln N_{TOT} = \frac{1}{N} \ln 2^N \equiv S_{T=\infty}$$

WE CAN USE THE SADDLE POINT IN x: WE HAVE TO MINIMIZE

$$a(x) = -xm - \frac{1}{\beta} \ln\cosh(\beta x)$$

HENCE

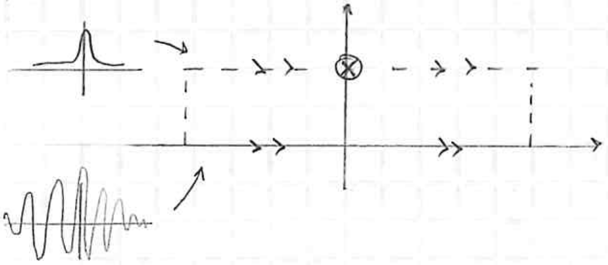
$$0 = \frac{\partial}{\partial x} a(x) = -m - \frac{1}{\beta} \frac{\mathrm{SINH}(\beta x)}{\mathrm{COSH}(\beta x)} \beta$$



$$m = -\mathrm{TANH}(\beta x) \qquad \Rightarrow \qquad \underline{\beta x_{SP} = -\mathrm{ATANH}(m)}$$

BUT WE WERE INTEGRATING ON THE COMPLEX AXIS!

WELL, IT'S THE POWER OF THE SADDLE POINT METHOD.



THEN

$$g(m) = -\frac{J}{p!} m^p + \frac{m}{\beta} \mathrm{ATANH}(m) - \frac{1}{\beta} \ln \mathrm{COSH}(\mathrm{ATANH}(m)) - \frac{1}{\beta} \ln 2$$

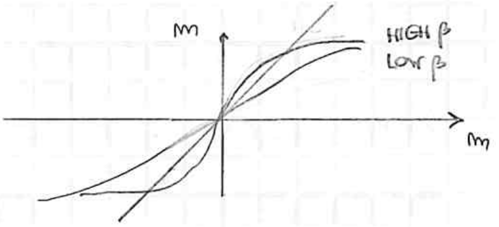
* <u>ISING, $P=2$, $h=0$</u>

$$g(m) = -\frac{J m^2}{2} + \frac{m}{\beta} \mathrm{ATANH}(m) - \frac{1}{\beta} \ln \mathrm{COSH}\left(\mathrm{ATANH}(m)\right)$$

$$0 \equiv \frac{\partial g}{\partial m} = -Jm + \frac{1}{\beta} \left\{ \mathrm{ATANH}(m) + m\,\mathrm{ATANH}'(m) - m\,\mathrm{ATANH}'(m) \right\} = -Jm + \frac{1}{\beta} \mathrm{ATH}($$

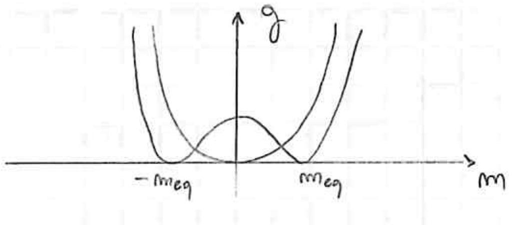
$$\underline{m_{eq} = \mathrm{TANH}(\beta J m_{eq})}$$ MEAN FIELD ISING EQ. FOR m_{eq}

WHICH IS A SELF-CONSISTENT EQUATION.



CLEARLY, THIS CANNOT BE THE REAL GIBBS FREE ENERGY $g_G(m)$, WHICH IS ALWAYS CONVEX. IT'S THE ONE THAT APPEARS IN

$$P(m) = \frac{1}{Z} e^{-\beta N g(m)}$$



THE CRITICAL TEMPERATURE AT WHICH THE PHASE TRANSITION APPEARS (II ORDER) IS

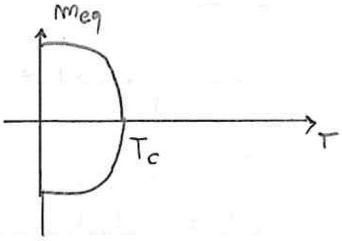
$$\beta J = 1 \qquad \Rightarrow \qquad T_c = J$$

YOU COULD VERIFY THAT

$$\left.\frac{\partial m_{eq}}{\partial T}\right|_{T_c} = \infty$$

$$g''(m_{eq})\Big|_{T_c} = 0 \quad \rightarrow \quad \chi = \infty$$

EVEN IF THERE'S NO CORRELATION FUNCTION.



NOTE:

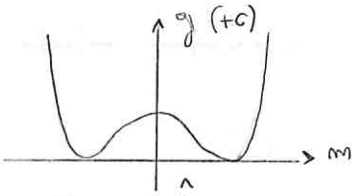
$$g''(m) = -J + \frac{1}{\beta}\frac{1}{1-m^2} \overset{\overset{m_{eq}^{T_c}}{\downarrow}}{=} 0$$

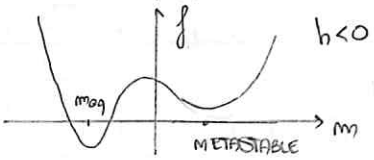
$$\frac{\partial}{\partial T} m_{eq} = -\beta^2 \frac{\partial}{\partial \beta} m_{eq} = ?$$

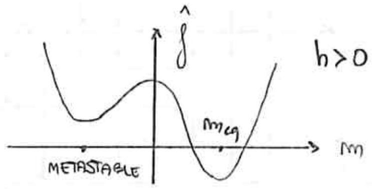
★ ISING, $P=2$, $h \neq 0$, $T < T_c$

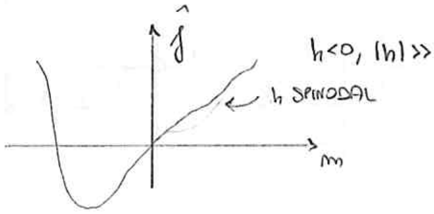
$$\hat{f}(h, m) = g(m,\beta) - hm$$

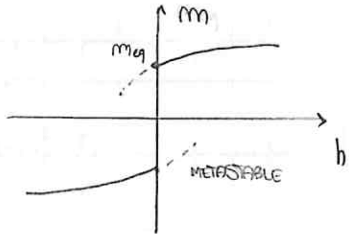
THERE IS A I ORDER P.T. DRIVEN BY THE ENERGY (NOT BY THE ENTROPY): IT'S TUNED BY THE FIELD h.











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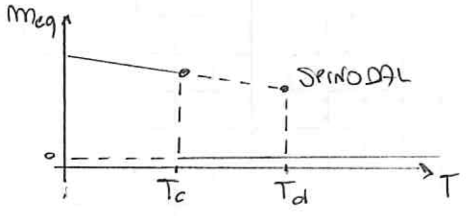
★ $P=3$ $(P>0)$, $h=0$

$$g(m) = -\frac{J}{6}m^3 + \frac{m}{\beta}\text{ATANH}(m) - \frac{1}{\beta}\ln\text{CH ATH}(m)$$

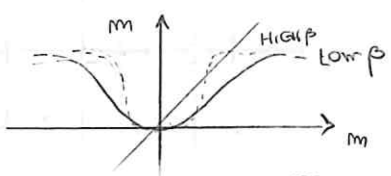
$$\frac{\partial g}{\partial m} = 0 \quad \Rightarrow \quad m = \text{TANH}\left(\frac{1}{2}J\beta m^2\right)$$

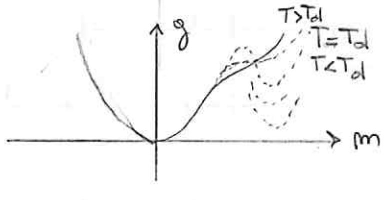
IT'S TUNED BY g'', WHERE g IS NOT THE REAL FREE ENERGY.

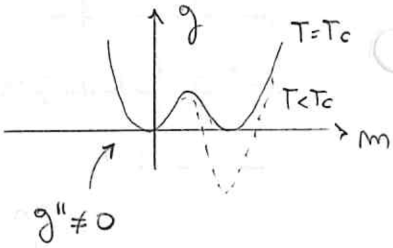
BY CHANGING T (AND NOT AN EXTERNAL FIELD) WE DRIVE THE TRANSITION:



THE LEFT MINIMUM NEVER DISAPPEARS (IT'S THE MAXIMUM OF S, ALWAYS METASTABLE), WHILE THE MINIMUM OF E CHANGES ITS STABILITY.







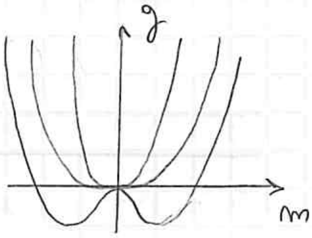
THERMODYNAMIC SPINODAL: IT'S THE VALUE OF THE "TUNING" PARAMETER AT WHICH A LOCALLY STABLE MINIMUM OF THE FREE ENERGY APPEARS (BUT IT'S NOT THE GLOBALLY STABLE ONE).

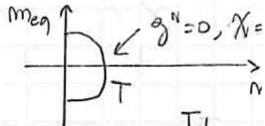
NOTICE AT $T=T_c$ IN A I^{ST} ORDER P.T., $g''(m_{eq}) \neq 0$: THERE'S NO DIVERGENCE.

- RECAP

i) $P=2$, $h=0$, T TUNING PARAMETER: II ORDER, S DRIVEN

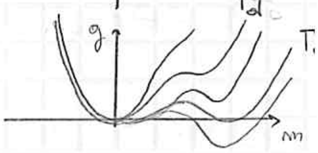
IT'S ENTROPY DRIVEN.



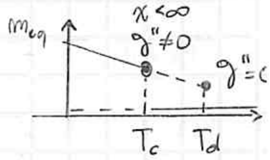


ii) $P=3$, $h=0$, T TUNING PARAMETER: I ORDER, S DRIVEN

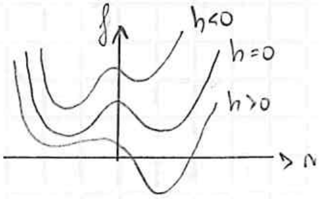
AT $T=T_d$, $g''|_{META\ m}=0 \rightarrow \chi=\infty$!

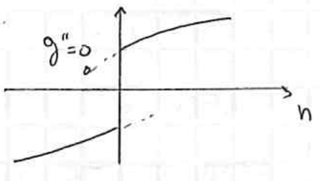


BUT IF THE SYSTEM IS AT EQUILIBRIUM, NOTHING HAPPENS. WE NEVER SEE THE SPINODAL POINT, WE ALWAYS COLLAPSE ONTO THE STABLE SOLUTION.



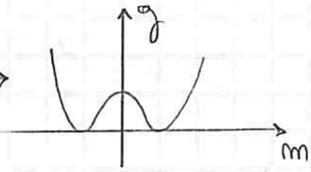
iii) $P=2$, $T<T_c$, h TUNING PARAMETER, I ORDER, E DRIVEN





- HOMEWORK

ISING MODEL. IF $g(m)$ HAS THIS SHAPE $\longrightarrow$ WHAT IS THE SHAPE OF THE REAL FREE ENERGY? ($P=2$, $h=0$). LARGE N, BUT FINITE.



REMEMBER THAT:

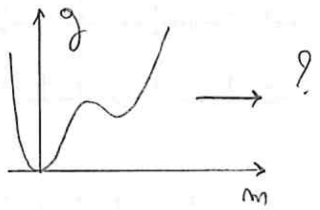
$$g_G(m) \overset{L.T.}{\rightleftarrows} f(h) \qquad (f(h) \text{ IS CONCAVE, } g_G(m) \text{ IS CONVEX})$$

$$g(m) = -\frac{1}{\beta N} \ln \int \mathcal{D}\sigma \, e^{-\beta H(\sigma)} \delta\left(m - \frac{1}{N}\sum_i \sigma_i\right)$$

$$\underset{\overset{\uparrow}{h=0}}{P(m)} = \frac{1}{Z} e^{-\beta N g(m)}, \qquad f(h) = -\frac{1}{\beta N} \ln \int dm \, e^{-\beta N (g(m) - hm)}$$

RECALL

$$\min_m g(m) - hm = \min_m \hat{f}(m,h)$$



IF YOU'RE GOOD ENOUGH, DO THE SAME WITH

USE h! WORK AT h SMALL, BUT FINITE.

DON'T USE AN EXPLICIT SHAPE OF g (NOT EXACTLY THAT OF THE I.M.)

· LESSON 08/03/2019

IN A FAIR WORLD, $P_{FAIR} = \frac{1}{2}$. IN OUR CLASS

$$\begin{array}{cc} XX & XY \\ 4 & 16 \end{array} \rightarrow P \sim 5 \cdot 10^{-4}$$

(PROBABILITY OF THIS REALIZATION).

· CORRECTION OF LAST HOMEWORK

PROVE THAT $g''(m=0)$ CAN CHANGE SIGN AT $N < \infty$.

LAST TIME WE DERIVED

$$g(m) = -\frac{1}{2} J m^2 - \frac{1}{\beta} S(m)$$

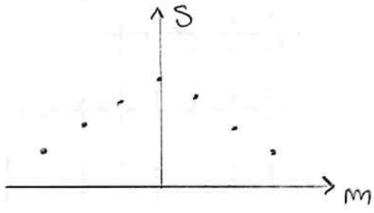
$$g''(m) = -J - T S''(m)$$

IF WE CAN PROVE THAT, FOR SOME FINITE N,

$$S''(0) < 0$$

THEN WE HAVE A COMPETITION BETWEEN J AND T.

FOR FINITE N,



$$S(m) = \frac{1}{N} \ln \sum_{\{\sigma_i\}} \delta\left(m, \frac{1}{N} \sum_i \sigma_i\right)$$

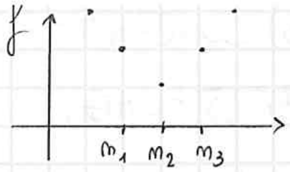
BUT THIS IS A SUM, NOT AN INTEGRAL: SO THAT CANNOT BE A DIRAC-δ, BUT A KRONECKER-δ.

$S(m)$ IS ACTUALLY AN HISTOGRAM: HOW DO WE TAKE THE 2^{ND} DERIVATIVE?

FIRST, WE COMPUTE $S(m)$:

$$S(m) = \frac{1}{N} \log \Omega(m) \qquad \Omega(m) = \binom{N}{N\left(\frac{1+m}{2}\right)}$$

1) CONCAVITY, FIRST WAY



$$f: \mathbb{N} \to \mathbb{R}$$

IT'S CONVEX IF

$$t^* f(m_1) + (1-t^*) f(m_3) \geq f(m_2)$$

WHERE t^* IS S.t.

$$m_2 = t^* m_3 + (1-t^*) m_1$$

WE CAN EVALUATE

$$\Omega(0) = \binom{N}{N/2}$$

$$\Omega\left(-\frac{2}{N}\right) = \Omega\left(\frac{2}{N}\right) \leq \Omega(0)$$

2) CONCAVITY, SECOND WAY

$$\Omega(m) = \frac{N!}{\left(N\left(\frac{1+m}{2}\right)\right)!\left(N\left(\frac{1-m}{2}\right)\right)!}$$

THE FACTORIAL CAN BE EXTENDED TO REAL VALUES BY MEANS OF THE Γ FUNCTION: IT'S NOT THE ONLY WAY, BUT IT'S A REASONABLE ONE (i.e. THE CORRESPONDENCE IS NOT UNIQUE).

$$m! \equiv \Gamma(m+1) = \int_0^\infty dt\, e^{-t} t^m \qquad \Gamma(m) = m\Gamma(m-1)$$

WITH THE PROPERTY THAT

$$\frac{d^2}{dz^2} \ln \Gamma(z) > 0 \quad \forall z > 0$$

HENCE REWRITE

$$\frac{d^2}{dm^2} \ln \Omega(m) = \frac{d^2}{dm^2} \left\{ \frac{\Gamma(N+1)}{\Gamma\left(N\left(\frac{1+m}{2}\right)+1\right)\Gamma\left(N\left(\frac{1-m}{2}\right)+1\right)} \right\}$$

$$= -\frac{N^2}{4}\left[\frac{d^2}{dz^2} \ln\Gamma(z)\Big|_{z_+} + \frac{d^2}{dz^2} \ln\Gamma(z)\Big|_{z_-} \right]$$

$$z_\pm \equiv \left(\frac{1 \pm m}{2}\right) N + 1$$

3) CONCAVITY, THIRD WAY

BY STIRLING,

$$\frac{1}{N}\log N! = f(N) + \overset{\sim O(\frac{1}{N})}{R(N)} \qquad = -1 + \frac{1}{N}(N\log N) + R(N)$$

$$\ln\Omega = N\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + N\left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right) + R(N)$$

$$\frac{d^2}{dm^2}\ln\Omega = \frac{N}{(1-m^2)} + R''(N)$$

(IT'S OK TO USE STIRLING FOR $m \sim 0$).

4) WHOLE EXERCISE, "A CAZZO DI CANE"

$$S(m) = \frac{1}{N}\ln\sum_{\{\sigma\}}\delta\left(m - \frac{1}{N}\sum_i\sigma_i\right)$$

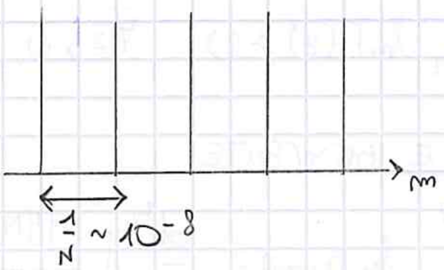
$$= \frac{1}{N}\ln\sum_{\{\sigma\}}\int_{\mathbb{R}} dx\, e^{ixm}\, e^{-ix\frac{1}{N}\sum_i\sigma_i}$$

$$= \frac{1}{N}\ln\int dx\, e^{ixm}\cos\left(\frac{x}{N}\right)^N 2^N \equiv \frac{1}{N}\ln Z(m)$$

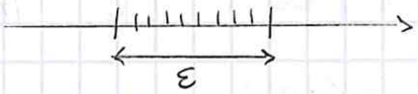
$$S'(m) = \frac{1}{NZ}\int dx\, ix\, f(x,m) \qquad f(x,m) = e^{ixm}\cos\left(\frac{x}{N}\right)^N$$

$$S''(m) = -\frac{1}{N}\left\{\langle x^2\rangle - \langle x\rangle^2\right\} \qquad \langle A\rangle = \frac{1}{Z}\int dx\, A\, f(x,m)$$

$$S''(m=0) = -\frac{1}{N}\frac{\int dx\, x^2\cos\left(\frac{x}{N}\right)^N}{\int dx\,\cos\left(\frac{x}{N}\right)^N}$$

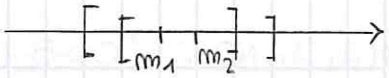


IMAGINE YOUR RESOLUTION IS 10^{-4}: HOW CAN THE DISCRETE NATURE OF THE PROBLEM AFFECT US? LET'S REGULARIZE IT.



WE FIX A RESOLUTION ε AND COUNT THE CONFIGURATIONS INSIDE.

ONE WAY IS BY BINNING WITH FIXED BINS, BUT THIS WAY I GET A DISCRETE $\int$ AGAIN. THEN WE ADOPT A MOBILE BINNING:



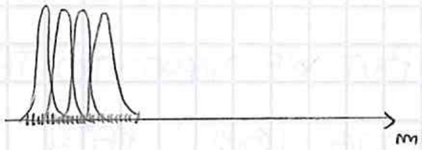
$$\int dm \, \Omega(m) = 2^N$$

(THE BINS OVERLAP, BUT THIS ISN'T A BIG DEAL).

DEFINE A δ-FUNCTION AS A SHARP GAUSSIAN,

$$\delta_\varepsilon(p) = \frac{1}{\varepsilon} e^{-\frac{1}{2}\frac{p^2}{\varepsilon^2}} \qquad [\delta_\varepsilon] = \left[\frac{1}{p}\right]$$

$$= \int dx \, e^{-\frac{1}{2}x^2\varepsilon^2 + ixp}$$

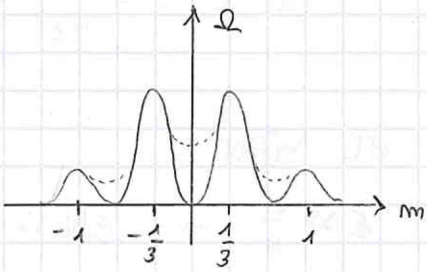


SO THE CALCULATION LOOKS THE SAME AS BEFORE, BUT WITH A NICE EXPONENTIAL CONVERGENCE FACTOR. INSTEAD OF A MONOCHROMATIC WAVE, THAT IS

$$\delta(p) = \int dx \, e^{ixp}$$

WE'RE ACTUALLY USING A WAVEPACKET.

LET'S TRY WITH $N = 3$:



$m =$	1	$\frac{1}{3}$	$-\frac{1}{3}$	-1
$\# =$	1	3	3	1

HOW DO WE TAKE THE LIMIT

$$\varepsilon \to 0, \; N \to \infty \; ?$$

WE NEED $\varepsilon \gg \frac{1}{N}$.

NOW WE CAN CALCULATE

$$S_\varepsilon(m) = \frac{1}{N} \ln \sum_{[\sigma]} \int dx \, e^{-\frac{1}{2}\varepsilon^2 x^2 + ixm - ix\frac{1}{N}\sum_i \sigma_i}$$

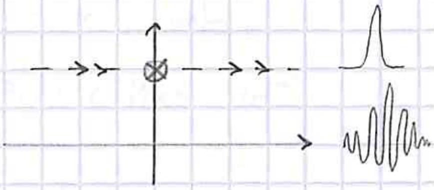
$$= \frac{1}{N} \ln \int dx \, e^{-\frac{1}{2}\varepsilon^2 x^2} e^{ixm} \cos\left(\frac{x}{N}\right)^N \cdot 2^N \equiv \frac{1}{N} \ln \Omega(m)$$

DEFINE

$$\mathcal{Z}(m) = \int dx\, e^{-\frac{1}{2}\varepsilon^2 x^2}\, e^{ixm} \cos\left(\frac{x}{N}\right)^N$$

WHICH NOW CONVERGES, BUT IT'S STILL NOT POSITIVE: IT OSCILLATES. HOWEVER,

$$f(x,m) = e^{-\frac{1}{2}\varepsilon^2 x^2}\, e^{ixm} \cos\left(\frac{x}{N}\right)^N$$



HAS A STATIONARY POINT ON THE COMPLEX AXIS!

TRY TO PLOT ON GNUPLOT, FOR SOME a,

plot Real(f(x+ia))

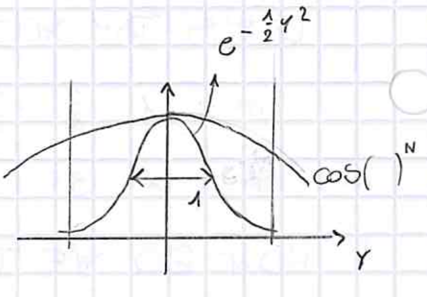
BUT WE WANT TO TAKE THE 2ND DERIVATIVE: WE'RE STUCK TO OUR (BAD) PATH. WE FIND

$$S''(m=0) = -\frac{1}{N}\,\frac{\int dx\, x^2 e^{-\frac{1}{2}\varepsilon^2 x^2} \cos\left(\frac{x}{N}\right)^N}{\int dx\, e^{-\frac{1}{2}\varepsilon^2 x^2} \cos\left(\frac{x}{N}\right)^N}$$

CHANGE COORDINATES TO

$$x\varepsilon = y$$

$$S''(m=0) = -\frac{1}{N\varepsilon^2}\,\frac{\int dy\, y^2 e^{-\frac{1}{2}y^2} \cos\left(\frac{y}{\varepsilon N}\right)^N}{\int dy\, e^{-\frac{1}{2}y^2} \cos\left(\frac{y}{\varepsilon N}\right)^N}$$



WE WANT

$$\varepsilon \gg \frac{1}{N} \quad \rightarrow \quad \varepsilon N \gg 1$$

SO NOW WE CAN EXPAND THE $\cos\left(\frac{y}{\varepsilon N}\right)$:

$$\cos\left(\frac{y}{\varepsilon N}\right)^N = \exp\left\{N \ln \cos\left(\frac{y}{\varepsilon N}\right)\right\} \simeq e^{N \ln\left(1 - \frac{y^2}{(\varepsilon N)^2}\right)} \simeq e^{-\frac{y^2}{\varepsilon^2 N}}$$

FIX $\varepsilon = \frac{A}{N}$, WITH $A \gg 1$: 'A' DOESN'T EVEN DEPEND ON 'N'.

$$S''(0) = -\frac{1}{N\varepsilon^2}\,\frac{\int dy\, y^2 e^{-\frac{1}{2}y^2\left(1+\frac{1}{N\varepsilon^2}\right)}}{\int dy\, e^{-\frac{1}{2}y^2\left(1+\frac{1}{N\varepsilon^2}\right)}} = -\frac{1}{1+N\varepsilon^2}$$

(SAME COMBINATION $N\varepsilon^2$ IN TWO PLACES: IT DOESN'T BREAK THE "BALANCE OF STOCAZZO").

THERE'S STILL A DEPENDENCE ON ε^2. AT FIXED ε, $N\varepsilon \gg 1$ AND SENDING $N \to \infty$ WE FIND, TRIVIALLY,

$$S''(0) \to 0$$

SO WE NEED

$$\varepsilon = \frac{A}{N} \longrightarrow 0$$

$$S''(0) = -\frac{1}{1 + N\frac{A^2}{N^2}} \xrightarrow[N\to\infty]{} -1$$

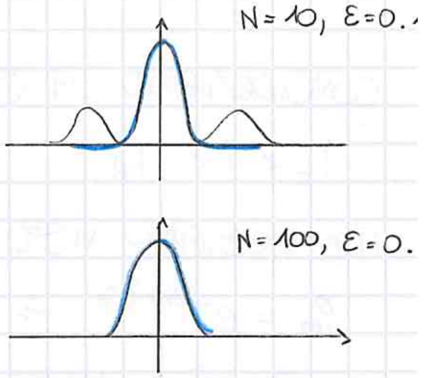
THIS IS THE THERMODYNAMIC LIMIT.

YOU CAN CHECK THAT

$$f(y) = e^{-\frac{1}{2}y^2} \cos\left(\frac{y}{\varepsilon N}\right)^N \quad —$$

$$g(y) = e^{-\frac{1}{2}y^2\left(1+\frac{1}{N\varepsilon^2}\right)} \quad —$$

LOOK LIKE THE GRAPHS ON THE RIGHT.



· CODA

$$S''(0) = -\frac{1}{N}\frac{\int dx\, x^2 \cos\left(\frac{x}{N}\right)^N}{\int dx \cos\left(\frac{x}{N}\right)^N} \simeq -\frac{1}{N}\frac{\int dx\, x^2 e^{-\frac{x^2}{2N}}}{\int dx\, e^{-x^2/2N}} = -1$$

WITH NO JUSTIFICATION WHATSOEVER... BUT THINK ABOUT THAT!

- LESSON 12.03.2019

- METASTABILITÀ E TEORIA DELLA NUCLEAZIONE

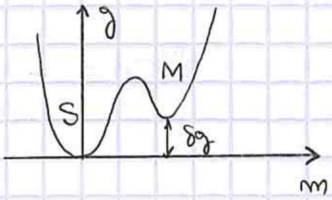
COSA SUCCEDE QUANDO g HA UN MINIMO LOCALE? ($N = \infty$, $d < \infty$ (NON MF))

1) UNO STATO METASTABILE HA SEMPRE VITA MEDIA FINITA

2) ESISTONO STATI METASTABILI CHE HANNO VITA MEDIA INFINITA

A) LA VITA MEDIA DELLO STATO METASTABILE ↑ SE $\delta g \uparrow$

B) LA VITA MEDIA DELLO STATO METASTABILE ↓ SE $\delta g \uparrow$



LE RISPOSTE GIUSTE SONO ① E Ⓑ.

CONSIDERIAMO UN SISTEMA ALLA ISING CON g COME IN FIGURA,

$T_c < T < T_d$

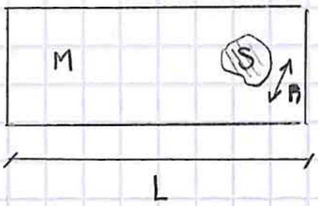
NOI ABBIAMO VISTO CHE

$P_M = e^{-\beta N \delta g} \to 0$

QUINDI È SOPPRESSO DAL PUNTO DI VISTA STATICO. MA, DINAMICAMENTE, POSSO PREPARARE IL SISTEMA VICINO A 'M' E QUELLO VI RIMARRÀ INTRAPPOLATO PER UN PO': T_d STA PER DINAMICA. VEDREMO CHE LA BARRIERA RILEVANTE NON È PERÒ δg. FUORI DAL MF QUESTA BARRIERA È FINITA, MENTRE DIVENTA INFINITA IN MF (PER QUESTO LO SI USA PER STUDIARE GLI STATI METASTABILI).

* PREPARO UN SISTEMA DI TAGLIA L NELLO STATO METASTABILE.

VOLENDO, USO LE PBC COSÌ DA IGNORARE LA SUPERFICIE (NUCLEAZIONE OMOGENEA).



PER FLUTTUAZIONI TERMICHE, SI PUÒ FORMARE UNA DROPLET DI TAGLIA 'R' NELLO STATO STABILE. IL NUOVO STATO È FAVOREVOLE?

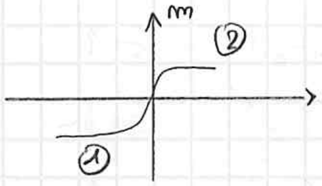
- HO UN GUADAGNO DI ENERGIA LIBERA DI VOLUME
- PERDO IN ENERGIA LIBERA DI SUPERFICIE

AVREMO

$$\text{GAIN} \simeq -\delta g \cdot R^d$$

(g È UNA DENSITÀ). INVECE

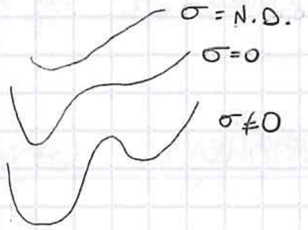
$$\text{LOSS} \simeq +\sigma \cdot R^{d-1}$$



DOVE σ È LA SURFACE TENSION ED È UNA MISURA DELLA DISTORSIONE DEL PARAMETRO D'ORDINE PER PASSARE DA UNO STATO ALL'ALTRO (È UN'ENERGIA LIBERA PER UNITÀ DI SUPERFICIE). NOTIAMO CHE

$\sigma \neq 0 \iff \exists$ PIÙ STATI $\iff \exists$ UN'ENERGIA DI INTERFACCIA

E CHE POSSO AVERE $\sigma \neq 0$ ANCHE SE GLI STATI HANNO LA STESSA g (VEDI CAPPELLO MESSICANO).

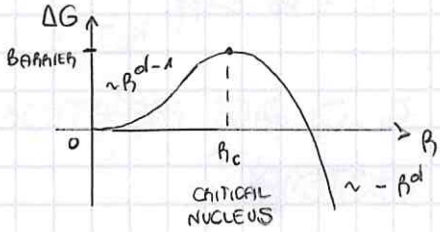


T_d È IL PUNTO IN CUI $\sigma \to 0$.

SI È TROVATO IL BILANCIO

$$\Delta G = \sigma R^{d-1} - \delta g \cdot R^d$$

E QUESTO DÀ LA VERA NUCLEATION BARRIER.



SI NOTI CHE "NON CAMPO MEDIO" SI VEDE IN

$$\text{LOSS} \simeq \sigma R^{d-1}$$

CHE RICHIEDE IL CONCETTO DI LOCALITÀ SPAZIALE (CHE SI PERDE IN CAMPO MEDIO).

*NOTA: SE $\sigma=0$ NON C'È ALCUN PERDITA A PASSARE NELLO STATO STABILE, QUINDI C'È SOLO QUELL

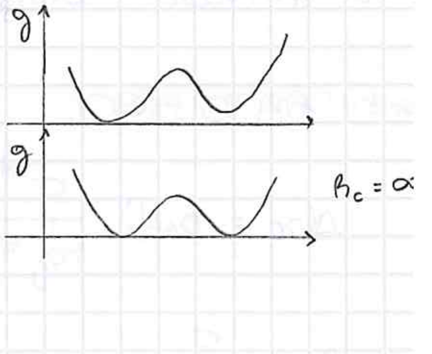
CERCHIAMO R_c:

$$0 = \frac{\partial}{\partial R}\Delta G = (d-1)\sigma R^{d-2} - d \cdot \delta g \cdot R^{d-1} \implies R_c = \frac{d-1}{d} \cdot \frac{\sigma}{\delta g}$$

QUINDI

$R_c \uparrow \quad \sigma \uparrow$

$R_c \uparrow \quad \delta g \downarrow$



LA BARRIERA VALE

$$\Delta G_c = \Delta G(R_c) = a(d) \cdot \frac{\sigma^d}{(\delta g)^{d-1}}$$

$$a(d) = \left(\frac{d-1}{d}\right)^{d-1} - \left(\frac{d-1}{d}\right)^{d}$$

DA CUI SI VEDE CHE

$\Delta G_c \uparrow \qquad \sigma \uparrow$

$\Delta G_c \uparrow \qquad \delta g \downarrow$

$\Delta G_c \downarrow \qquad \delta g \uparrow$

ESSENDO σ LOCALE (~ QUANTA ENERGIA PAGO A FLIPPARE GLI SPIN),

$\sigma < \infty$

QUINDI $\Delta G_c < \infty$ A MENO CHE $\delta g = 0$ (NEL QUAL CASO NON È PIÙ METASTABILITÀ, MA BISTABILITÀ).

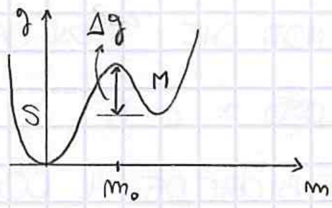
QUINDI, SE $\delta g > 0$, LA BARRIERA ΔG_c È SEMPRE FINITA (ANCHE SE $N \to \infty$)

· ARRHENIUS LAW

$$\tau = \tau_0 \exp\left(\frac{\Delta G_c}{k_B T}\right) \qquad\qquad \Delta G_c \gg k_B T$$

τ_0 SI DICE PREFATTORE CINETICO (LA SCALA MICROSCOPICA DEL SISTEMA).

SE $\Delta G_c < \infty$, $\tau < \infty$.



SI NOTI CHE $\Delta g \sim O(1)$ FA PENSARE $\Delta G_c \sim O(N)$, MA IL FATTO È CHE Δg NON È LA BARRIERA.

IL PUNTO È CHE NON PASSO DAVVERO PER IL MASSIMO m_0. È UNA SORTA DI TUNNELING: UNA FRAZIONE DEL SISTEMA STA IN S E UN'ALTRA IN M, IN MODO CHE LA MEDIA SIA m_0 (UN MODO ANALOGO È LA MEDIA TEMPORALE).

NOTA: OCCHIO, QUESTO τ È PER UNITÀ DI TEMPO E DI VOLUME (È UN RATE). SE IL SISTEMA È PIÙ GRANDE, CHIARAMENTE NUCLEO PRIMA.

* RICAPITOLANDO,

$$\Delta G_c = a(d) \, \frac{\sigma^d}{\delta g^{d-1}}$$

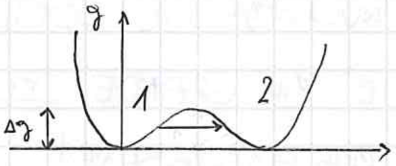
$$R_c = \frac{\sigma}{\delta g}$$

- **BISTABILITÀ** (COESISTENZA): $\delta g = 0$

LO SI TROVA IN

$P = 3, \quad T = T_c, \quad h = 0$

$P = 2, \quad T < T_c, \quad h = 0$



ATTRAVERSANDO UNA TRANSIZIONE DEL I ORDINE.

$\delta g = 0 \rightarrow \Delta G_c = \infty$

- **ESERCIZIO**

SE $N < \infty$, LA BARRIERA È INFINITA?

SE $\delta g = 0$, RIPARTO DA

$\Delta G = \sigma R^{d-1}$

IL CUI MASSIMO È IN L, DOVE

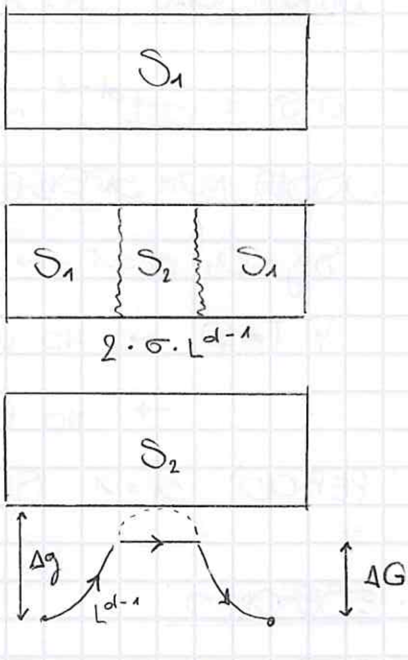
$\Delta G_c = \sigma L^{d-1} = \sigma N^{\frac{d-1}{d}}$

DA CONFRONTARSI CON

$\Delta g^{(TOT)} \sim O(N)$

SE $N \to \infty$, IN EFFETTI LA BARRIERA DIVERGE (ERGODICITY BREAKING).

NOTA: UNA VOLTA CHE SI FORMA UN NUCLEO S_2, NEL DISEGNO NON DE PAGARE PIÙ NULLA PER ESPANDERLO. HO PAGATO QUINDI ΔG_c E NON Δg, CHE CORRISPONDE INVECE AL CASO PIÙ SFAVOREVOLE, IN CUI PAGO PER OGNI SPIN FLIPPATO (SE ASPETTO ABBASTAN IL SISTEMA FA ANCHE QUEL CAMMINO MA INSOMMA...).



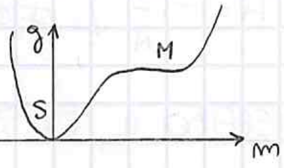
- **SPINODALE**

$\sigma = 0$

$\Delta G_c \sim 0$

CI INDUCE IN ERRORE IL FATTO CHE IN QUESTO PUNTO

"Δg" $= 0$ E $\Delta G_c = 0$

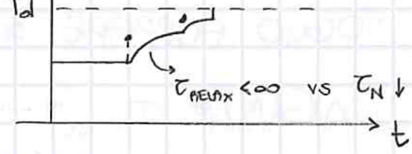


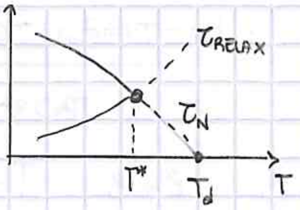
NON OSSERVIAMO MAI LO SPINODALE TERMODINAMICO. SE ALZO LA TEMPERATURA "A SCALINI" FINO A

$T = T_d - \varepsilon$

$\tau_N = \tau_0 \, e^{\frac{\Delta G_c}{k_B T}}$ (TEMPO CHE CI METTO A DECADERE IN S)

È PICCOLO. RISPETTO A COSA?





TEMPO DI RILASSAMENTO (DECORRELAZIONE) E DI NUCLEAZIONE COMPETONO; VERSO T_d, τ_{RELAX} È PIÙ GRANDE DEL TEMPO CHE CI METTE IL SISTEMA PER ANDARSENE!

MA C'È DI PIÙ. A $T=T_d$ LO STATO È PIATTO, QUINDI

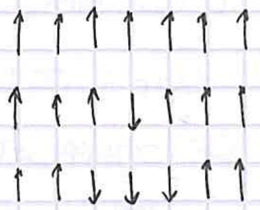
$\chi \uparrow$

ABBIAMO VISTO CHE C'È UN COLLEGAMENTO TRA χ, ξ, τ. QUINDI

$\tau_{RELAX} \uparrow$

SI DEFINISCE PERCIÒ UNO SPINODALE CINETICO (T^*), CHE È CIÒ CHE SI OSSERVA.

· DIMENSIONE $d=1$



$COST \simeq \sigma L^{d-1} \sim \sigma$

OSSIA NON DIPENDE PIÙ DA L. A BASSA T, $\sigma = J$.

NOTA: A T PIÙ ALTE SUBENTRA UN CONTRIBUTO ENTROPICO A σ.

$\delta g = 0,\ d=1 \rightarrow \Delta G_c \sim \sigma \sim O(1)$

$\forall T \neq 0 \rightarrow$ NO LONG RANGE ORDER

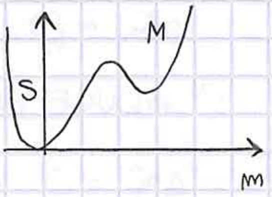
$\rightarrow$ NO PHASE TRANSITION

PERCIÒ $d=1$ SI DICE LOWER CRITICAL DIMENSION.

· ESERCIZIO

MEAN FIELD, ISING $P=2$

$$H = -\frac{J}{N}\sum_{ij}\sigma_i\sigma_j + (\text{CAMPO } h)$$



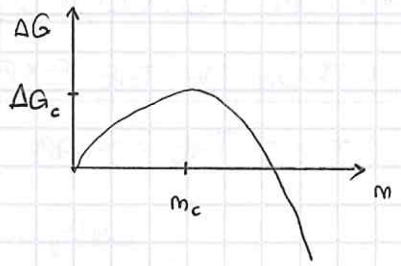
PREPARO IL SISTEMA IN M E MI CHIEDO COME E SE VA NELLO STATO STABILE. ASSUMI σ DOMINATA DALL'ENERGIA (LOW T).

SOLUZIONE

VOGLIO FLIPPARE m SPIN.

$$\Delta G(m) = \sigma \cdot \underbrace{(N-m)\cdot m}_{COUPLINGS} - \delta g \cdot m = (\sigma N - \delta g)m - \sigma m^2$$

POICHÉ (BASSA T, MEAN FIELD)



$$\sigma \sim \frac{J}{N}$$

$$\Delta G(m) \sim (J-\delta g)m - \frac{J}{N}m^2$$

$$m_c = \left(\frac{J-\delta g}{2J}\right)N$$

QUINDI

$$\Delta G_c = \Delta G(m_c) = \frac{(J-\delta g)^2}{2J}N - \frac{J}{N}\frac{(J-\delta g)^2}{4J}N^2 \sim O(N) \qquad \left(=(J-\delta g)^2\frac{N}{4J}\right)$$

E SE $N \to \infty$, $\Delta G_c \to \infty$.

IN CAMPO MEDIO, GLI STATI METASTABILI HANNO VITA MEDIA INFINITA.

PUOI VEDERE CHE, IN ISING, IL PUNTO h_{sp} È QUELLO IN CUI $(\tilde{\sigma} - \delta g) = 0$

- ESERCIZIO (NO CALCOLI)

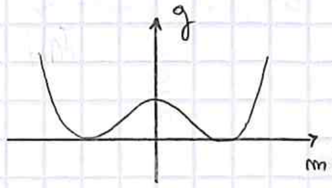
VOGLIO STUDIARE LE PROPRIETÀ DI UNO STATO METASTABILE (e.g. LIQUIDO SOTTORAFFREDDATO): SONO UNO SPERIMENTALE.
DI CHE TAGLIA PRENDO IL MIO SAMPLE?

1) ∞ PICCOLO?
2) ∞ GRANDE?
3) ∞ MEDIO?

(VOGLIO EVITARE LA FASE CRISTALLO, CHE GIÀ CONOSCO).

LESSON 19.03.19

SOLUTION OF THE EXERCISE



WHAT IS THE SHAPE OF THE REAL $g_G(m)$?

$$P(m,h) = \frac{e^{-\beta N(g(m)-hm)}}{\underbrace{\int dm\, e^{-\beta N(g(m)-hm)}}_{\equiv Z}}$$

$$\hat{f}(m,h) = g(m) - h\,m$$

$$f(h) = -\frac{1}{\beta N} \ln \int dm\, e^{-\beta N(g(m)-hm)} = -\frac{1}{\beta N} \ln Z(h)$$

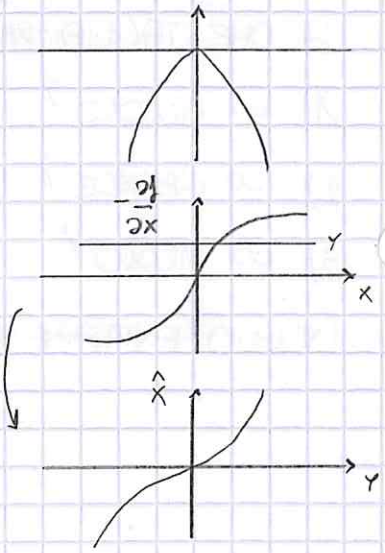
THIS DEFINES $g(m)$. NOW DEFINE m_{eq} AS

$$-\frac{\partial f}{\partial h}(h) = \frac{1}{Z} \int dm\, m\, e^{-\beta N(g(m)-hm)} \equiv m_{eq}(h) \qquad (I)$$

(THIS IS NOT THE MINIMUM OF $\hat{f}$ IF $N<\infty$).

NOTE: IF $N<\infty$, A FINITE FRACTION OF THE SYSTEM IS IN THE OTHER MINIMUM AS WELL.

ABOUT LEGENDRE



$$f(x) \xrightarrow{LT} g(y) = f(\hat{x}(y)) + \hat{x}(y) \cdot y$$

WHERE THE FUNCTION $\hat{x}(y)$ IS DEFINED VIA

$$-\frac{\partial f}{\partial x}(\hat{x}(y)) = y$$

INVERTING SO AS TO WRITE

$$f(\hat{x}(y)) = g(y) - \hat{x}(y) \cdot y$$

IS POSSIBLE IF $\hat{x}(y)$ IS MONOTONIC; THEN WE DEFINE $\hat{y}(x)$ AS

$$\hat{x}(\hat{y}(x)) = x \qquad \hat{y}(\hat{x}(y)) = y$$

HENCE

$$f(x) = g(\hat{y}(x)) - x \cdot \hat{y}(x)$$

IN FACT

$$\frac{\partial g}{\partial y}\left(\hat{y}(x)\right) = \frac{\partial f}{\partial x}\left(\hat{x}(\hat{y}(x))\right) \cdot \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{x}}{\partial y} \cdot \hat{y}(x) + \hat{x}(\hat{y}(x))$$

$$= -\hat{y}(x) \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{x}}{\partial y} \hat{y}(x) + X \quad = \quad X$$

AND WE RECOVER

$$\frac{\partial g}{\partial y}\left(\hat{y}(x)\right) = X$$

WHERE $\hat{x}(y)$, $\hat{y}(x)$ ARE THE INVERSE OF EACH OTHER.

* LET'S APPLY THIS MACHINERY TO OUR PROBLEM.

$$f(h) \rightarrow \quad g_G(m) \equiv f(h_{eq}(m)) + h_{eq}(m) \cdot m$$

WITH

$$-\frac{\partial f}{\partial h}\left(h_{eq}(m)\right) = m$$

WHICH DEFINES $h_{eq}(m)$. THEN

$$f(h) = g_G(m_{eq}(h)) - h \cdot m_{eq}(h)$$

$$\frac{\partial g_G}{\partial m}\left(m_{eq}(h)\right) = h \qquad (II$$

WHICH DEFINES $m_{eq}(h)$, AND $h_{eq}(m)$ AND $m_{eq}(h)$ ARE INVERSE OF ONE ANOTHER,

$$m_{eq}(h_{eq}(m)) = m \qquad\qquad h_{eq}(m_{eq}(h)) = h$$

SO THAT IN FACT

$$-\frac{\partial f}{\partial h}\left(h_{eq}(m_{eq})\right) = m_{eq}(h)$$

NOTE: $-\frac{\partial f}{\partial h}\left(h_{eq}(m_{eq}(h))\right) = m_{eq}(h)$. (III

THIS IS UTTERLY IMPORTANT: IT IS THIS, AND NOT (I), THE RELATION WE'LL USE TO CALCULATE $m_{eq}(h)$.

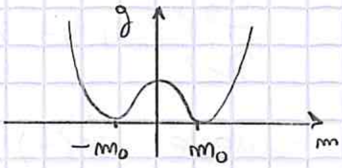
VIA (II), WE SEE THAT

$$\frac{\partial g_G}{\partial m}(m) = h_{eq}(m)$$

SO WE CAN GET $g_G(m)$ BY INTEGRATING $h_{eq}(m)$, WHICH IN TURN IS THE INVERSE OF $m_{eq}(h)$; BUT CALCULATING $m_{eq}(h)$ INVOLVES, VIA (I)/(III), THE FUNCTION $g(m)$.

1) COMPUTE $m_{eq}(h)$.

WE ASSUME AS LITTLE AS POSSIBLE FOR $g(m)$:



$$g'(\pm m_0) = 0$$

$$g''(\pm m_0) \neq 0$$

* IF $h = 0$,

$$m_{eq} = \frac{e^{-\beta N g(m_0)} m_0 + (-m_0)\, e^{-\beta N g(-m_0)}}{e^{-\beta N g(m_0)} + e^{-\beta N g(-m_0)}} = 0$$

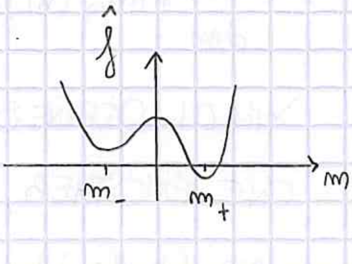
NOTE: IT'S A SADDLE POINT,

$$\int dx\, f(x) e^{-N g(x)} = \sqrt{\frac{2\pi}{N|g''(x_0)|}} \cdot f(x_0) e^{-N g(x_0)} \left(1 + O\left(\frac{1}{N}\right)\right)$$

WHICH WE COULD HAVE ARGUED BY SYMMETRY. IF WE CHOSE m_1, m_2 INSTEAD OF $\pm m_0$, WE WOULD FIND

$$m_{eq} = \frac{m_1 + m_2}{2}$$

* IF $h = O(\varepsilon)$, i.e. SMALL, BUT NOT DEPENDENT ON N (WHEN $N \to \infty$), WE STUDY $\hat{f}$. ITS MINIMA ARE WHERE



$$g'(m_\pm) = h$$

TO FIND THEM, WE EXPAND g AROUND $\pm m_0$.

SETTING $g(\pm m_0) = 0$ W.L.O.G.,

$$g(m_\pm) = \frac{1}{2}(m_\pm \mp m_0)^2 g''(m_0)$$

NOTE: ACTUALLY $g''(\pm m_0) = g''(m_0)$ IF $h \neq 0$, $\hat{f}$ CHANGES ITS SHAPE, BUT g DOES NOT.

$$g'(m_\pm) = (m_\pm \mp m_0)\, g''(m_0) \equiv h$$

$$\rightarrow m_\pm = \pm m_0 + \frac{1}{g''(m_0)} \cdot h = \pm m_0 + O(h)$$

NOTICE INSTEAD

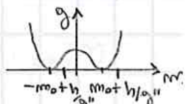
$$g(m_+) = \frac{1}{2}(m_+ - m_0)^2 g''(m_0) = \frac{h^2}{2(g'')^2} g'' = O(h^2)$$

SO WE'LL ASSUME FOR SIMPLICITY

$$g(m_+) \simeq g(m_-)$$

NOTE: IN THE PARABOLIC APPROXIMATION THIS IS EXACT

$$g(m_+) = g(m_-) = \frac{h^2}{2g''(m_0)}$$



(YOU CAN CARRY ON WITHOUT THIS ASSUMPTION, THE CONCLUSIONS DON'T CHANGE). THEN

$$m_{eq}(h) = \frac{m_+ e^{-\beta N g(m_+)} e^{\beta N h m_+} - |m_-| e^{-\beta N g(m_-)} e^{-\beta N h |m_-|}}{e^{-\beta N g(m_+)} e^{\beta N h m_+} + e^{-\beta N g(m_-)} e^{-\beta N h |m_-|}}$$

$$\equiv m_+ P_+ - |m_-| P_-$$

WHERE WE SET

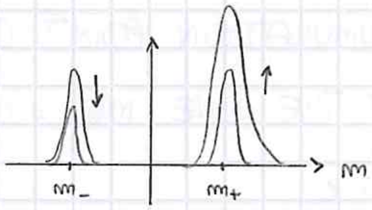
$$P_+ = \frac{1}{\hat{Z}} e^{\beta N h m_+} \qquad P_- = \frac{1}{\hat{Z}} e^{-\beta N h |m_-|}$$

$$\hat{Z} = e^{\beta N h m_+} + e^{-\beta N h |m_-|}$$

NOTICE

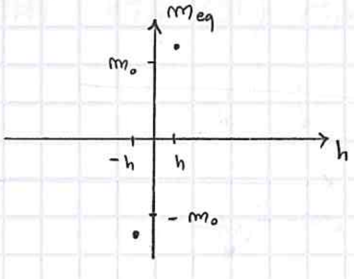
$P_+ \to 1$, $N \to \infty$

$P_- \to 0$, $N \to \infty$



SO AS $N \to \infty$

$$m_{eq}(h) = m_+(h) \simeq m_0 + \frac{h}{g''(m_0)}$$



* IF $h \to 0$ AND $N \to \infty$,

$$\frac{1}{N} \ll h \ll 1 \qquad (h > 0)$$

$$hN \to \infty$$

THIS MEANS $h \to 0$ AFTER $N \to \infty$, SO THAT THEIR PRODUCT IS ALWAYS LARGE.

LOOKING AT $P_\pm$, WE SEE $Nh \to \infty$ AND THE CONCLUSION IS

THE SAME AS BEFORE (EVEN IF $h \to 0$) :

$$\rho_+ \to 1$$

$$\rho_- \to 0$$

THUS

$$m_{eq}(h) = m_+ \rho_+ + m_- \rho_- \xrightarrow[N\to\infty]{} m_+ = m_0 + \frac{h}{g''(m_0)} \overset{h\to 0}{\simeq} m_0$$

BECAUSE NOW $h \to 0$.

* IN THE GENERAL CASE,

$$m_{eq}(h) = m_+ \rho_+ + m_- \rho_- = \left[\left(m_0 + \frac{h}{g''}\right) e^{\beta m_+ h N} + \left(-m_0 + \frac{h}{g''}\right) e^{-\beta |m_-| h N}\right] \frac{1}{Z}$$

$$= m_0 \,\text{TANH}\left(\beta m_0 h N\right) + \frac{h}{g''(m_0)} + O\left(\frac{1}{N}; h^2\right)$$

i.e. A FINITE VALUE OF m EVEN IN ZERO FIELD: THIS IS S.S.B..

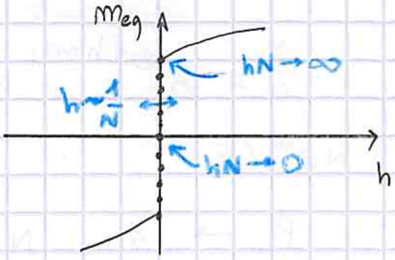
* IF INSTEAD $h \to 0$ BEFORE $N \to \infty$, SO THAT

$$hN \to 0$$

THE ACCUMULATION POINT CHANGES.

WE VISIT THE LINE $m_{eq} = 0$ IF hN IS FINITE, i.e.

$$h \sim \frac{1}{N} \qquad\qquad hN \sim O(1)$$



TO SEE THIS (TO VISIT THE COEXISTENCE LINE), TAKE

$$h = \frac{\mathscr{h}}{N} \qquad\qquad \mathscr{h} = O(1)$$

THEN

$$m_{eq} = m_0 \,\text{TANH}\left(\beta m_0 \mathscr{h}\right) + \underbrace{\frac{h}{g''(m_0)}}_{=0}$$

WHILE IN GENERAL THE SLOPE IS $O(N)$:

$$\frac{\partial}{\partial h} \text{TANH}(\beta m_0 h N) \sim \left.\frac{\beta m_0 N}{\text{COSH}^2(\beta m_0 h N)}\right|_{h=0} \sim N$$

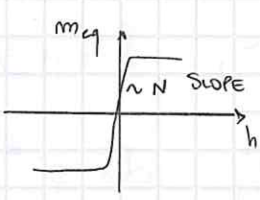
★ TO SUM UP,

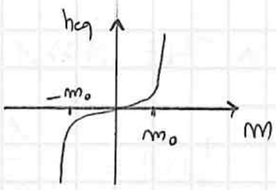
$m_{eq}(h) \rightarrow m_0$ $\quad N \rightarrow \infty,\ h \rightarrow 0,\ hN \rightarrow \infty$

$m_{eq}(h) = 0$ $\quad h \rightarrow 0,\ N \rightarrow \infty,\ hN \rightarrow 0$

$m_{eq}(h) = m_0 \, \mathrm{TANH}(\beta m_0 h)$ $\quad h = \frac{h}{N},\ hN \sim 1$

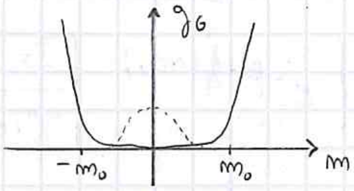
2) INVERT





3) INTEGRATE

$$g_G(m) = \int_0^m dm' \, h_{eq}(m')$$

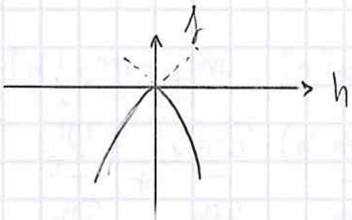


THE NICE THING IS YOU DON'T REALLY NEED TO BE IN THE THERMODYNAMIC LIMIT TO SEE THIS: IT WORKS EXACTLY IN THE SAME WAY IF

$N \gg 1, \quad h \ll 1$

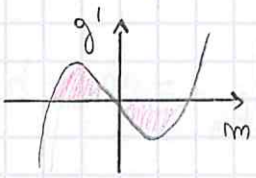
i.e. IN REAL EXPERIMENTS OR SIMULATIONS.

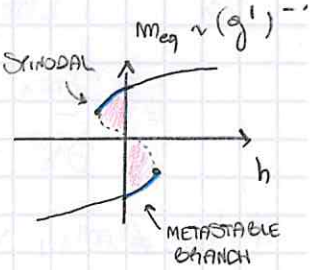
★ WHAT HAPPENS TO $f(h)$?



★ WE COULD EVEN DRAW $g'(m)$ AND ITS INVERSE, HENCE LOCATE THE SPINODAL.

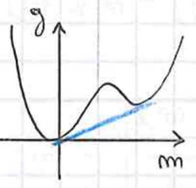
NOTE: RECALL $\frac{\partial g_G}{\partial m}(m) = h_{eq}(m)$.

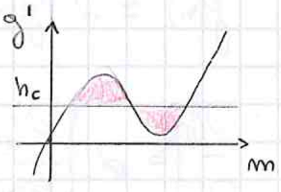




· EXERCISE (MAXWELL'S CONSTRUCTION)

PROVE THAT THE COEXISTENCE VALUE h_c IS S.t. THE TWO AREAS ARE THE SAME.





NOTE: h_c s.t. $g - h_c m$ IS BISTABLE.

· ALTERNATIVE SOLUTION (FEDERICA)

$$\hat{f}(h,m) = g(m) - hm$$

$$e^{-\beta N f(h)} = \int dm \, e^{-N\beta[g(m)-hm]}$$

$$f(h) = -\frac{1}{\beta N} \ln \int dm \, e^{-\beta N \underbrace{[g(m)-hm]}_{\hat{f}(h,m)}}$$

IN THE $N \gg 1$ LIMIT, IF $\hat{f}$ HAS 2 MINIMA m_1, m_2

$$f(h) \simeq -\frac{1}{\beta N} \ln\left\{ \int_{I(m_1)} dm \, e^{-\beta N[\hat{f}(m_1)+g''(m_1)(m-m_1)^2]} + \int_{I(m_2)} dm \, e^{-\beta N[\hat{f}(m_2)+g''(m_2)(m-m_2}\right.$$

$$\simeq -\frac{1}{\beta N} \ln\left\{ e^{-\beta N\hat{f}(m_1)} \sqrt{\frac{2\pi}{g''(m_1)\beta N}} + e^{-\beta N \hat{f}(m_2)} \sqrt{\frac{2\pi}{g''(m_2)\beta N}} \right\}$$

$$\simeq \hat{f}(m_1) + O\left(\frac{\ln N}{N}\right) + (?)$$

HOW DO I GROUP $\hat{f}(m_1), \hat{f}(m_2)$?

DEFINE THE CONVEX COMBINATION

NOTE: I THINK IT'S LIKE SAYING THE REAL AVG LIES SOMEWHERE IN BETWEEN.

$$\hat{f} \equiv \alpha \hat{f}(m_1) + (1-\alpha)\hat{f}(m_2) \qquad \alpha \in [0,1]$$

BUT $m_1 = m_1(h)$, $m_2 = m_2(h)$. IF $\hat{f}$ HAD A SINGLE MINIMUM, WE WOULD HAVE

$$\frac{\partial f}{\partial h} = \frac{\partial \hat{f}(m,h)}{\partial m} \frac{\partial m_1}{\partial h} + \frac{\partial \hat{f}}{\partial h}$$

$$= \frac{\partial m_1}{\partial h} \frac{\partial g}{\partial m}\bigg|_{h_{eq}} - m_1 - h \frac{\partial m_1}{\partial h}\bigg|_{h_{eq}} = -m_1$$

IF $m_1 \neq m_2$, WE FIND INSTEAD (USING LAPLACE'S METHOD)

$$f(h) \simeq \alpha\, g(m_1(h)) - \alpha h m_1(h) + (1-\alpha) g(m_2(h)) - (1-\alpha) h\, m_2(h)$$

$$\frac{\partial f}{\partial h}\bigg|_{h_{eq}} = \alpha \left\{ \frac{\partial g}{\partial m} \cdot \frac{\partial m_1}{\partial h}\bigg|_{h_{eq}} - m_1 - h \frac{\partial m_1}{\partial h} \right\} + \ldots = -[\alpha m_1 + (1-\alpha) m_2] \equiv -m_{eq}(h)$$

NOW WE CAN COMPUTE

$$g_G(m) = \alpha\left[\hat{f}(m_{eq}(h_{eq}(m_1))) + h_{eq}(m)\underbrace{m(h_{eq}(m_1))}_{m_1}\right] + (1-\alpha)\left[\hat{f}(m_{eq}(h_{eq}(m_2)) + h\underbrace{m(h_{eq}(m}_{m_2}\right.$$

$$= \alpha\, g(m_{eq}(h_{eq}(m_1))) + (1-\alpha)\, g(m_{eq}(h_{eq}(m_2)))$$

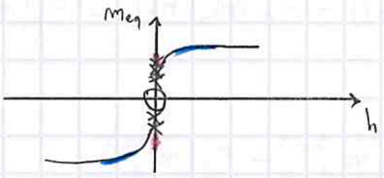
WHICH IS A CONVEX HULL.

- LESSON 22.03.19
- SUMMARY OF LAST TIME

$$m_{eq} = P_+ m_+ + P_- m_-$$

$$P_\pm = \frac{1}{\hat{2}} e^{\pm\beta N h |m_\pm|} \qquad h>0$$

$$m^{(\pm)} = \pm m_0 + \frac{h}{g''(m_0)}$$



IF h IS INDEPENDENT OF N, •

$P_+ \to 1$, $P_- \to 0$

$$m_{eq} \to m_+ = m_0 + \frac{h}{g''}$$

IF $h \to 0$, $N \to \infty$, $hN \to \infty$, •

$P_+ \to 1$, $P_- \to 0$

$$m_{eq} \to m_+ = m_0 + \frac{h}{g''} \to m_0$$

IF $h \to 0$, $N \to \infty$, $hN \sim 1$, ×

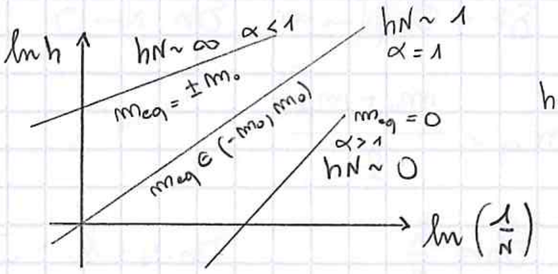
$h = \frac{\underline{h}}{N}$, $P_+ \sim 1$, $P_- \sim 1$

$$m_{eq} = P_+ m_0 - P_- m_0 = m_0 \text{TANH}(\beta m_0 h$$

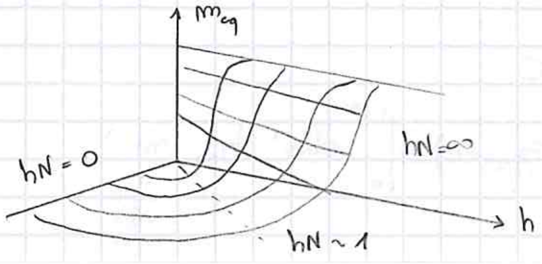
IF $h \to 0$, $N \to \infty$, $hN \to 0$, ⊙

$P_+ \to \frac{1}{2}$, $P_- \to \frac{1}{2}$

$$m_{eq} = 0$$

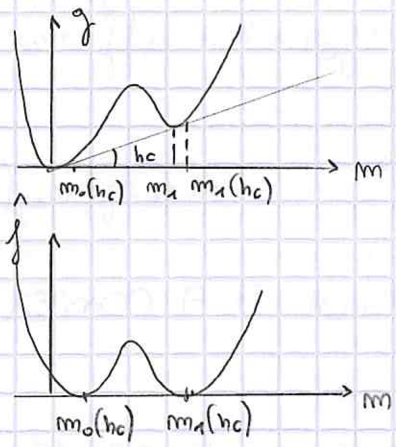


$$h \sim \frac{1}{N^\alpha}$$



- MAXWELL'S CONSTRUCTION (ISING $P=3$, $T_c < T < T_d$)

$$g'(m_0) = g'(m_1)$$
$$g(m_0) < g(m_1)$$



WE WANT TO CONSTRUCT

$$\hat{f}(m,h) = g(m) - hm$$

AND FIND h_c SUCH THAT

$$\hat{f}'(m_0(h_c)) = 0 = \hat{f}'(m_1(h_c))$$
$$g'(m_0(h_c)) = h_c$$
$$g'(m_1(h_c)) = h_c$$

THE COEXISTENCE CONDITION IS

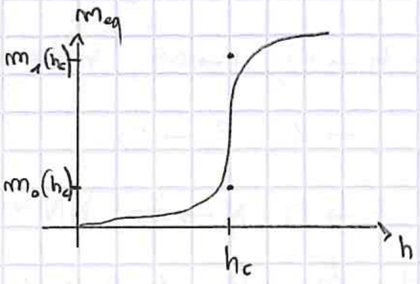
$$\hat{f}(m_0(h_c)) = \hat{f}(m_1(h_c))$$
$$g(m_0(h_c)) - h_c\, m_0(h_c) = g(m_1(h_c)) - h_c\, m_1(h_c)$$

HENCE

$$h_c = \frac{g(m_1^{h_c}) - g(m_0^{h_c})}{m_1^{h_c} - m_0^{h_c}}$$

1) $h = h_c$

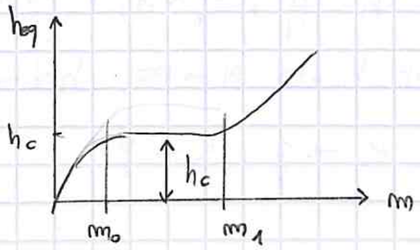
$$m_{eq} = \frac{m_0 + m_1}{2}$$



2) $h = h_c + \delta h$

$\delta h \to 0$, $N \to \infty$, $\delta h \cdot N \to \infty$

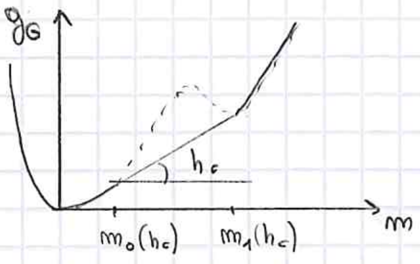
$$m_{eq} \to m_1(h_c)$$



3) $\delta h \to 0$, $N \to \infty$, $\delta h \cdot N \to 0$

$$m_{eq} = \frac{m_0 + m_1}{2}$$

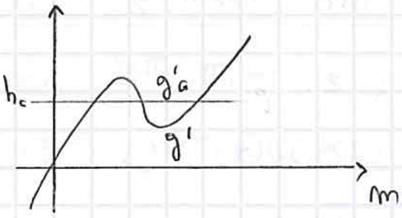
4) $\delta h = \frac{h}{N}$ $\qquad \delta h \cdot N \sim h \sim O(1)$



So

$$g_G(m) = \int_0^m dm'\, h_{eq}(m')$$

BY LOOKING AT THE DERIVATIVE,



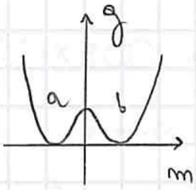
$$0 = \int_{m_0(h_c)}^{m_1(h_c)} dm \left(g'(m) - h_c \right)$$

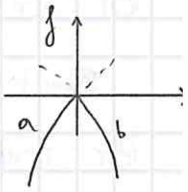
$$= \left[g(m_1(h_c)) - g(m_0(h_c)) \right] - h_c \left(m_1(h_c) - m_0(h_c) \right)$$

NOTE: AGAIN, $\frac{\partial g_G}{\partial m}(m) = h_{eq}(m)$

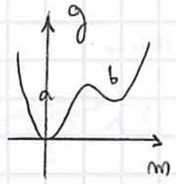
SOME OTHER EXAMPLES:

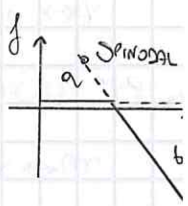
- ISING $p = 2$, $T < T_c$





- ISING $p = 3$, $T < T_d$



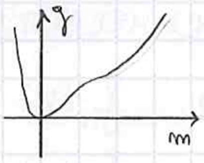


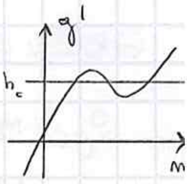
* GIVEN A CERTAIN $g(m)$, IS IT ALWAYS POSSIBLE TO FORM A METASTABLE STABLE BY ADDING AN EXTERNAL FIELD?

IT DEPENDS ON WHETER $g(m)$ HAS FLEXES OR NOT.

LOOKING AT g', WE SEE $\hat{f}$ MAY ADMIT 2 MINIMA IF

$$\frac{\partial \hat{f}}{\partial m} = 0 \quad \Rightarrow \quad \frac{\partial g}{\partial m} = h \quad \text{HAS MORE THAN A ROOT}$$

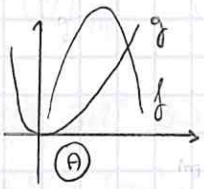


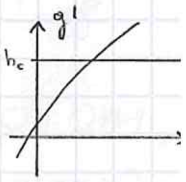


SO ONLY IF g' IS NOT MONOTONIC.

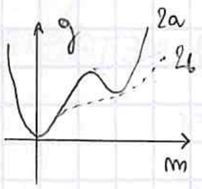
IF g' IS MONOTONIC, f LOOKS LIKE Ⓐ

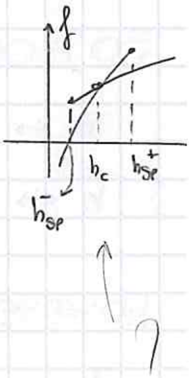
AND THERE'S NO METASTABILITY.





THERE DOESN'T NEED TO BE A SECOND MINIMUM: THERE HAS TO BE A REGION OF NONCONVEXITY (BOTH 2a, 2b ARE OK; THEY GIVE AN f WITH TWO BRANCHES).





· THE STRANGE CONNECTION WITH THE PISTON

(i.e. WITH THE LIQUID - VAPOUR TRANSITION)

CONJUGATED FIELDS ↔ LEGENDRE PAIRS

m, h

V, P

BY USING COEXISTENCE, WE'RE TEMPTED TO MATCH

$P \leftrightarrow h, \quad m \leftrightarrow V$

BY USING THE CONTROL PARAMETER, INSTEAD,

$V \leftrightarrow h, \quad m \leftrightarrow P$

BY USING EXTENSIVITY,

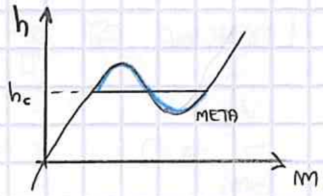
$m \leftrightarrow V, \quad P \leftrightarrow h$

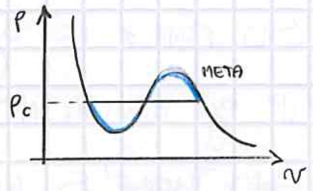
THE CONTROL PARAMETER IS THE ONE THAT DEFINES THE ENERGY LEVEL IN THE HAMILTONIAN (h, $V \leftrightarrow$ SIZE OF THE BOX).

GIBBS AND HELMOTZ THEMSELVES DEFINED

$g_G(m), \quad f_H(h)$

$g_G(P), \quad f_H(v)$





AND NOTICE THE SIMILARITY BETWEEN THE TWO GRAPHS $h(m)$, $P(v)$ $\left(v = \frac{V}{N}\right)$.

SO FROM METASTABILITY (COEXISTENCE),

$m \leftrightarrow v$

$h \leftrightarrow P$

HOWEVER,

$$f(x) = -\frac{1}{\beta N} \ln Z = -\frac{1}{\beta N} \ln \int D\sigma \, e^{-\beta H(x)}$$

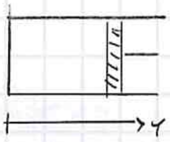
SO IT DEPENDS ON THE EXTERNAL CONTROL PARAMETER, WHICH SUGGESTS

$h \leftrightarrow v$

$m \leftrightarrow P$

ALBEIT CONFUSING, THE RIGHT VIEW IS THE FIRST.

* HOW STRONG IS THE EXTERNAL/INTERNAL VS CONTROL CORRESPONDENCE?



1) CONTROL V, MEASURE P

→ CONTROL V, MEASURE FORCE

2) CONTROL P, MEASURE V

→ FEEDBACK: YOU TUNE THE FORCE, IN ORDER TO GET THE DESIRED P

MATEMATHICALLY, THEY CORRESPOND TO

① → $f_H(V) = -\frac{1}{\beta N} \ln \int Dx \, e^{-\beta H_V(x)}$

BY ADDING PV,

② → $g_G(P) = \min_{V} \left\{ -\frac{1}{\beta N} \ln \int Dx \, e^{-\beta H_V(x) + \beta PV} \right\}$

SO V ACTS AS A LAGRANGE MULTIPLIER.

NOTE: WE WANT TO OBTAIN A CONSTANT P, IT'S OUR CONSTRAINT

IN THE MAGNETIC CASE,

1) CONTROL h, MEASURE M

2) CONTROL m, MEASURE h

→ WHAT IS THE VALUE OF h THAT I NEEDED TO GET THIS m?

① → $f_H(h) = -\frac{1}{\beta N} \ln \int D\sigma \, e^{-\beta H(h)}$

NOTA: TRY! YOU GET
$g_G(m) = \min_{h} \left\{ f_H(h) - \underset{\sim}{\frac{1}{\beta}} hm \right\}$

② → $g_G(m) = \min_{h} \left\{ -\frac{1}{\beta N} \ln \int D\sigma \, e^{-\beta H + hM} \right\}$

MULTIPLIER

AND THIS FEEDBACK IS NOTHING ELSE THAN THE LEGENDRE TRANSFORM

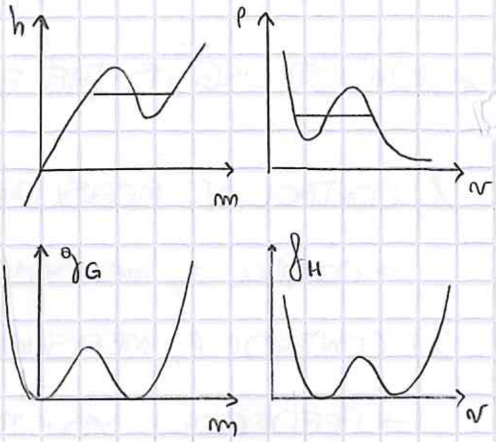
THE SECOND CORRESPONDENCE IS THUS FUZZY, BECAUSE IT RELIES ON THE WAY THE EXPERIMENT IS PERFORMED.

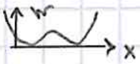
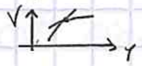
THE ONLY REAL CORRESPONDENCE IS

$h \leftrightarrow p$ INTENSIVE

$m \leftrightarrow v$ EXTENSIVE

(M) (V)



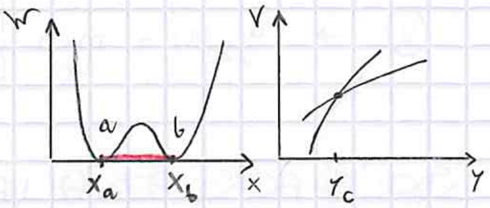
	EXTENSIVE	INTENSIVE
INTERNAL	$g_G(m)$	$g_G(p)$
EXTERNAL	$f_H(v)$	$f_H(h)$
		

→ IF YOU HAVE COEXISTENCE, THAT POTENTIAL IS EXTENSIVE.

· <u>HOMEWORK</u>:

1) PROVE THAT, IF $W(x)$ HAS A DOUBLE WELL AT $Y = Y_c$, THEN X IS EXTENSIVE.

$$x = \frac{X}{N}$$



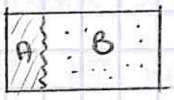
STANDARD EXPLANATION: IF I HAVE TWO PHASES IN CONTACT, THAN THE INTENSIVE QUANTITY MUST BE THE SAME.
BUT TRY TO ELABORATE ON THAT. NOTICE THAT BOTH W AND V ARE EXTENSIVE (FREE ENERGIES ALWAYS ARE). RECALL THE RED ONE IS THE "REAL POTENTIAL".

IN GENERAL, THE <u>ORDER PARAMETER</u> IS EXTENSIVE.

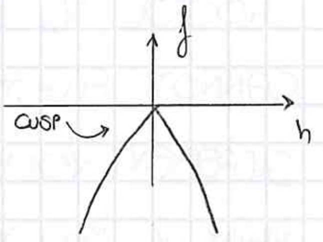
2) IN FERROMAGNETS (ISING), YOU HAVE FLIP-FLOPS AS

$$\Delta G \sim L^{d-1}$$

IN LIQUID-VAPOUR SYSTEMS YOU NEVER SEE THEM, BUT YOU SEE PHASE SEPARATION (ABSENT IN FERROMAGNETS!): HOW IS THAT?



3) IN A 2^{ND} ORDER P.T., χ DIVERGES, WHILE IT DOESN'T IN A 1^{ST} ORDER ONE... RIGHT?



$$\chi = \frac{\partial^2 g_H}{\partial h^2} = \left(\frac{\partial^2 g_G}{\partial m^2} \right)^{-1} \overset{?}{=} \infty$$

BUT THE REAL $g_G(m)$ IS FUCKING FLAT!

TO SUMMARIZE:

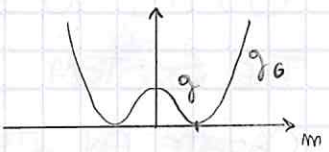
1) PROVE THAT, IF $W(x)$ IS DOUBLE-WELLED AT Y_c, THEN X IS EXTENSIVE AND Y IS INTENSIVE.

2) IF FERROMAGNETS AND PISTON ARE THE SAME, THEN WHY DO WE ALWAYS SEE FLIP-FLOP IN THE FORMER, PHASE SEPARATION IN THE LATTER, AND NEVER VICE-VERSA?

3) IS $\chi = \infty$, OR $\chi = 1$ AT A FIRST ORDER P.T.?

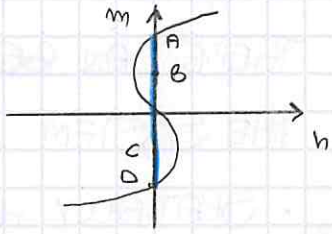
LESSON 26.03.19

PHASE SEPARATION VS FLIP-FLOP



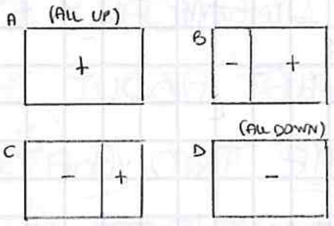
AT $h=0$, THERE'S A BARRIER

$$\Delta G_c = \sigma L^{d-1}$$

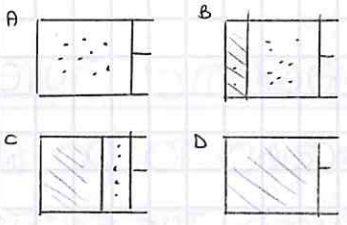


ONCE YOU CREATE THE INTERFACE, IT CAN EXTEND ACROSS THE WHOLE SYSTEM: WE CALL IT PHASE SEPARATION (B, C).

YOU BASICALLY WALK UP THE BLUE LINE.



SIMILARLY IN THE VAPOUR-LIQUID CASE (NEGLECTING GRAVITY); BUT WHY ARE WE TALKING OF PHASE-SEPARATION FOR BOTH SYSTEMS? ACTUALLY, WE'LL SEE THE BEHAVIOUR IS DIFFERENT DEPENDING ON THE WAY THE EXPERIMENT IS PERFORMED.



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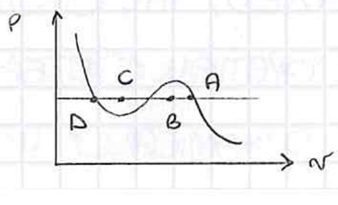
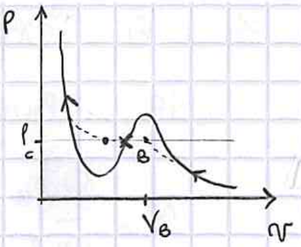


(*) NOTE: I THINK THE TOTAL VOLUME OUGHTS TO VARY.

* <u>LIQUID-VAPOUR: CONTROL V, MEASURE P</u>

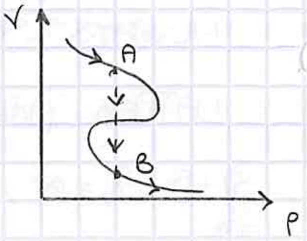


CHOOSE V_B (THE EXTENSIVE VALUE): THIS CANNOT FLUCTUATE. I CAN ONLY COMBINE DIFFERENT VOLUMES OF THE TWO PHASES. AS YOU VARY V, YOU MEASURE P; IN PRACTICE, YOU FIND

$$P = P_c \pm O\left(\frac{1}{N}\right)$$

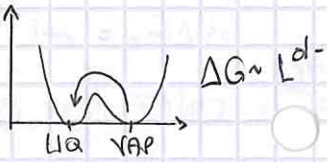
DYNAMICALLY, THE SYSTEM IS HOMOGENEOUS (IT DOESN'T CHANGE IN TIME) AND SPATIALLY ETEROGENEOUS (PHASE SEPARATION).

* <u>L-V: CONTROL P, MEASURE V</u>



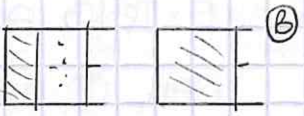
SIMULATING THIS IS HARDER THAN JUST FIXING THE VOLUME, BUT YOU CAN DO IT THROUGH SOME FEEDBACK (YOU CHANGE V SO AS TO GET THE DESIRED P).

THE SYSTEM CAN STAY SPATIALLY HOMOGENEOUS: NO NEED TO CREATE AN INTERFACE. HOWEVER, NUCLEATION THEORY TELLS US THERE CAN BE FLUCTUATIONS THAT CREATE ONE.



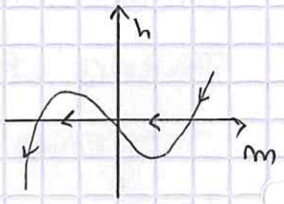
THE SYSTEM IS

- SPATIALLY HOMOGENEOUS
- DYNAMICALLY ETEROGENEOUS → FLIP-FLOP



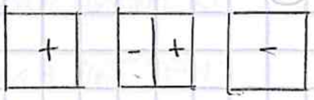
WHAT ABOUT THE LATENT HEAT? WE'RE IN A HEAT BATH. THE TWO PHASES HAVE DIFFERENT E AND S, AND THE TWO COMPENSATE (SEE PROBLEM LATER).

* <u>MAGNETIC: TUNE m, MEASURE h</u>

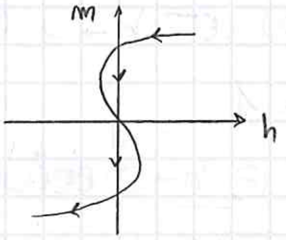


WEIRD TO DO IN A SIMULATION. ONCE m IS FIXED, THE SYSTEM CANNOT FLIP-FLOP AND GETS

- SPATIALLY ETEROGENEOUS
- DYNAMICALLY HOMOGENEOUS



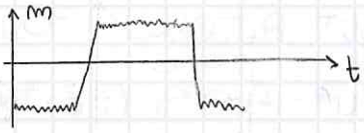
★ MAGNETIC: TUNE h, MEASURE m



YOU GET THE USUAL FLIP-FLOPS:

- SPATIALLY HOMOGENEOUS
- DYNAMICALLY ETEROGENEOUS

TO SUM UP, WHEN THE VARIABLE YOU TUNE IS

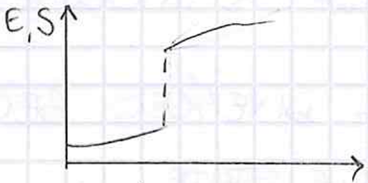


- INTENSIVE → FLIP-FLOP → { SPATIALLY HOMOGENEOUS / DYNAMICALLY ETEROGENEOUS
- EXTENSIVE → PHASE SEP. → { SPATIALLY ETEROGENEOUS / DYNAMICALLY HOMOGENEOUS

WHAT IS THE ROLE OF TIME? IN THE FIRST CASE, FOR LARGE ENOUGH L YOU'LL NEVER SEE FLIP-FLOPS IF t IS SMALL.

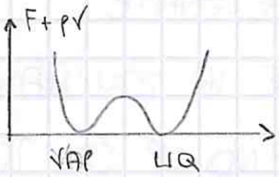
· HOMEWORK

IN THE LIQUID-VAPOUR CASE, 'E' AND 'S' ARE DIFFERENT IN THE TWO PHASES (≠ MAGNETS).



WHAT HAPPENS DURING THIS SPONTANEOUS TRANSITION IN TERMS OF

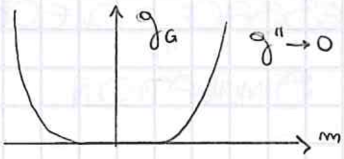
$\Delta E, \Delta S$, LATENT HEAT, $P\Delta V$?



TIP: USE 1ST AND 2ND PRINCIPLE.

· SUSCEPTIBILITY IN A 1ST ORDER P.T.

$$\chi = \frac{\partial m_{eq}}{\partial h}(h_{eq}) = -\frac{\partial^2 f}{\partial h^2} = \left(\frac{\partial^2 g_G}{\partial m^2}\right)^{-1}$$



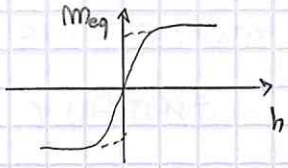
WE FEEL IT SHOULD BE FINITE: IT'S THE BIG DIFFERENCE WITH 2ND ORDER P-Ts.

THE SOLUTION TO THIS CONUNDRUM IS 2-FOLD.

FIRST, FROM WHICH SIDE ARE WE TAKING THE LIMIT?

IF YOU LET $h \to 0$ AFTER $N \to \infty$, BECAUSE OF SYMMETRY

$\chi \sim 1$ $\qquad Nh \to \infty$

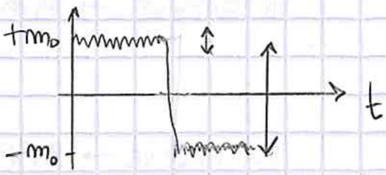


BUT IF $h \to 0$ BEFORE $N \to \infty$,

$\chi \sim \infty$ $\qquad Nh \to 0$

$m_{eq} = 0$

BUT ACTUALLY YOU GET $m_{eq} = 0$ (AT $h=0$) IN A SIMULATION THROUGH FLIP-FLOPS (i.e. TIME AVERAGE). RECALL



$$\chi = \frac{\partial m_{eq}}{\partial h} = \beta N \left(\langle m^2 \rangle - \langle m \rangle^2 \right)$$

AND THERE ARE ACTUALLY 2 SCALES OF FLUCTUATIONS: FROM $-m_0$ TO $+m_0$ ($\chi_{OUT} \sim O(1) \cdot N$) AND WITHIN A SINGLE STATE ($\chi_{IN} \sim O(1)$). WE COULD SAY

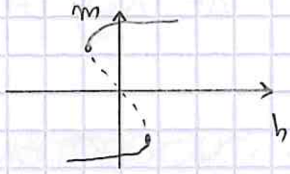
NOTE: $\langle \delta M^2 \rangle_{IN} \sim O(N)$, $\langle \delta m^2 \rangle_{IN} \sim O(\frac{1}{N}$

- χ_{IN} : HOMOGENEOUS SUSCEPTIBILITY, $\chi_{IN} \sim O(1)$, SINGLE STATE
- χ_{OUT} : ETEROGENEOUS SUSCEPTIBILITY, $\chi_{OUT} \sim O(N)$, ALL STATES

χ_{OUT} DIVERGES BECAUSE YOU WEREN'T ABLE TO SELECT A SINGLE STATE.

NOTICE IF YOU HAVE A LARGE SYSTEM AND CHANGE h CONTINUOUSLY, YOU END UP ON THE METASTABLE BRANCH UP TO $h_{SPINODAL}$; IN GOING THROUGH $h=0$ YOU SEE NO DIVERGENCE.

IN $h = h_{SPIN}$ YOU COLLAPSE ONTO $h=0$.



MESSAGE: CHECK WHAT'S GOING ON DURING A SIMULATION.

NOTICE (ARRHENIUS)

$$\tau_{FLIP\text{-}FLOP} \sim e^{\beta \sigma L^{d-1}} \simeq 1000 \text{ MCS} \quad \text{(MONTE CARLO STEPS)}$$

SO YOU GET RANDOM NUMBERS FOR $t_{OBS} \overset{?}{\approx} \tau_{FF}$; YOU HAVE TO CHECK* IF

- $t_{OBS} \ll \tau_{FF} \quad \to \quad m_{eq} = +m_0, \ \chi = 1$ SYMMETRY BREAKING
- $t_{OBS} \gg \tau_{FF} \quad \to \quad m_{eq} = 0, \ \chi \gg 1$ ERGODIC

*NOTE: i.e. COUNT THE NUMBER JUMPS DURING THE SIMULATION. IT' NOT NECESSARILY TRUE THAT THE LONGER, THE BETTER.

* IF YOU'RE NOT AT COEXISTENCE,

$$P(m) \sim \frac{1}{Z} e^{-\beta N \Delta g}$$

BUT YOU CAN STILL SEE 'M' DURING A SIMULATION, WITH A TIME RATIO GIVEN BY $P(m)$.
IF YOU CHOOSE TOO LONG A TIME, YOU GET A BIAS (UNPHYSICAL, THAT JUMP IS A FINITE SIZE EFFECT).

USING MAXWELL'S CONSTRUCTION, CAN $\chi = \infty$ $\left(\text{SINCE } \frac{\partial^2 g_G}{\partial m^2} = 0\right)$ EVEN IF THERE'S A SINGLE STABLE STATE?
WELL, AT $m = 0$ THE SLOPE IS NOT ZERO! THE STRAIGHT LINE STARTS NOT AT $m = 0$, BUT AT $m = O(h)$.

- EXTENSIVE – INTENSIVE (PICCIOLI)

$$X = X(\underline{q})$$

$$g(x) := -\frac{1}{\beta N} \ln \int \mathcal{D}q \, e^{-\beta H(q)} \, \delta(x - X(q))$$

BY ABSURD, TAKE $m < N$ AND ASSUME

$$X = X(q_1, \dots, q_m)$$
$$\underline{q}_m = (q_1, \dots, q_m)$$
$$\underline{q}_N = (q_{m+1}, \dots, q_N)$$

SO THAT

$$H(q) = \underbrace{H(\underline{q}_m)}_{O(N)} + \underbrace{H(\underline{q}_m, \underline{q}_N)}_{O(m)}$$

AND

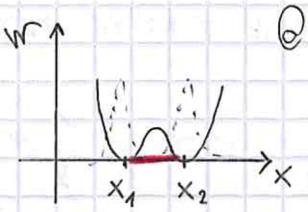
$$g(x) = -\frac{1}{\beta N} \ln \int \mathcal{D}\underline{q}_m \, \delta(x - X(\underline{q}_m)) \underbrace{\int \mathcal{D}\underline{q}_N \, e^{-\beta H(\underline{q}_N)} e^{-\beta H(\underline{q}_m, \underline{q}_N)}} \equiv Z_N \cdot \langle e^{-\beta H(q}$$

$$= -\frac{1}{\beta N} \ln Z_N \underbrace{- \frac{1}{\beta N} \ln \int \mathcal{D}\underline{q}_m \, \delta(x - X(\underline{q}_m)) \langle \underbrace{e^{-\beta H(q_m, q_N)}}_{O(m)} \rangle_N}_{\text{AT MAX. } \sim h(x) \, e^{m f(x)} \text{ (m NUMBER OF INTERACTIONS)}} = \text{CONST.}$$

@ Y_c

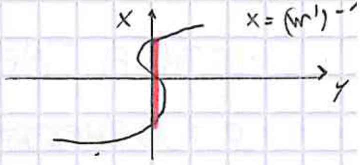
- <u>HINT</u>: EXPERIMENTALLY, WE MEASURE

$$P(x) = e^{-\beta N w(x)}$$

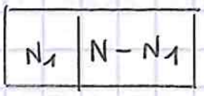


- FROM THIS WE GET THE GRAPH, WHICH UNFORTUNATELY DEPENDS ON $N \times Y$. BUT

1) w IS EXTENSIVE
2) THERE IS PHASE SEPARATION
3) "CONVEXITY" OF $\underline{\hat{w}}$



NOTE: $x = (w')^{-1}$



$\alpha = \frac{N_1}{N}$ $1 - \frac{N_1}{N} \equiv (1-\alpha)$

TRY TO TILT w A LITTLE, $w(x) \to w(x) - xY$, AND RECALL THE DEFINITION OF CONVEXITY.

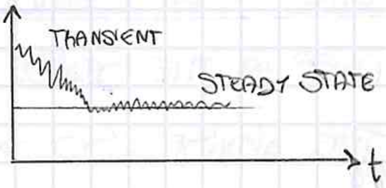
LESSON 29.03.19

DYNAMICS

WE'VE DONE SO FAR ONLY STATIC CALCULATIONS,

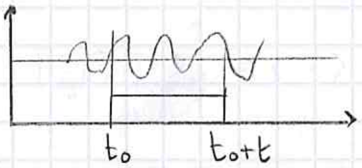
$$\langle f(\sigma) \rangle = \frac{1}{Z} \int D\sigma \, f(\sigma) e^{-\beta H(\sigma)}$$

BUT ACTUALLY THIS SERVES FOR FINDING



$$\overline{f}(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int^{T} dt \, f(\sigma(t))$$

WE'LL USE THINGS LIKE



$$\overline{f(t_0) f(t_0+t)} = \lim_{T \to \infty} \frac{1}{T} \int^{T} dt_0 \, f(\sigma(t_0)) f(\sigma(t_0+t))$$

BY STUDYING FLUCTUATIONS, WE HAVE

$$\overline{\delta f(t_0) \delta f(t_0+t)} \equiv C(t) = \lim_{T \to \infty} \frac{1}{T} \int^{T} dt_0 \, \delta f(\sigma(t_0)) \delta f(\sigma(t_0+t))$$

WHICH IS THE DYNAMICAL CORRELATION FUNCTION.

★ LET'S START FROM NEWTON EQUATIONS,

$$m \ddot{x} = F$$

IMAGINE WE'RE IN A HEAT BATH AND WE WANT TO CONDENSE THE EFFECT OF VERY MANY INTERACTIONS. WE KNOW THAT

1) ALL THE UNKNOWN INTERACTIONS CREATE MOTION OF THE PARTICLE
2) THEY ALSO SLOW DOWN THE PARTICLE

SO WE ADD TWO TERMS TO THE EQUATION:

1) NOISE
2) FRICTION (DISSIPATION)

$$m \ddot{x} = F + \xi(t) - \eta \dot{x}$$

WHICH IS KNOWN AS LANGEVIN EQUATION.

IN ORDER NOT TO BIAS THE PARTICLE, WE CHOOSE

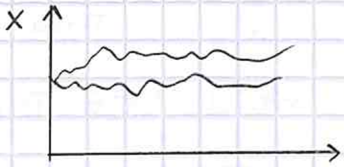
$$\langle \xi(t) \rangle = 0$$

SO THE IMPORTANT THING IS ITS VARIANCE:

$$\langle \xi(t')\xi(t'') \rangle = \Gamma \delta(t'-t'')$$

· ONE "FREE" PARTICLE

FREE IN THE SENSE THAT $F=0$; THE TRAJECTORY STILL DEPENDS ON THE NOISE, SO WE AVERAGE OVER IT:



$$\langle \cdot \rangle = \int D\xi \, P(\xi) \, (\cdot)$$

WE REWRITE

$$m\ddot{x} + \eta \dot{x} = \xi(t)$$

AS A SYSTEM OF 1^{ST} ORDER ODES,

$$\begin{cases} v = \dot{x} \\ m\dot{v} + \eta v = \xi(t) \end{cases}$$

WHICH WE SOLVE VIA G.F.M.

GREEN FUNCTION METHOD

$$\hat{A} \cdot f(t) = h(t)$$

$\hat{A}$ DIFFERENTIAL OPERATOR, $f(t)$ UNKNOWN, $h(t)$ EXTERNAL SOURCE (FIELD).

1) SOLVE THE HOMOGENEOUS,

$$\hat{A} f_0(t) = 0$$

2) FIND THE GREEN FUNCTION,

$$\hat{A} G(t-t') = \delta(t-t')$$

3) BUILD

$$f(t) = f_0(t) + \int dt' \, G(t-t') h(t')$$

1) HOMOGENEOUS

$$m\dot{v} + \eta v = 0$$

WE NOTICE, BY DIMENSIONAL ANALYSIS, A TIME SCALE:

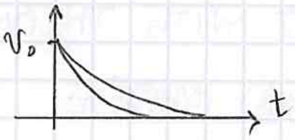
$$\tau \simeq \frac{m}{\eta}$$

$\tau \uparrow \quad m \uparrow \quad \eta \downarrow$

$\tau \downarrow \quad m \downarrow \quad \eta \uparrow$

(ALWAYS TRY TO DO THIS WHEN YOU HAVE A DIFFERENTIAL EQUATION)

$$v(t) = v_0 e^{-t/\tau}$$



2) GREEN FUNCTION (PROPAGATOR)

$$\left(m\frac{d}{dt} + \eta\right) G(t-t') = \delta(t-t') \qquad [G] = \left[\frac{1}{m}\right]$$

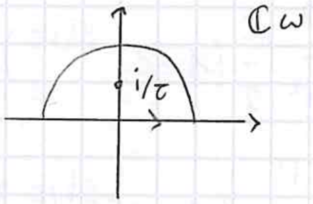
GOING TO FOURIER SPACE,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \, e^{i\omega t} \hat{f}(\omega)$$

$$(i\omega m + \eta) \hat{G}(\omega) = \frac{1}{\sqrt{2\pi}} \quad \Rightarrow \quad \hat{G}(\omega) = \frac{1}{\sqrt{2\pi}(i\omega m + \eta)}$$

$$G(t-t') = \frac{1}{2\pi} \int d\omega \frac{e^{i\omega(t-t')}}{(i\omega m + \eta)}$$

$$= \frac{1}{2\pi} \frac{1}{im} \int d\omega \frac{e^{i\omega(t-t')}}{(\omega - \frac{i}{\tau})}$$



IF $(t-t') > 0$, CLOSE UP → $\int \neq 0$

IF $(t-t') < 0$, CLOSE DOWN → $\int = 0$

THIS GIVES A $\theta(t-t')$, WHICH ENSURES CAUSALITY. HENCE

$$G(t-t') = \theta(t-t') \frac{1}{2\pi} \frac{1}{im} 2\pi i \, e^{-\frac{(t-t')}{\tau}} = \theta(t-t') \frac{1}{m} e^{-\frac{(t-t')}{\tau}}$$

AND THIS IS THE GREEN FUNCTION FOR VELOCITY, SO THAT

$$v(t) = v_0 e^{-t/\tau} + \frac{1}{m} \int dt' \, e^{-\frac{1}{\tau}(t-t')} \theta(t-t') \xi(t')$$

CHOOSE THE BOUNDARY CONDITION

$$v(t=0) = v_0$$

AND REWRITE

$$\underline{v_\xi(t) = v_0 e^{-t/\tau} + \frac{1}{m}\int_0^t dt' e^{-\frac{(t-t')}{\tau}} \xi(t)}$$

WHICH STILL DEPENDS ON ξ.

TO MAKE CONTACT WITH THERMODYNAMICS, WE EVALUATE THE KINETIC ENERGY AND IMPOSE DYNAMICS ≡ STATICS. THAT IS

$$K = \frac{1}{2} m \langle v^2 \rangle$$

$$\langle v(t) v(t) \rangle = \underbrace{v_0^2 e^{-2t/\tau}}_{\text{DETERMINISTIC PART}} + \frac{1}{m^2}\int_0^t dt' \int_0^t dt'' e^{-\frac{t-t'}{\tau}} e^{-\frac{(t-t'')}{\tau}} \underbrace{\langle \xi(t')\xi(t'') \rangle}_{=\Gamma\delta(t-t')}$$

$$= v_0^2 e^{-2t/\tau} + \frac{1}{m^2}\int_0^t dt' \, \Gamma e^{-\frac{1}{\tau}(t-t'+t-t')}$$

$$= v_0^2 e^{-2t/\tau} + \frac{\Gamma}{m^2} e^{-2t/\tau} \int_0^t dt' \, e^{2t'/\tau}$$

$$= v_0^2 e^{-2t/\tau} + \frac{\Gamma}{m^2}\frac{\tau}{2}(1-e^{-2t/\tau}) \qquad \tau = \frac{m}{\eta}$$

$$= v_0^2 e^{-2t/\tau} + \frac{\Gamma}{2m\eta}(1-e^{-2t/\tau}) \xrightarrow[t \gg \tau]{} \frac{\Gamma}{2m\eta}$$

LET'S USE EQUIPARTITION, i.e.

$$\rho(p) = e^{-\beta\frac{p^2}{2m}}$$

$$K = \frac{1}{2m}\langle p^2 \rangle = \frac{1}{2m}\frac{\int dp \, e^{-\beta\frac{p^2}{2m}} p^2}{\int dp \, e^{-\beta p^2/2m}} = \frac{1}{2m}\frac{m}{\beta} = \frac{1}{2}k_B T$$

HENCE WE REQUIRE, FOR $t \gg \tau$,

$$\Gamma = 2\eta k_B T$$

NOTE: THIS IS 0^{TH} ORDER FdT.
THIS IS A VERY IMPORTANT 2, REMEMBER IT

EINSTEIN RELATION

· DIMENSIONALITY

$$m\ddot{x}_\alpha + \eta\dot{x}_\alpha = \xi_\alpha \qquad \alpha, \beta = x, y, z$$

$$\langle \xi_\alpha \xi_\beta \rangle = \delta_{\alpha\beta}\,\delta(t-t')\,\underbrace{2\eta T}_{=\Gamma} \qquad k_B \equiv 1$$

WHICH IS SOMETIMES WRITTEN AS

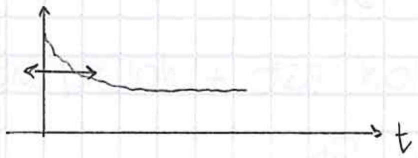
$$\langle \underline{\xi} \cdot \underline{\xi} \rangle = \delta(t-t')\, d\, 2\eta T \qquad T = \frac{\Gamma}{2\eta}$$

KICK

SLOWING DOWN

· OVERDAMPED LIMIT

$$\langle v^2 \rangle = v_0^2 e^{-2t/\tau} + \frac{T}{m}\left(1 - e^{-2t/\tau}\right)$$



FOR $t \gg \tau$, ALL TRANSIENTS DIE. RECALL

$\tau = \frac{m}{\eta}$, SO FOR

"$\eta \gg m$", $\tau \to 0$ (!)

SO, AT THE END OF THE DAY, PEOPLE JUST TAKE AWAY

$$\underline{m\ddot{x}} + \eta\dot{x} = F + \xi \qquad \text{OVERDAMPED LIMIT}$$

THEN I CAN RESCALE

$$t \to \eta t \qquad \dot{x} \to \frac{1}{\eta}\dot{x}, \quad \delta(t) \to \frac{1}{\eta}\delta(t)$$

SO AS TO GET

$$\dot{x} = F + \xi \qquad \langle \xi\xi \rangle = 2T\delta(t-t')$$

BUT THIS IS NOT COMPLETELY SATISFACTORY. BY DOING THE SAME RESCALING IN THE ORIGINAL EQUATION, WE WOULD FIND

$$\frac{m}{\eta^2}\ddot{x} + \dot{x} = F + \xi \qquad \langle \xi\xi \rangle = 2T\delta(t-t')$$

SO NOW IT SEEMS LIKE WE'RE ASKING FOR

"$\frac{m}{\eta^2} \to 0$"

WHY DO THEY BOTH SUCK? IN ORDER TO WRITE $x \ll 1$, x HAS TO BE ADIMENSIONAL AND NEITHER $\frac{m}{\eta}$, NOR $\frac{m}{\eta^2}$ ARE.

SO WHAT YOU'RE ACTUALLY COMPARING IS

$$t \quad \text{vs} \quad \tau = \frac{m}{\eta}$$

(PRACTICALLY SPEAKING, YOU END UP IN THE SAME PLACE).

- **SMALL PIPPOTTO**

$$\begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} - \eta \dot{x} + \xi \end{cases}$$

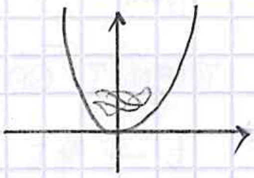
HAMILTON EQS + NOISE/DISSIPATION. WHEN YOU GO OVERDAMPED,

$$\dot{x} = -\frac{\partial H}{\partial x} + \xi$$

THIS IS "BEFORE NEWTON" VS "AFTER NEWTON": THEY USED TO THINK THE FORCE ACTS UPON VELOCITY. BEAR IN MIND WHAT IT REALLY MEAN

- **HOMEWORK**: STOCHASTIC HARMONIC OSCILLATOR

$$m\ddot{x} + \eta\dot{x} + kx = \xi \qquad \langle \xi \xi \rangle = 2T\eta\,\delta(t-t')$$



GOING TO ($\dot{x} = v$) DOESN'T HELP...

1) SOLVE THE HOMOGENEOUS (DETERMINISTIC H.O.) WITH

$$\begin{cases} x_0(0) = 0 \\ \dot{x}_0(0) = v_0 \end{cases}$$

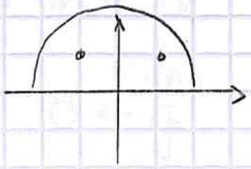
DO IT IN FOURIER:

$$x_0 = e^{iwt}$$

AND SOLVE FOR ω. CALL $\tau = \frac{m}{\eta}$ AND $\omega_0^2 = \frac{K}{m}$. WORK IN THE REGIME

$$\omega_0^2 \tau^2 < 1$$

(NOT $\ll 1$, DON'T EXPAND, IT'S FOR $\sqrt{\ldots}$)



2) FIND THE GREEN FUNCTION

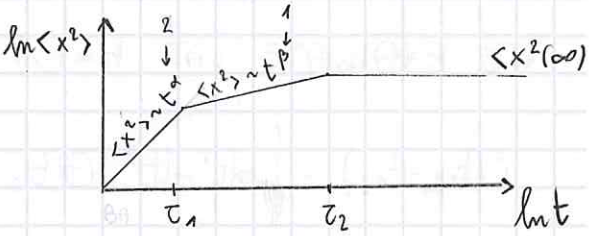
$$(-mw^2 + i\eta w + k)G(w) = \frac{1}{\sqrt{2\pi}}$$

3) ONCE YOU HAVE

$$x(t) = x_0(t) + \int dt'\, G(t-t')\,\xi(t')$$

COMPUTE THE MEAN SQUARE DISPLACEMENT

$\langle x^2(t) \rangle$



AND STUDY THE REGIMES BALLISTIC, DIFFUSIVE, SATURATION WITH THEIR RELEVANT EXPONENTS AND TIME SCALES.

GREEN FUNCTION AND RESPONSE

ASSUME WE ADD A FIELD $h(t)$ TO

$$\begin{cases} A\,x(t) = \delta(t) \\ x(t) = \int dt'\, G(t-t')\,\delta(t') \end{cases}$$

$AG(t-t') = \delta(t-t')$

$X_0 = 0$

SO THAT

$$A\,x(t) = \xi(t) + h(t)$$

IT'S LIKE HAVING ADDED A FORCE,

$$f = -\frac{\partial H}{\partial x}$$

$$H \rightarrow H - h(t)\,x(t)$$

$$-\frac{\partial H}{\partial x} \longrightarrow -\frac{\partial H}{\partial x} + h(t)$$

WE FIND

$$X(t) = \int dt'\, G(t-t')\,[\,\xi(t') + h(t')\,]$$

$$\langle x(t)\rangle = \int dt'\, G(t-t')\cdot h(t')$$

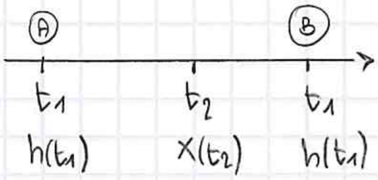
$\langle \xi \rangle = 0$

AND WE'RE INTERESTED IN THE RESPONSE

$$R(t_2,t_1) = \frac{\delta\langle x(t_2)\rangle}{\delta h(t_1)} = \frac{\delta}{\delta h(t_1)} \int^{t_2} dt'\, G(t_2-t')\,h(t')$$

IF $t_1 < t_2$, Ⓐ

$$\frac{\delta}{\delta h(t_1)} \langle x(t_2)\rangle = G(t_2 - t_1)$$



IF $t_1 > t_2$, Ⓑ

$$R(t_2,t_1) = 0$$

(CAUSALITY)

* LET'S NOW ASSUME TIME TRANSLATIONAL INVARIANCE (TTI) AND $\langle x \rangle = 0$.

WE EVALUATE (AT $h=0$)

$$C(t_1-t_2) = \int dt'\, dt''\, G(t_1-t')G(t_2-t'') \underbrace{\langle \xi(t')\xi(t'') \rangle}_{} = 2T\eta\, \delta(t'-t'')$$

$$= 2T\eta \int dt'\, G(t_1-t')G(t_2-t')$$

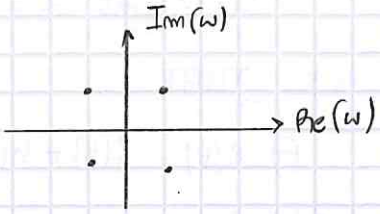
$$\int \frac{d\omega}{\sqrt{2\pi}} e^{i\omega(t_1-t_2)} C(\omega) = 2T\eta \int dt' \frac{d\omega'}{\sqrt{2\pi}} \frac{d\omega''}{\sqrt{2\pi}} e^{i\omega'(t_1-t')} e^{i\omega''(t_2-t')} G(\omega')G(\omega'')$$

USING

$$\int dt'\, e^{-it'(\omega'+\omega'')} = 2\pi\, \delta(\omega'+\omega'')$$

SO THAT

$$\underline{C(\omega) = \sqrt{2\pi}\, 2T\eta\, G(\omega)G(-\omega)}$$



WE SEE THE POLE STRUCTURE IS SYMMETRICAL : $G(\omega)$ IS CAUSAL, BUT $C(\omega)$ IS NOT!

NOTE: WITH MY USUAL CONVENTION

$$\int \frac{d\omega}{2\pi} \delta(\omega) = 1$$

$$C(\omega) = 2T\eta\, G(\omega)G(-\omega)$$

LESSON 02.04.19

S.H.O.

$$m\ddot{x}+\eta\dot{x}+kx=\xi \qquad\qquad \langle\xi\xi\rangle=2T\eta\,\delta(t-t')$$

1) HOMOGENEOUS

$$X=e^{i\omega t}$$

$$-m\omega^2+i\eta\omega+k=0$$

$$\omega_\pm=i\frac{\eta}{2m}\pm\sqrt{\omega_0^2-\frac{\eta}{2m}}\equiv i\gamma\pm\sqrt{\omega_0^2-\gamma^2} \qquad \gamma:=\frac{\eta}{2m}$$

IF $\gamma^2>\omega_0^2$,

$$\omega_\pm=i\gamma\pm i\sqrt{\gamma^2-\omega_0^2}:=i\gamma\pm i\hat{\omega}$$

THE HOMOGENEOUS SOLUTION IS

$$X_0(t)=ae^{i\omega_+t}+be^{i\omega_-t}$$

INITIAL CONDITIONS:

$$\begin{cases} X_0(0)=a+b=0 \\ \dot{X}_0(0)=v_0=i\omega_+a+i\omega_-b=-2\hat{\omega}a \end{cases}$$

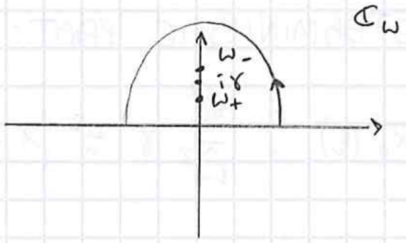
WHENCE

$$X_0(t)=\frac{v_0}{2\hat{\omega}}\left(e^{i\omega_-t}-e^{i\omega_+t}\right)=\frac{v_0}{2\hat{\omega}}e^{-\gamma t}\left(e^{\hat{\omega}t}-e^{-\hat{\omega}t}\right)$$

$$=\frac{v_0}{\hat{\omega}}e^{-\gamma t}\sinh(\hat{\omega}t)$$

2) GREEN FUNCTION

$$\left(-m\omega^2+i\eta\omega+k\right)G(\omega)=\frac{1}{\sqrt{2\pi}}$$



$$G(\omega)=-\frac{1}{m\sqrt{2\pi}(\omega-\omega_+)(\omega-\omega_-)}$$

$$G(t-t')=-\frac{1}{m}\int\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{(\omega-\omega_-)(\omega-\omega_+)}$$

SO

$$G(t-t') = -\frac{2\pi i}{2\pi m}\left\{ \frac{e^{i\omega_+(t-t')}}{\omega_+-\omega_-} + \frac{e^{i\omega_-(t-t')}}{\omega_--\omega_+} \right\}\theta(t-t')$$

$$= -\frac{i}{m}\frac{1}{(\omega_+-\omega_-)}\left\{ e^{i\omega_+(t-t')} - e^{i\omega_-(t-t')} \right\}\theta(t-t')$$

$$= \frac{e^{-\gamma(t-t')}}{m\hat{\omega}} \text{SINH}\left(\hat{\omega}\cdot(t-t')\right)\theta(t-t')$$

HENCE

$$x(t) = x_0(t) + \int dt' \, G(t-t')\xi(t')$$

3) M.S.D.

$$\langle x^2 \rangle = x_0^2(t) + \frac{2T\eta}{4m^2\hat{\omega}^2}\int_0^t dt' \, e^{-2\gamma(t-t')}\left[e^{\hat{\omega}(t-t')} - e^{-\hat{\omega}(t-t')} \right]^2$$

$$= \dots = x_0^2(t) + \frac{2T\eta}{4m^2\hat{\omega}^2}\left\{ \frac{1-e^{-2(\gamma-\hat{\omega})t}}{2(\gamma-\hat{\omega})} + \frac{1-e^{-2(\gamma+\hat{\omega})t}}{2(\gamma+\hat{\omega})} - 2\,\frac{1-e^{-2\gamma t}}{2\gamma} \right\}$$

$$= \frac{v_0^2}{\hat{\omega}^2} e^{-2\gamma t}\,\text{SINH}^2(\hat{\omega}t) + \frac{T\eta}{4m^2\hat{\omega}^2}\left\{ \frac{1-e^{-2(\gamma-\hat{\omega})t}}{\gamma-\hat{\omega}} + \frac{1-e^{-2(\gamma+\hat{\omega})t}}{\gamma+\hat{\omega}} - 2\,\frac{1-e^{-2\gamma t}}{\gamma} \right\}$$

WITH

$$\omega_\pm = i\gamma \pm i\hat{\omega} \qquad\qquad \hat{\omega} = \sqrt{\gamma^2 - \omega_0^2}$$

4) STUDY OF REGIMES

A) BALLISTIC $\qquad 2\gamma t \ll 1, \quad t \ll \frac{1}{2\gamma} = \frac{m}{\eta} \equiv \tau$

$$\text{SINH}(\hat{\omega}t) = \text{SH}\left(\gamma t\sqrt{1-\omega_0^2/\gamma^2}\right) \simeq \gamma t\sqrt{1-\frac{\omega_0^2}{\gamma^2}}$$

$$e^{-2\gamma t} \simeq 1$$

DETERMINISTIC PART:

$$x_0^2(t) = \frac{v_0^2}{\bar{\omega}^2}\gamma^2 t^2\left(1-\frac{\omega_0^2}{\gamma^2}\right) = v_0^2 t^2$$

STOCHASTIC PART:

$$\frac{T\eta}{4m^2\hat{\omega}^2}\left\{\frac{2(\gamma-\hat{\omega})t}{\gamma-\hat{\omega}}+\frac{2(-)(\gamma+\hat{\omega})t}{\gamma+\hat{\omega}}-\frac{2}{\gamma}2\gamma t\right\}$$

$$=-\frac{T\eta}{4m^2\hat{\omega}^2}4\gamma t \ll 1 \qquad t \ll \frac{1}{2\gamma}=\tau$$

B) DIFFUSIVE

IT $\gamma t \gg 1$

ASSUME $\omega_0^2 \ll \gamma^2$

$$\hat{\omega}=\sqrt{\gamma^2-\omega_0^2}=\gamma\sqrt{1-\frac{\omega_0^2}{\gamma^2}}\overset{\downarrow}{\simeq}\gamma\left(1-\frac{\omega_0^2}{2\gamma^2}\right)$$

HENCE

$$\gamma+\hat{\omega}\simeq 2\gamma$$

$$\gamma-\hat{\omega}\simeq\frac{\omega_0^2}{2\gamma}$$

WE WANT

$$2(\gamma-\hat{\omega})t \ll 1$$

SO WE REQUIRE

$$\frac{1}{2\gamma} \ll t \ll \frac{\gamma}{\omega_0^2}$$

THE ASSUMPTION WE MADE JUST ENSURES THERE IS SPACE BETWEEN THE 2 LIMITS (OTHERWISE I DON'T SEE THE DIFFUSIVE REGIME). THEN

$$e^{-\gamma t}\sinh(\hat{\omega}t)=\underset{\sim}{e^{-\gamma t+\hat{\omega}t}}-e^{-\gamma t-\hat{\omega}t}\simeq e^{-\frac{\omega_0^2}{2\gamma}t}\simeq 1$$

SO THE DETERMINISTIC PART AMOUNTS TO

$$x_0^2(t)\simeq\frac{v_0^2}{\hat{\omega}^2} \quad \text{CONST.}$$

THE STOCHASTIC PART READS

$$e^{-2\gamma t}\to 0$$

$$e^{-2(\gamma+\hat{\omega})t}\to 0$$

$$e^{-2(\gamma-\hat{\omega})t}\simeq 1-\frac{\omega_0^2}{\gamma}t$$

SO THAT

$$\langle x^2 \rangle = \frac{v_0^2}{\hat{\omega}^2} + \frac{T\eta}{4m^2\hat{\omega}} \left\{ \frac{1}{\gamma-\hat{\omega}} \frac{\omega_0^2 t}{\gamma} + \frac{1}{\gamma+\hat{\omega}} - \frac{2}{\gamma} \right\} = \frac{T\eta\, t}{2m^2\gamma^2} = \frac{2T}{\eta} t$$

THIS IS EINSTEIN RELATION

$$\langle x^2 \rangle = 2Dt \qquad \longrightarrow \qquad D = \frac{k_B T}{\eta}$$

NOTICE

NOTE: WE FOUND IT WITH VULPIANI FOR $\ddot{x} = -\frac{\dot{x}}{\tau} + \xi$

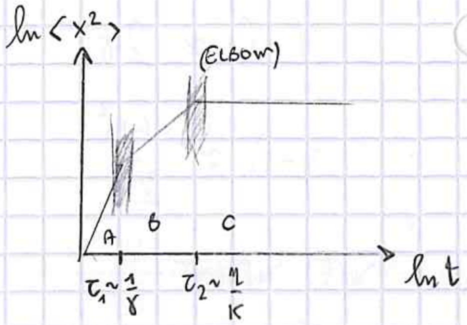
$$\tau \sim \frac{1}{2\gamma} << t << \frac{\gamma}{\omega_0^2} \sim \frac{\eta}{2k}$$

← VISCOSITY

← STIFFNESS OF THE SPRING

WHICH DOESN'T KNOW ANYTHING ABOUT m.

C) SATURATION



$t >> \tau_2$

$$\langle x^2 \rangle = \frac{T\eta}{4m^2\hat{\omega}^2} \left\{ \frac{1}{\gamma-\hat{\omega}} + \frac{1}{\gamma+\hat{\omega}} - \frac{2}{\gamma} \right\} = \frac{T}{K}$$

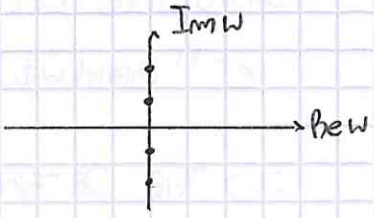
WHICH IS OK: THIS IS A DYNAMICAL AVG OVER ξ, BUT THE STATIC

$$\langle x^2 \rangle_{STAT} = \frac{1}{Z} \int dx\, x^2 e^{-\frac{1}{2}\beta K x^2} = \frac{T}{K}$$

SO

$$\langle x^2 \rangle_{DYN} \xrightarrow[t\to\infty]{} \langle x^2 \rangle_{STAT}$$

- CORRELATION FUNCTION OF THE SHO



$$C(\omega) = \sqrt{2\pi}\, 2T\eta\, G(\omega) G(-\omega)$$

$$C(t) = \ldots = C_0 e^{-\gamma t} \left[\cos(\hat{\omega} t) + \frac{\gamma}{\hat{\omega}} \sin(\hat{\omega} t) \right]$$

WHERE C_0 IS A CONSTANT AND

$$\hat{\omega} = \left[\omega_0^2 - \gamma^2 \right]^{1/2}$$

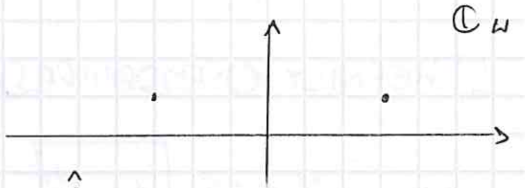
NOTICE $t = t_1 - t_2$ IS A DIFFERENCE OF TIMES, NOT AN ABSOLUTE ONE

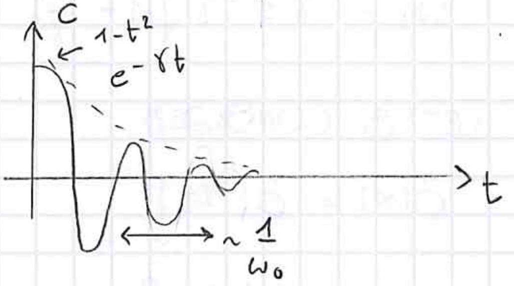
WHAT IS THE SHAPE OF C(t)? IT'S QUITE GENERAL: IT'S THE GAUSSIAN/LINEAR APPROXIMATION OF ANYTHING. LET'S STUDY

$$\hat{C}(t) = \frac{C(t)}{C_0} = e^{-\gamma t}\left\{ \cos(\hat{\omega} t) + \frac{\gamma}{\hat{\omega}} \sin(\hat{\omega} t)\right\}$$

1) UNDERDAMPED $\gamma^2 < \omega_0^2$

$$\omega_{\pm} = i\gamma \pm \sqrt{\omega_0^2 - \gamma^2}$$





IT'S CLEAR THAT

$$\dot{C}(t=0) = 0$$

i.e. NON-EXPONENTIAL, BECAUSE $e^{-t/\tau}$ HAS A NONZERO FIRST DERIVATIVE.

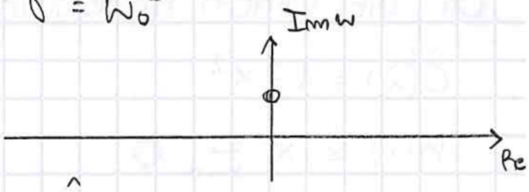
IF YOU GO OVERDAMPED IN

$$\underset{\sim}{m}\ddot{x} + \eta\dot{x} + Kx = f$$

YOU GET A SINGLE POLE, i.e. A SIMPLE EXPONENTIAL IN TIME.
YOU NEED 2 POLES TO HAVE SIN/COS.
A FIRST ORDER DYNAMICS ONLY GENERATES EXPONENTIALS, EVEN IN C(t) (YOU CAN CHECK IT'S TRUE).

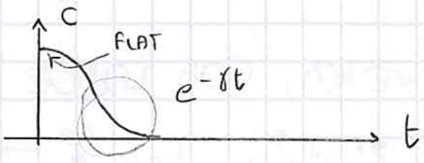
2) CRITICALLY DAMPED $\gamma^2 = \omega_0^2$

THE 2 ROOTS ARE DEGENERATE.



$$\omega_{\pm} = i\gamma$$

$$\hat{C}(t) = e^{-\gamma t}(1+\gamma t)$$



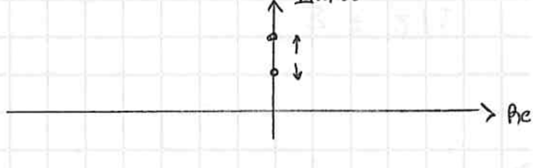
AND STILL

$$\dot{C}(t=0) = 0$$

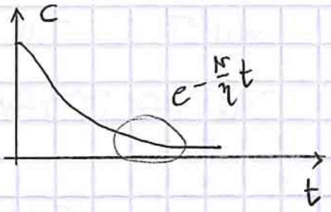
3) OVERDAMPED $\gamma^2 > \omega_0^2$

$$\omega_{\pm} = i\gamma \pm i\sqrt{\gamma^2 - w^2}$$

THE TWO ROOTS MOVE APART.



IT REALLY SEEMS TO BE EXPONENTIAL NOW,



BUT STILL

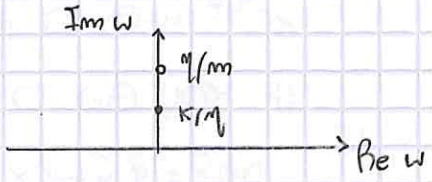
$$\dot{C}(0) = 0$$

THIS CAN BE PROVEN TO BE TRUE WHENEVER THERE ARE 2 POLES IN $C(\omega)$.

THIS IS A POWERFUL CHECK ON EXPERIMENTAL DATA.

4) HEAVILY OVERDAMPED

$$\omega_{\pm} = i\gamma \pm i\gamma\sqrt{1-\frac{\omega_0^2}{\gamma^2}} \simeq i\gamma \pm i\gamma\left(1-\frac{\omega_0^2}{2\gamma^2}\right) = \begin{cases} 2i\gamma & = i\,\frac{\eta}{m} \\ i\,\frac{\omega_0^2}{2\gamma} & = i\,\frac{K}{\eta} \end{cases}$$

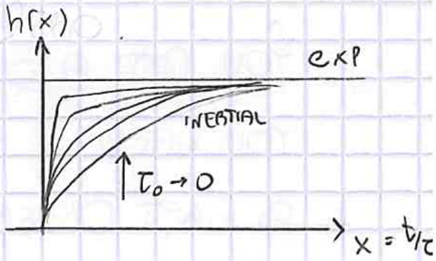


LET'S CONSIDER

$$\hat{C}(x) = \hat{C}(t/\tau)$$

$$h(x) = -\frac{1}{x}\ln\hat{C}(x)$$

WHICH IS A TRICK TO STUDY THE DERIVATIVE IN ZERO. IF YOU HAVE AN EXPONENTIAL,



$$\hat{C}(x) = e^{-x} \simeq 1 - x$$

$$h(x) = 1$$

ON THE OTHER HAND, IF YOU HAVE AN INERTIAL FORM SUCH AS

$$\hat{C}(x) = 1 - x^2$$

$$h(x) = x \xrightarrow[x\to 0]{} 0$$

HOWEVER, FOR LARGE DISSIPATIONS,

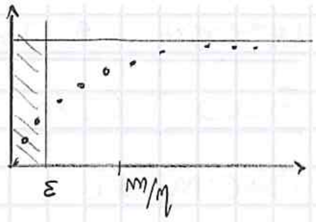
$$\eta^2 \gg \omega_0^2, \quad \tau_0 = \frac{m}{\eta} \to 0$$

THE CURVE $h(x)$ BECOMES STEEPER AND STEEPER NEAR $x=0$.

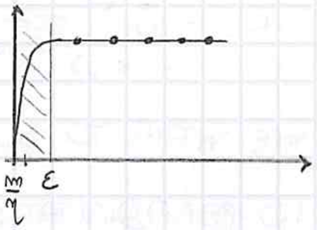
IN A REAL EXPERIMENTS THERE'S A TIME RESOLUTION ε. THEN

$$t/\tau \geq \varepsilon$$

IF $\varepsilon \ll \frac{m}{\eta}$, YOU CAN DETECT EXPERIMENTALLY THAT THE DYNAMIC WAS INERTIAL (YOU SEE THE RELICK OF MASS).



IF $\varepsilon \gg \frac{m}{\eta}$, YOU'RE FUCKED. FOR ALL PRACTICAL PURPOSES, YOUR DYNAMICS IS DISSIPATIVE.



IN SOME CASES YOU HAVE AN INHERENT TIME SCALE UNDER WHICH IT MAKES NO SENSE TO GO: THEN YOU ARE IN THE SECOND CASE.

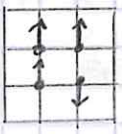
- LESSON 05.04.2019

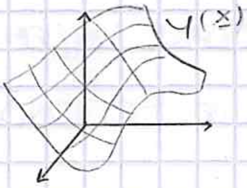
FIELD THEORY

ISING MODEL

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

WE WANT TO BUILD A CONTINUOUS FIELD THEORY FOR THIS MODEL, WHICH IS PARADIGMATIC (AND SIMPLE) FOR THE DESCRIPTION OF CRITICAL PHENOMENA.





$$i \rightarrow \underline{x}$$

$$\sigma_i \rightarrow \varphi(\underline{x})$$

$$H_{ISING} \longrightarrow \quad H_{LG} = \int d^d x \, \mathcal{L}(\varphi(\underline{x}))$$

1) WHY?

- CALCULATIONS ARE EASIER
- EXPAND AROUND MEAN FIELD
- UNIVERSALITY: WE CAN IMAGINE THAT THE SAME CONTINUOUS MODEL CAN CORRESPOND TO MANY DIFFERENT MICROSCOPIC MODELS WHICH DIFFER FROM EACH OTHER BY IRRELEVANT DETAILS, AS LONG AS WE'RE CONCERNED WITH THE MACROSCOPIC BEHAVIOUR OF THE SYSTEM. THE R.G. WILL PROVIDE A FORMAL JUSTIFICATION.

2) HOW?

COARSE GRAINING (LOCAL AVERAGES) $\rightarrow \begin{cases} \text{ARBITRARY} \\ \text{HEURISTIC} \end{cases}$

BEFORE R.G., THE JUSTIFICATION FOR ALL OF THIS WAS THAT, SINCE WE'RE INTERESTED IN STUDYING THE BEHAVIOUR OF THE SYSTEM CLOSE TO T_c, WE HAVE SCALE INVARIANCE (POWER-LAW CORRELATIONS), SO THAT THE BEHAVIOUR OF THE SYSTEM LOOKS VERY SIMILAR AT EVERY SCALE:

$$T \simeq T_c \rightarrow \xi \gg a$$

ξ: CORRELATION LENGTH

a: MICROSCOPIC LENGTH SCALE

HOPEFULLY, THEN, MICROSCOPIC DETAILS SHALL BE IRRELEVANT. HOWEVER, SCALE INVARIANCE IN ITSELF IS NOT ENOUGH; WHAT IS REALLY NEEDED IS THAT

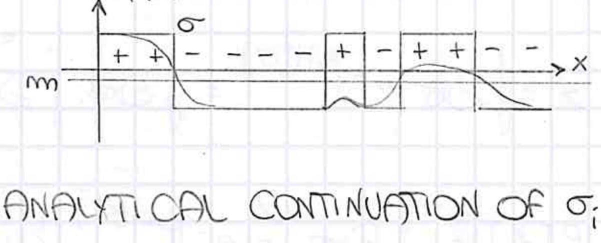
SHORT RANGE INTERACTIONS → LONG RANGE CORRELATIONS

DETAILS OF THE MICROSCOPIC MODEL DO MATTER IF, ON THE CONTRARY, WE ARE IN ONE OF THE FOLLOWING CASES:

- $\xi \simeq a$
- LONG RANGE INTERACTIONS: $r >> a$ ($\xi >> a$ EVENTUALLY)

IF THE PHYSICS WE'RE INTERESTED IN HAS TYPICAL SCALES MUCH LARGER THAN THE MICROSCOPIC SCALE AND CORRELATION LENGTHS ARE LARGE AS WELL, THEN HOPEFULLY THE NEW COARSE-GRAINED THEORY SHOULD DESCRIBE THE ORIGINAL SYSTEM AND EVEN THE WAY WE BUILD THE NEW THEORY SHOULDN'T BE RELEVANT.

SINCE THE SOLUTION FOR M.F. IS KNOWN, WE CAN USE IT AS A COMPASS! BY USING THE COARSE-GRAINED THEORY WE SHOULD RECOVER APPROXIMATELY THE USUAL M.F. (WHICH IS NOT SHORT RANGED, HOWEVER).



- COARSE GRAINING

$\{\sigma_i\}$ IN $d = 1$

NO COARSE GRAINING → TAKE THE ANALYTICAL CONTINUATION OF σ_i IN

FULL COARSE GRAINING → FULL AVERAGE, $m = \frac{1}{N} \sum_i \sigma_i$

REASONABLE COARSE GRAINING:

$$\varphi(x) = \frac{1}{v} \sum_{i \in v} \sigma_i$$ $\varphi(x)$ LOCAL MAGNETIZATION

WHERE v IS A "SMALL" VOLUME AROUND x:

$a^3 << v << L^3$

$a << l << L$

MICROSCOPIC MESOSCOPIC MACROSCOPIC

(MESOSCOPIC SCALE THEORY).

* WHAT IS THE PROBABILITY DENSITY FUNCTIONAL OF $\varphi(x)$?

THE GOAL IS TO FIND $P(\varphi(x))$ KNOWING THAT

$$P[\sigma] = \frac{1}{Z} e^{-\beta H[\sigma]} \qquad \text{(MICROSCOPIC)}$$

$$P(m) = \frac{1}{Z} \int \mathcal{D}\sigma \, e^{-\beta H[\sigma]} \, \delta\left(m - \frac{1}{V} \sum_i \sigma_i\right) = \frac{1}{Z} e^{-\beta N g(m)} \qquad \text{(MACROSCOPIC)}$$

FINDING THE NEW THEORY MEANS FINDING

$$P(\varphi) = \frac{1}{Z} e^{-H^{\beta}_{eff}[\varphi]} \qquad \text{(MESOSCOPIC)}$$

NOTE THAT IN ALL OF THESE CASES Z IS THE SAME PARTITION FUNCTION, SINCE IT'S INDEPENDENT OF THE DEGREES OF FREEDOM AND ONLY DEPENDS ON β AND THE EXTERNAL FIELDS:

$$Z = \int \mathcal{D}\sigma \, e^{-\beta H(\sigma)} = \int dm \, e^{-\beta N g(m)} = \int \mathcal{D}\varphi(x) \, e^{-H^{\beta}_{eff}[\varphi(x)]}$$

LET'S TRY TO DERIVE HEURISTICALLY $H[\varphi(x)]$. NOTE WE'RE TRYING TO PASS FROM A SCALAR σ_i TO A FUNCTION $\varphi(x)$. THE EXTREME COARSE GRAINING CORRESPONDS TO THE FULL AVERAGE m, WHICH IS A SCALAR, i.e. BASICALLY A CONSTANT FUNCTION WITH NO STRUCTURE.

$$Z = \int \mathcal{D}\sigma \, e^{-\beta H[\sigma]} = \int \mathcal{D}\varphi \int \mathcal{D}\sigma \, e^{-\beta H[\sigma]} \, \delta\left(\varphi(x) - \frac{1}{V_x} \sum_{i \in V_x} \sigma_i\right)$$

$$P[\varphi] = \int \mathcal{D}\sigma \, \frac{1}{Z} e^{-\beta H[\sigma]} \, \delta\left(\varphi(x) - \frac{1}{V_x} \sum_{i \in V_x} \sigma_i\right) \overset{def}{\equiv} \frac{e^{-H^{\beta}_{eff}[\varphi]}}{\int \mathcal{D}\varphi \, e^{-\beta H_{eff}[\varphi]}}$$

1) THE STATISTICAL WEIGHT OF A REALIZATION $\varphi(x)$ DEPENDS ON THE ENERGY $H[\sigma]$ (BOLTZMANN WEIGHT OF MICROSCOPIC CONFIGURATION).

2) THE NUMBER OF DIFFERENT MICROSCOPIC WAYS IN WHICH WE CAN REALIZE THE SAME $\varphi(x)$ DEPENDS ON THE ENTROPY ONLY.

ANY PECULIAR ARRANGEMENT OF THE SPINS WILL GIVE A SPECIFIC CONTRIBUTION IN TERMS OF ENERGY, BUT THE SAME VALUE OF $\varphi(x)$ (MESOSCOPIC FIELD) CORRESPONDING TO A GIVEN MICROSCOPIC CONFIGURATION CAN BE OBTAINED IN MANY DIFFERENT WAYS.

NOTE THAT, AT VARIANCE WITH THE HELMOTZ FREE ENERGY (OR GIBBS' IN M.F.), WE DON'T HAVE THE SAME FUNCTION IN THE exp AND IN THE δ, SO WE DON'T HAVE AN EXPLICIT EXPRESSION FOR $H_{eff}[$

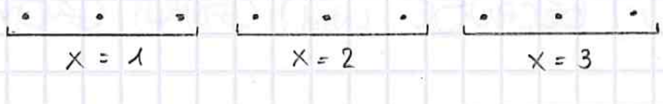
LET'S SAY THAT

$$Z \simeq \underbrace{\int \mathcal{D}\varphi \, e^{-W(\varphi)}}_{\text{STATISTICAL WEIGHT (NOT NORMALIZED) OF A REALIZATION } \varphi(x)} \underbrace{\int \mathcal{D}\sigma \, \delta\Big(\varphi(x) - \frac{1}{v_x} \sum_{i \in v_x} \sigma_i\Big)}_{\text{NUMBER OF DISTINCT WAYS TO REALIZE } \varphi(x)} \simeq \int \mathcal{D}\varphi \, e^{-H_{eff}[\varphi]}$$

THE SEPARATION BETWEEN ENTROPIC AND ENERGETIC CONTRIBUTION FOR EACH CONFIGURATION $\varphi(x)$ IS BY A LEAP OF FAITH, BUT THERE IS AT LEAST A SITUATION WHERE THIS WORKS.
FOR $\beta \to 0$ ($T \to \infty$) WE DON'T HAVE THE STATISTICAL WEIGHT, BUT ONLY THE ENTROPY (OF COURSE, SUCH A SITUATION IS FAR FROM T_c):

$$Z(\beta = 0) = \int \mathcal{D}\varphi \mathcal{D}\sigma \, \delta\Big(\varphi(x) - \frac{1}{v_x} \sum_{i \in v_x} \sigma_i\Big)$$

SUPPOSE $d = 1$, $v = 3$ (COARSE GRAINING IN GROUPS OF 3 SPINS):



$$1 = \int d\varphi_1 d\varphi_2 d\varphi_3 \, \delta\Big(\varphi_1 - \frac{1}{3} \sum_{i \in 1} \sigma_i\Big) \delta\Big(\varphi_2 - \frac{1}{3} \sum_{i \in 2} \sigma_i\Big) \delta\Big(\varphi_3 - \frac{1}{3} \sum_{i \in 3} \sigma_i\Big)$$

$$Z(\beta = 0) = \int d\sigma_1 \dots d\sigma_9 \int d\varphi_1 d\varphi_2 d\varphi_3 \, \delta(\dots)\delta(\dots)\delta(\dots)$$

$$= \int d\varphi_1 \int d\sigma_1 d\sigma_2 d\sigma_3 \, \delta\Big(\varphi_1 - \frac{1}{3} \sum_{i \in 1} \sigma_i\Big) \cdot \int d\varphi_2 \int d\sigma_4 d\sigma_5 d\sigma_6 \, \delta(\dots) \cdot \Big(\int d\varphi_3 \, \dots\Big)$$

THOSE 3 TERMS ARE IDENTICAL:

$$\int d\sigma_1 d\sigma_2 d\sigma_3 \, \delta\left(\varphi_1 - \frac{1}{3}\sum_{i\in 1}\sigma_i\right) = \mathcal{N}(\varphi_i) =$$ NUMBER OF MICROSCOPICALLY DISTINCT WAYS OF REALIZING φ_1

$\mathcal{N}(\varphi_1) \equiv e^{\Omega(\varphi_1)}$ $\qquad$ $\Omega(\varphi_1) \equiv$ ENTROPY OF φ_1

SO AT T=∞ ONLY ENTROPY MATTERS AND THE PARTITION FUNCTION IS

$$Z(\beta=0) = \int \left(\prod_{x=1}^{3} d\varphi_x\right) \prod_{x=1}^{3} e^{\Omega(\varphi(x))}$$

LET'S ASSUME THAT WE HAVE AN <u>EXTENSIVE</u> ENTROPY:

$$\Omega(\varphi(x)) = v \cdot w(\varphi(x))$$

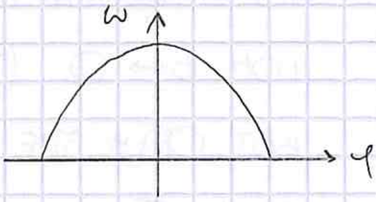
$$Z(\beta=0) = \int \prod_{x=1}^{3} d\varphi(x) \, e^{\sum_x v \cdot w(\varphi(x))}$$

$$\longrightarrow \int \mathcal{D}\varphi(x) \, e^{\int d^d x \, w(\varphi(x))}$$ (FUNCTIONAL INTEGRAL)

NOW WE HAVE TO HAND-WAVE AGAIN TO GUESS THE FORM OF $w(\varphi(x))$.

φ IS THE LOCAL AVERAGE OF (MANY) SPINS.

FOR COMBINATORIAL REASONS WE EXPECT IT TO HAVE A MAXIMUM AT $\varphi(x) = 0$ AND THAT IT IS SYMMETRIC. HENCE, UP TO THE LOWEST ORDERS, WE CAN EXPAND



$$w(\varphi) = w_0 - w_2 \varphi^2 - w_4 \varphi^4$$ (ENTROPY DENSITY OF THE FIELD $\varphi(x)$)

THE MINUS SIGN ARISES $(w_2 > 0)$ BECAUSE $w(\varphi)$ CANNOT GROW AROUND $\varphi(x) = 0$.

N.B.: THE SYMMETRY OF $H[\sigma]$ HAS NOTHING TO DO WITH THE FACT THAT $\varphi = 0$ IS THE SYMMETRY POINT OF $w(\varphi)$, SINCE THIS JUST COMES FROM THE FACT THAT $\pm\varphi$ CAN BE OBTAINED BY INVERTING UP AND DOWN SPINS IN EACH MICROSCOPIC CONFIGURATION.

NOTE: $\pm\varphi$ MIGHT AS WELL HAVE DIFFERENT ENERGY, $H(\pm\varphi)$

WITH THIS SUPPOSED FUNCTIONAL FORM FOR $w(\varphi)$,

$$\mathcal{Z}(\beta=0) = \int \mathcal{D}\varphi(x)\, e^{\int d^dx\,[-w_2\varphi^2(x) - w_4\varphi^4(x)]}$$

$$\Omega[\varphi] = \int d^dx\,[-w_2\varphi^2(x) - w_4\varphi^4(x)]$$

* ALL THIS IS DERIVED AND WORKS IN THE SIMPLE SCENARIO OF $\beta=$

LET'S CONSIDER NOW $T<\infty$, $\beta>0$. REINSTATE THE ENERGY:

$$\mathcal{Z}(\beta>0) = \int \mathcal{D}\varphi(x)\, e^{-\beta W(\varphi)}\, e^{\Omega(\varphi)}$$

$\Omega(\varphi)$ IS ANALOGOUS TO AN ENTROPIC WEIGHT OF THE MESOSCOPIC FIELD CONFIGURATION; $W(\varphi)$ IS ANALOGOUS TO THE ENERGETICAL WEIGHT OF THE FIELD. THEN WE NEED TO WORK WITH IT AT THIS STAGE:

$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j \qquad \text{(ALIGNMENT)}$$

THE MAIN DIFFICULTY IN WORKING WITH THE ISING HAMILTONIAN COMES FROM THE CONSTRAINT $\sigma_i = \pm 1$ (EVEN IN M.F.). WE'LL SEE HOW THIS IS TAKEN INTO ACCOUNT IN LANDAU-GINZBURG THEORY

LANDAU-GINZBURG HAMILTONIAN

LET'S START FROM THE ISING HAMILTONIAN,

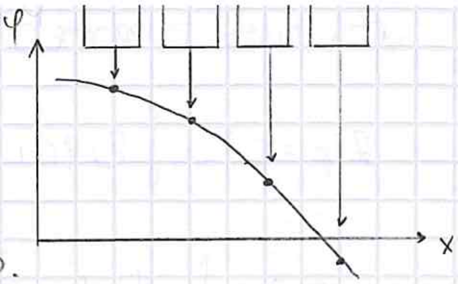
$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j = J\sum_{\langle ij\rangle}(\sigma_i-\sigma_j)^2 - \text{CONST.}$$

(UPON A RESCALING OF J). WE CAN NEGLECT THE CONSTANT AND TAKE THE CONTINUOUS LIMIT OF H:

$$H \simeq Ja^2\sum_{\langle ij\rangle}\frac{(\sigma_i-\sigma_j)^2}{a^2} \longrightarrow \frac{1}{2}\hat{a}^2\int d^dx\,(\underline{\nabla}\varphi(x))^2$$

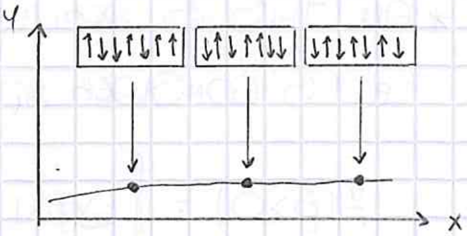
$\langle ij\rangle$ WAS A SUM OVER NEAREST NEIGHBOURS WITH LATTICE SPACING a, NOW $\hat{a}$ IS A CONSTANT THAT TAKES CARE OF THE DIMENSIONS AND J

THE TERM $(\underline{\nabla}\varphi)^2$ RESULTING FROM THE CONTINUOUS LIMIT OF THE HAMILTONIAN IS SOMETHING THAT DISCOURAGES DISTORSIONS OF THE FIELD AND ENHANCES ITS SMOOTHNESS.



IT COMES FROM THE ALIGNMENT MECHANISM BETWEEN ADJACENT BLOCKS:

$(\underline{\nabla}\varphi)^2$ → INTER-BLOCK INTERACTION



SINCE SPINS DO ALIGN EVEN WITHIN THE BLOCK, WE NEED TO QUANTIFY THIS THROUGH SOME TERM WHICH ENHANCES HIGH VALUES OF φ AND DISCOURAGES A SITUATION LIKE THAT IN THE GRAPH ABOVE, WHERE $\varphi(x)$ IS VERY SMOOTH BUT VERY SMALL. WE ADD THE SIMPLEST REGULAR TERM OF THIS KIND, THAT IS

$-b\varphi^2(x)$ → ALIGNMENT WITHIN BLOCKS

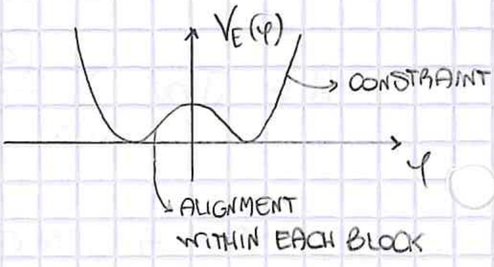
FINALLY, WE NEED TO CONSIDER THE CONSTRAINT $\sigma_i = \pm 1$; AS A RESULT, $\varphi(x)$ OVER A FINITE ν IS LIMITED. IF WE STOP AT THE φ^2 TERM, THE GROUND STATE WOULD BE $|\varphi| \to \infty$, SO WE NEED TO INTRODUCE A CONSTRAINT. WHAT WE GET IS

$$w(\varphi) \sim \int d^d x \left[\frac{1}{2}\hat{\alpha}^2 (\underline{\nabla}\varphi)^2 - b\varphi^2(x) + c\varphi^4(x) \right]$$

C MUST BE A GEOMETRICAL CONSTANT WHICH DEPENDS ON THE CONSTRAINT ON φ; THIS MAY BE LINKED TO THE ORIGINAL CONSTANTS OF THE MACROSCOPIC VARIABLES, THE COARSE GRAINING PROCEDURE, ETC.. ON THE CONTRARY, b HAS A PHYSICAL MEANING, SINCE IT COMES FROM THE ALIGNMENT WITHIN THE BLOCK.

$J\uparrow \quad b\uparrow$

$J\downarrow \quad b\downarrow$



WE CAN DRAW AN ENERGETIC POTENTIAL FOR φ.

THEN

$$\mathcal{Z}(\beta) = \int \mathcal{D}\varphi(x)\, e^{-\beta W(\varphi)} e^{\Omega(\varphi)}$$

$$= \int \mathcal{D}\varphi(x)\, e^{-\beta \int d^dx \left[\frac{1}{2}\hat{\alpha}^2 (\nabla\varphi)^2 - b\varphi^2(x) + c\varphi^4(x)\right]} e^{\int d^dx \left[-\omega_2\varphi^2(x) - \omega_4\varphi^4(x)\right]}$$

$$= \int \mathcal{D}\varphi(x)\, e^{-\int d^dx \left\{\frac{1}{2}\beta\hat{\alpha}^2(\nabla\varphi)^2 - (\beta b - \omega_2)\varphi^2(x) + (\beta c + \omega_4)\varphi^4(x)\right\}}$$

NOTE THAT IT'S CONVENIENT TO REWRITE AN EFFECTIVE HAMILTONIAN THAT IS DIMENSIONLESS (β IS INCLUDED), SINCE THERE IS NO WAY TO DISENTANGLE THE NEW HAMILTONIAN AFTER THE COARSE GRAINING: SMALL FREE ENERGIES DEPENDING ON β ALWAYS APPEAR WHEN WE COARSE GRAIN. MOREOVER,

$\omega_4 + \beta c > 0$

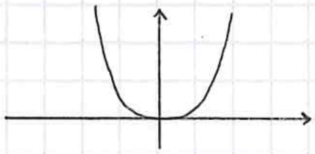
$\beta b - \omega_2$ CAN CHANGE SIGN!

IN THIS COMBINATION OF SIGNS THERE IS THE WHOLE THEORY OF CRITICAL PHENOMENA. LET'S CALL <u>BARE MASS</u> THE QUANTITY

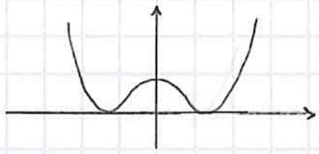
$\mu^2 \equiv \omega_2 - \beta b$

WE CAN DISTINGUISH TWO REGIMES:

$\begin{cases} T\uparrow & \beta\downarrow \\ J\downarrow & b\downarrow \end{cases}$ (HIGH T OR WEAK INTERACTION): $\mu^2 > 0$



$\begin{cases} T\downarrow & \beta\uparrow \\ J\uparrow & b\downarrow \end{cases}$ (LOW T OR STRONG INTERACTION): $\mu^2 < 0$



SO WE EXPECT SOME SECOND ORDER PHASE TRANSITION AT T_c s.t. $\mu^2 \equiv 0$ (BARE LEVEL). OUR HAMILTONIAN BECOMES

$$H_{LG} = \int d^dx \left\{\frac{1}{2}\hat{\alpha}^2(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2(x) + \frac{\lambda}{4!}\varphi^4(x)\right\}$$

$$P(\varphi(x)) = \frac{1}{\mathcal{Z}} e^{-H_{LG}(\varphi)} \qquad \mathcal{Z} = \int \mathcal{D}\varphi(x)\, e^{-H_{LG}(\varphi)}$$

NOTE WE DIDN'T HAVE TO IMPOSE THE PRESENCE OF A 2^{ND} ORDER PHASE TRANSITION TO BUILD THE THEORY: THE ENERGY - ENTROPY COMPETITION GIVING RISE TO THE EMERGENCE OF 2 EQUIVALENT MINIMA UNDER PROPER CONDITIONS RESULTS FROM THE WAY WE BUILT THE ENTROPY DENSITY, WHICH DOESN'T IMPLY ANY AD HOC ASSUMPTION.

- LEZIONE 10.04.19

- RICAPITOLANDO

$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$ $\qquad \sigma_i = \pm 1$ (FERROMAGNET ISING)

$a \ll l \ll L$

PASSIAMO, CON UN COARSE GRAINING,

$\{\sigma_i\}$ (MICROSCOPICHE) $\rightarrow$ $\varphi(x) = \frac{1}{v_x} \sum_{i \in v_x} \sigma_i$ (MESOSCOPICO)

$\hookrightarrow m = \frac{1}{V} \sum_i \sigma_i$ E' MAC

SE $P(\sigma) = \frac{1}{Z} e^{-\beta H}$, QUANTO VALE $P(\varphi)$?

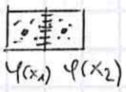
$$P(\varphi) = \frac{1}{Z} e^{-H_{LG}(\varphi)} = \frac{1}{Z} e^{-H_{LG}(\varphi; \alpha(\beta), \mu(\beta), \lambda(\beta))}$$

NON HA DAVANTI IL β, E' GIA' NEI COEFFICIENTI. H_{LG} NON E' UN'ENERGIA, MA UNA ENERGIA LIBERA.

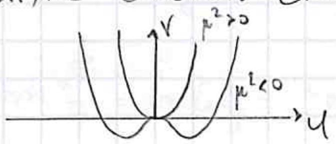
$$H_{LG}(\varphi) = \int d^d x \left\{ \frac{1}{2} \alpha^2 (\underline{\nabla}\varphi)^2 + \underbrace{\mu^2 \varphi^2 + \lambda \varphi^4}_{V(\varphi)} \right\}$$

- IL GRADIENTE E' L'EREDE MORALE DI $\qquad (\underline{\nabla}\varphi)^2 \rightleftarrows \frac{1}{2} J \sum_{\langle ij \rangle} (\sigma_i - \sigma_j)^2$

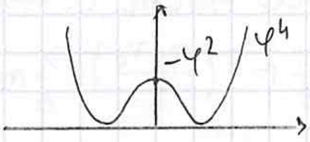
E' PERO' UN GRADIENTE TRA I BLOCCHI.

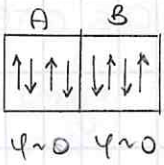


- $\lambda \varphi^4$: ALL ENERGIES MUST BE BOUNDED!
- $\mu^2 \varphi^2$: E' IL PIU' INTERESSANTE, PERCHÉ PUO' CAMBIARE SEGNO. CIO' BASTA PER COSTRUIRE UNA HAMILTONIANA AB INICIO.



MA UN ALTRO MOTIVO E' CHE μ^2 CONTIENE IL TRADE OFF TRA ENTROPIA ED ENERGIA.





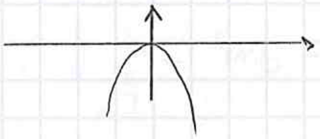
NEL DISEGNO, A E B SONO CONFIGURAZIONI SFAVORITE, MA CIO' NON SI VEDE IN $P(\varphi) = e^{-W(\varphi)}$.

METTO A MANO UN $-\varphi^2$, i.e. "stay off $\varphi = 0$" (E' IL FERROMAGNETISMO DENTRO AI BLOCCHI):

$W \sim -b(J)\varphi^2 + \lambda\varphi^4$

MA CI SONO MOLTI MODI DI OTTENERE CONFIGURAZIONI CON $\varphi = 0$:

$P(\varphi) = e^{S(\varphi)}$ $\qquad S \simeq -w_2 \varphi^2 - w_4 \varphi^4$

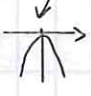


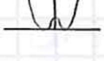
ORA LE METTO INSIEME E, MORALMENTE,

$$H \sim \beta F \sim \beta(E - TS) = \beta E - S$$

$$\sim \beta(-b\varphi^2 + \hat{\lambda}\varphi^4) - (-w_2\varphi^2 - w_4\varphi^4)$$

$$= \underbrace{\varphi^4(\beta\hat{\lambda} + w_4)}_{\equiv \lambda > 0} + \varphi^2(w_2 - \beta b(J))$$





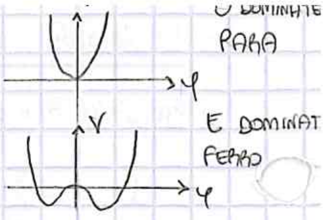
QUINDI IN EFFETTI μ^2 CAMBIA SEGNO:

$$\mu^2 \equiv \left(w_2 - \beta b(J) \right) \qquad (\leftrightarrow E - TS)$$

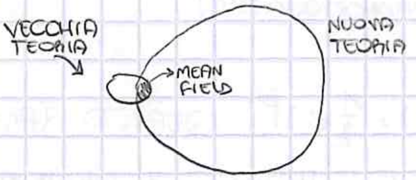
A $\mu = 0$ HO UNA TRANSIZIONE DI FASE.

T↑ β↓
J↓ b↓

T↓ β↑
J↑ b↑



E CHI MI DICE CHE NON CI SIANO TERMINI $\varphi^6, \varphi^8 \dots$? CE LO DIRÀ IL GdR.



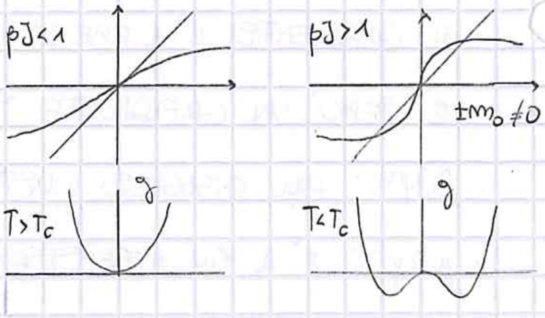
· CAMPO MEDIO (FULLY CONNECTED)

$$H^{fc} = -\frac{J}{N} \sum_{ij} \sigma_i \sigma_j = -\frac{J}{N} \left(\sum_i \sigma_i \right)^2$$

$$m = \frac{1}{N} \sum_i \langle \sigma_i \rangle \quad \Rightarrow \quad g(m) = -\frac{1}{2} J m^2 + \frac{m}{\beta} \operatorname{Ath}(m) - \frac{1}{\beta} \ln \operatorname{Ch} \operatorname{Ath}(m)$$

$$\frac{\partial g}{\partial m} \equiv 0 \quad \Rightarrow \quad \operatorname{Ath}(m) = \beta J m, \quad m = \operatorname{Th}(\beta J m)$$

$$(\beta J)_c = 1 \rightarrow T_c = J$$



PER $T \simeq T_c$, ESPANDO PER $m << 1$ $\left(\operatorname{Ath}(m) = \frac{1}{2} \ln \frac{1+m}{1-m} \right)$

$$\operatorname{Ath} m \simeq m + \frac{1}{3} m^3$$

$$m + \frac{1}{3} m^3 = \beta J m = \frac{T_c}{T} m \quad \Rightarrow \quad m \left(\frac{T - T_c}{T} \right) + \frac{1}{3} m^3 = 0$$

CHE OTTENGO ANCHE ESPANDENDO

$$g(m) = \frac{1}{2} m^2 \left(\frac{T - T_c}{T} \right) + \frac{1}{6} m^4 \qquad \frac{\partial g}{\partial m} \equiv 0$$

CHE È PRATICAMENTE LA $V(\varphi)$ DI L.G.! STICAZZI CHE MANCANO $\varphi^6, \varphi^8, \dots$

CHIARAMENTE MANCA IL GRADIENTE:

$$-\sum_{\langle ij \rangle} \sigma_i \sigma_j \simeq \sum_{\langle ij \rangle} (\sigma_i - \sigma_j)^2$$

$$\downarrow \text{MF}$$

$$\sum_{ij} \sigma_i \sigma_j \sim m^2$$

SI NOTI CHE PER OTTENERE UN FERROMAGNETE DA

$$g(m) = -Jm^2 - \frac{1}{\beta} (\dots)$$

CI VUOLE CHE IN (...) CI SIA UN m^2. È LA BUONA VECCHIA (E − TS).

· APPROSSIMAZIONE DI LANDAU (MEAN FIELD, TREE LEVEL, FREE THEORY)

PRENDIAMO L.G. E BUTTIAMO VIA IL GRADIENTE. STIAMO CALCOLANDO

Z E FACCIAMO UN PUNTO DI SELLA, i.e. φ_0 CHE MINIMIZZI $H(\varphi)$, ANCHE SE IN

$$Z = \int D\sigma \, e^{-H(\varphi)} \qquad H(\varphi) = \int d^dx \left\{ \frac{1}{2}\alpha (\nabla\varphi)^2 + \underbrace{\mu^2\varphi^2 + \lambda\varphi^4}_{V(\varphi)} \right\}$$

NON C'È UN PARAMETRO "GRANDE".

INTANTO φ_0 SARÀ UN CAMPO COSTANTE: MORALMENTE, CERCO IL MINIMO DI $V(\varphi)$. ME

$$\varphi(x) = \frac{1}{V_x} \sum_{i \in V_x} \sigma_i \equiv \varphi_0 \equiv \frac{1}{V} \sum_i \sigma_i = m$$

NOTA: i.e. QUESTA COSTANTE NON PUÒ CHE ESSERE m STESSO.

$$Z = e^{-H(\varphi_0)} \quad \Rightarrow \quad \beta g_L = \mu^2 \varphi_0^2 + \lambda \varphi_0^4$$

COME IN CAMPO MEDIO. INOLTRE*

$$\mu^2 \simeq \frac{T - T_c^{MF}}{T_c^{MF}}$$

T_c DI M.F.!

QUESTO GIÀ CI DICE CHE, SE RIMETTIAMO $(\nabla\varphi)^2$, AVREMO SÌ UNA TRANSIZION

MA ALLA $T_c^{MF} \equiv T_0$. INVECE $T_c^{VERA} \neq T_0$.

*NOTA: IN REALTÀ $\mu^2 = \frac{T - T_c}{T} = \frac{\beta_c - \beta}{\beta_c}$.

- ESERCIZIO

WHAT IS THE PHYSICAL SIGNIFICANCE (IF ANY) OF THE LANDAU APPROXIMATION?

DIPENDE DA QUANTO GRANDE PRENDO IL COARSE GRAINING. FINCHÉ È PIÙ GRANDE DELLA LUNGHEZZA DI CORRELAZIONE, $v^{1/d} \gg \xi$, LANDAU FUNZIONA, ALTRIMENTI NO. PERCIÒ SIGNIFICA

$$\xi \ll \underset{\uparrow}{v^{1/d}}$$

(ESPERIMENTO CON IL FOGLIO E LA MAGLIETTA)

MA ξ È LA SCALA ENTRO CUI HO VARIAZIONI: $\varphi(x) = \frac{1}{V_x} \sum_{i \in V_x} \sigma_i$.

È QUESTO L'UNICO SIGNIFICATO FISICO DEL CAMPO MEDIO. RICORDIAMO CHE

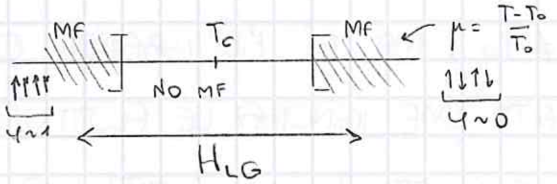
$$G(r) = \frac{1}{r^d} e^{-r/\xi}$$

NOTA: SE $\xi \ll v^{1/d}$, IN PRATICA $\varphi(x)$ È PIÙ O MENO COSTANT

SE $\xi \gg L$, NON SIGNIFICA CHE IL SISTEMA È TUTTO OMOGENEO, MA CH HO FLUTTUAZIONI SU TUTTE LE SCALE: ξ È LA TAGLIA MASSIMA DELLE FLUTTUAZIONI.

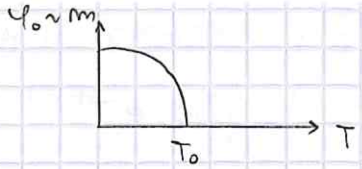
NOTA: i.e. DEI DOMINI ORIENTATI.

MA È PROPRIO PER $T \simeq T_c$ CHE $\xi \sim \infty$: È PER QUESTO CHE PER $T \simeq T_c$ IL CAMPO MEDIO NON FUNZIONA (ANCHE SE IL FIXING DEI PARAMETRI LO ABBIAMO FATTO LÌ).



IL CRITERIO DI GINZBURG CI DIRÀ QUANTO È GRANDE LA FINESTRA (IN PRATICA, INFATTI, IO NON CONOSCO IL COARSE GRAINING).

- <u>ESPONENTI CRITICI</u> (LANDAU APPROX.)



★ $T < T_0$, $\mu^2 < 0$

$$\frac{\partial g}{\partial \varphi_0} = 0 \quad \overset{(I)}{\downarrow}\Rightarrow \quad \varphi_0^2 = -\frac{6\mu^2}{\lambda} > 0$$

$$\mu^2 \simeq T - T_0 \quad \Rightarrow \quad \varphi_0 \sim (T_0 - T)^{\textcircled{1/2}} \rightarrow \beta_{MF} = \beta_L$$

★ $\varphi_0 = \varphi_0(h)$, @ T_0

$$\beta g = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4!}\lambda\varphi_0^4 - \beta h \varphi_0 \qquad (I)$$

$$\frac{\partial g}{\partial \varphi_0} = 0 \quad \Rightarrow \quad \mu^2\varphi_0 + \frac{1}{6}\lambda\varphi_0^3 = \beta h \qquad (II)$$

$$@ T_0,\ \mu^2 = 0 \quad \Rightarrow \quad \varphi_0^3 \sim h, \quad \varphi_0 \sim h^{\textcircled{1/3}} \rightarrow \frac{1}{\delta}, \quad \delta_{MF} = \delta_L = 3$$

★ $\chi = \dfrac{\partial \varphi_0}{\partial h}$

NOTA: LA PROSSIMA SI OTTIENE DERIVANDO LA (II) IN ∂_h.

$$\mu^2\chi + \frac{1}{2}\lambda\varphi_0^2\chi = \beta \quad \Rightarrow \quad \chi = \frac{\beta}{\mu^2 + \frac{1}{2}\lambda\varphi_0^2}$$

SE $T > T_c$,

$$\varphi_0 = 0 \quad \Rightarrow \quad \chi \sim \frac{1}{\mu^2} = \frac{1}{(T-T_0)}^{\textcircled{1}} \nearrow \gamma$$

SE $T < T_c$,

$$\varphi_0 = -\frac{6\mu^2}{\lambda} \ (\mu^2 < 0!) \quad \Rightarrow \quad \chi = \frac{\beta}{-2\mu^2} \sim \frac{1}{T_0 - T}$$

LANDAU NON CI DICE NIENTE SU η E ν:

$$G(r) = \frac{1}{r^{d-2+\eta}} e^{-r/\xi} \qquad \xi \sim (T - T_c)^{-\nu}$$

PERCHÉ NON SA NIENTE DELLO SPAZIO.

- <u>ESERCIZIO</u>

$$H_{LG} = \int d^dx \left\{ \frac{1}{2}\alpha^2 (\underline{\nabla}\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4!}\varphi^4 \right\}$$

$T_c \neq T_0$, MA È PIÙ GRANDE O PIÙ PICCOLA? $T_c < T_0$.

INFATTI MF IGNORA LE FLUTTUAZIONI DENTRO I BLOCCHI DI TAGLIA ξ, E LE FLUTTUAZIONI DISTRUGGONO L'ORDINE.

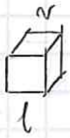
· LESSON 12.04.19

· RECAP OF LANDAU APPROX

$\varphi(x) \simeq \varphi_0$ MINIMUM OF H_{LG}

$$\mathcal{Z} = \int \mathcal{D}\varphi(x)\, e^{-H_{LG}(\varphi)} \simeq e^{-H(\varphi_0)}$$

$$\varphi(x) = \frac{1}{\mathcal{V}_x} \sum_{i \in \mathcal{V}_x} \sigma_i$$



WITH REFERENCE TO THE ATTACHED FIGURE,

•) $T > T_c$ NOTE: SEE ALSO BINNEY, P. 18-19.

UNIFORM GRAY (SALT AND PEPPER). FLUCTUATIONS WRT GREY HAVE ξ SMALL ($\xi \sim a$),

$\varphi - \langle\varphi\rangle$, $\sigma - \langle\sigma\rangle$

IN EACH BLOCK $\mathcal{V} \rightarrow \varphi_0 \simeq 0 \ \forall x$

•) $T < T_c$

UNIFORM MAGNETIZED (WHITE). FLUCTUATIONS WRT TO WHITE HAVE ξ SMALL ($\xi \sim a$),

$\varphi_0 \simeq +1 \ \forall x$

•) $T \simeq T_c$

NOT UNIFORM,

$\varphi(x_1) \neq \varphi(x_2) \neq \varphi_0$

· GAUSSIAN (FREE) THEORY

$$\mathcal{Z}_G = \int \mathcal{D}\varphi(x) \exp\left\{ - \int d^dx \left(\frac{1}{2}\alpha^2 (\underline{\nabla}\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 \right) \right\}$$

WE THROW AWAY WHAT WE'RE NOT ABLE TO DEAL WITH, i.e. $\lambda\varphi^4$

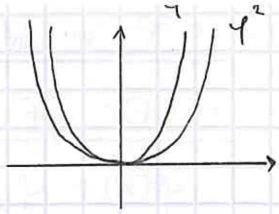
WHY?

1) IT'S OFTEN THE BEST YOU CAN DO!

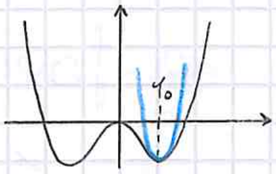
2) IT IS VERY INSTRUCTIVE.

3) IT'S THE BUILDING BLOCK OF THE DIAGRAMMATIC EXPANSION.

4) IT'S REASONABLE AT $T >> T_c$ AND $T << T_c$.

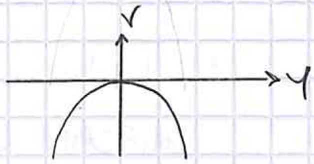


CLOSE TO T_c, $\mu \simeq 0$ SO THE $\sim \varphi^4$ TERM CANNOT BE DISCARDED. BUT EVEN BELOW T_c, YOU HAVE SYMMETRY BREAKING AND YOU CAN USE



$$\psi(x) = \varphi(x) - \varphi_0$$

$$\hat{\mu}^2 = \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi_0} \qquad \rightarrow \qquad \hat{\mu}^2 \psi^2(x)$$



BE CAREFUL!

$T \gtrsim T_0$

(CRITICAL TEMPERATURE OF LANDAU THEORY). YOU CAN STUDY THE CASE $T << T_c$, BUT YOU CAN'T GO ACROSS T_0.

LANDAU AND GAUSSIAN HAVE THE SAME CRITICAL TEMPERATURE

$T_0 : \mu^2(T = T_0) = 0$

- HOMEWORK

FORGET ABOUT FERROMAGNETS AND FORGET THAT $\mu^2 = \mu^2(\beta)$, i.e.

α, μ FIXED PARAMETERS

ASSUME THAT H IS A REAL POTENTIAL ENERGY.

WHAT PHYSICAL SYSTEM IS AN ACTUAL REALIZATION OF THE GAUSSIAN THEORY?

WRITE A MICROSCOPICAL MODEL OR GIVE A PICTURE.

★ ADD A SOURCE (FIELD)

$$H \rightarrow H - h \sum_i \sigma_i \qquad \rightarrow \qquad \frac{1}{\beta} \frac{\partial \ln Z}{\partial h} = \frac{1}{N} \sum_i \langle \sigma_i \rangle$$

$$H \rightarrow H - \sum_i h_i \sigma_i \qquad \rightarrow \qquad \frac{1}{\beta} \frac{\partial \ln Z}{\partial h_i} = \langle \sigma_i \rangle , \quad \frac{1}{\beta} \frac{\partial^2 \ln Z}{\partial h_i \partial h_j} = \langle \delta\sigma_i \delta\sigma_j \rangle$$

SIMILARLY,

$$H_{LG}(\varphi) \rightarrow H_{LG}(\varphi) - \int d^d x \; j(x)\varphi(x)$$

$$\langle \delta\varphi(x)\delta\varphi(y)\rangle = \frac{\delta}{\delta j(x)}\frac{\delta}{\delta j(y)} \ln Z \qquad \text{(CONNECTED)}$$

BUT IT'S GAUSSIAN, HENCE

$$\langle\varphi\rangle = 0 \rightarrow \langle\delta\varphi(x)\delta\varphi(y)\rangle = \langle\varphi(x)\varphi(y)\rangle$$

* TECHNICAL NOTE

$$\alpha^2(\underline{\nabla}\varphi)^2$$

WE WANT TO RESCALE IT (IT'S THE RELIC OF J) SO THAT $\alpha=1$

$$\int d^d x \; \alpha^2(\underline{\nabla}\varphi)^2 + \mu^2\varphi^2 = \int d^d x \; (\underline{\nabla}(\alpha\varphi))^2 + \mu^2\varphi^2$$

$$\alpha\varphi \equiv \varphi'$$

$$\int d^d x \; (\underline{\nabla}\varphi')^2 + \underbrace{\frac{\mu^2}{\alpha^2}}_{\rightarrow \mu'^2}\varphi'^2 \qquad \text{(FIELD NORMALIZATION)}$$

AT THIS LEVEL,

$$\mu'^2 = 0 \quad \leftrightarrow \quad \mu^2 = 0$$

SO WE'LL JUST DROP THE α AND THE PRIME FOR THE NEXT COUPLE OF LECTURES.

· GO IN FOURIER!

$$Z = \int \mathcal{D}\varphi(x) \; \exp\left\{-\int d^d x \; \frac{1}{2}(\underline{\nabla}\varphi)^2 + \frac{1}{2}\mu^2\varphi^2(x) - j\varphi\right\}$$

INTEGRATING BY PARTS,

$$\int dx \; \partial\varphi\,\partial\varphi = -\int dx \; \varphi\,\partial\partial\varphi$$

$$Z = \int \mathcal{D}\varphi(x) \exp\left\{-\frac{1}{2}\int d^d x \; \varphi(x)\left[-\nabla^2+\mu^2\right]\varphi(x)\right\} \exp\left(\int d^d x \; j(x)\varphi(x)\right)$$

NOW

$$\varphi(x) = \int d^dk\, e^{-ikx}\, \varphi(k) \qquad\qquad [\varphi(x)] \neq [\varphi(k)]$$

SO THAT

$$\frac{1}{2}\int d^dx\, \varphi(x)\left[-\nabla^2+\mu^2\right]\varphi(x) = \frac{1}{2}\int d^dk\, d^dq\, d^dx\, \varphi(k)\varphi(q)e^{-iqx}(k^2+\mu^2)e^{-ikx}$$

$$= \frac{1}{2}\int d^dk\, d^dq\, \varphi(k)\varphi(q)\, \delta^{(d)}(k+q)\, (k^2+\mu^2)$$

SIMILARLY,

$$\int d^dx\, j(x)\varphi(x) = \int d^dx\, d^dk\, d^dq\, j(q)\, \varphi(k)\, e^{-ikx}e^{-iqx}$$

$$= \int d^dk\, d^dq\, j(q)\, \varphi(k)\, \delta^{(d)}(k+q) = \int d^dk\, j(-k)\varphi(k)$$

AND WE CAN REWRITE

$$\underline{Z = \int D\varphi(x)\, \exp\left\{-\frac{1}{2}\int d^dk\, d^dq\, \varphi(k)\, \delta^{(d)}(k+q)\, (k^2+\mu^2)\varphi(q) + \int d^dk\, j(-k)\varphi(k)\right\}}$$

IN GENERAL, A GAUSSIAN INTEGRAL HAS SOLUTION

$$Z = \int d\varphi\, e^{-\frac{1}{2}\sum_{ij}\varphi_i A_{ij}\varphi_j + \sum_i b_i\varphi_i} = Z(0)\, e^{\frac{1}{2}\sum_{ij} b_i (A^{-1}_{ij})\, b_j}$$

SO MORALLY WE SUBSTITUTE

$$\sum_{ij} \to \int d^dk\, d^dq \qquad\qquad A_{ij} \to \delta^{(d)}(k+q)\, (k^2+\mu^2)$$

$$\varphi_i \to \varphi(k) \qquad\qquad b_i \to j(k)$$

ONLY, THE CONSTANT $Z(0)$ WILL BE INFINITE. WE FIND

$$\underline{Z_{GAUSS} = Z(0)\, \exp\left\{\frac{1}{2}\int d^dk\, d^dq\, j(k)\, \delta^{(d)}(k+q)\, (k^2+\mu^2)^{-1} j(q)\right\}} \qquad (I)$$

RIGHT? LET'S TRY DIMENSIONAL ANALYSIS:

$$\int d^dx\, (\underline{\nabla}\varphi)^2 = 1 \qquad \to \qquad [\varphi(x)] = x^{\frac{2-d}{2}}$$

$$\varphi(k) = \int d^dx\, \varphi(x) \qquad \to \qquad [\varphi(k)] = x^{\frac{d+2}{2}} = k^{-\frac{d+2}{2}}$$

SO WE FOUND

$$[\varphi_x \varphi_x] = \frac{1}{x^{d-2}} \qquad\qquad [\varphi_k \varphi_k] = \frac{1}{k^{d+2}}$$

AND DIMENSIONAL ANALYSIS IS OK. IT'S ACTUALLY CORRECT:

$$\delta^{(d)}(k_i + k_j) = \delta_{ij} \frac{1}{(d^d k)} \qquad \longrightarrow \qquad \delta^{(d)}(\,)^{-1} = \delta_{ij}\,(d^d k)$$

CHECK IT AS AN EXERCISE.

- GAUSSIAN PROPAGATOR

NOTE: TO SEE THIS, USE k' AND q' IN (I). THE $j($

IS NOT CAUSED BY THE FT CONVENTION.

$$\langle \varphi(k)\, \varphi(q) \rangle = \frac{\delta^2 \ln Z}{\delta j(-k)\, \delta j(-q)} = \frac{\delta^{(d)}(k+q)}{k^2 + \mu^2} \equiv \delta^{(d)}(k+q)\; G_0(k)$$

WHERE WE DEFINED THE GAUSSIAN OR FREE PROPAGATOR (IN FOURIER

$$G_0(k) = \frac{1}{\underset{\substack{\swarrow \\ \text{FROM} \\ (\nabla\varphi)^2,\ \text{FERROM. ALIGNMENT}}}{k^2} + \underset{\substack{\searrow \\ \text{FROM } \mu^2\varphi^2}}{\mu^2}}$$

IT HAS THE SAME FORM FOR ALL d, AND THIS IS WHY THE FOURIER VERSION IS MORE FAMOUS.

REMARK:

$$\langle \varphi(k)\, \varphi(-k) \rangle = \frac{V}{k^2 + \mu^2}$$

NOTE: i.e. $\delta^{(d)}(0) = V$.

NOW WE CAN CALCULATE

$$\langle \varphi(x)\, \varphi(y) \rangle = \int d^d q\, d^d k\; e^{-ikx}\, e^{-iqy} \langle \varphi(k) \varphi(q) \rangle$$

$$= \int d^d q\, d^d k\; e^{-ikx}\, e^{-iqy}\, \delta^{(d)}(k+q)\, G_0(k) = \int d^d k\; e^{-ik(x-y)}\, G_0(k)$$

$$\equiv G_0(|x-y|)$$

SO WE FOUND

$$G_0(r) = \int d^d k\; e^{-ikr}\, G_0(k)$$

WHICH MEANS

$$\langle \varphi(k)\varphi(q)\rangle = \delta^{(d)}(k+q)\, G_0(k) \qquad \sim \frac{1}{k^{d+2}}$$

$$\langle \varphi(x)\varphi(x+r)\rangle = G_0(r) \qquad \sim \frac{1}{x^{d-2}}$$

- CASE $d=3$

$$G_0(r) = \int d^3k \, \frac{e^{-ikr}}{k^2+\mu^2} \overset{(*)}{=} \int dk\, k^2 \int_0^{\pi} d\theta \, \sin\theta \, \frac{e^{-ikr\cos\theta}}{k^2+\mu^2}$$

$$= \int dk \, \frac{k^2}{k^2+\mu^2} \int_{-1}^{1} du \, e^{-ikru} = \int dk \, \frac{k^2}{k^2+\mu^2} \, \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$= \int_{\mathbb{R}} dk \, \frac{k}{k^2+\mu^2} \, \frac{1}{ir} \, e^{ikr} \underset{*}{=} 2\pi \, \frac{e^{-\mu r}}{r}$$

(*) NOTE: NONTRIVIAL, IT MEANS WE'RE CHOOSING THE $\hat{z}$ AXIS IN $\underline{k}$ SPACE SO THAT IT COINCIDES WITH $\underline{r}$.

(WHICH YOU CAN EASILY CHECK, IT'S A COMPLEX INTEGRAL). WE FOUND

$$\underline{G_0(r) = 2\pi \, \frac{e^{-\mu r}}{r}} \qquad d=3$$

AND THE CORRELATION LENGTH IS SIMPLY $\xi = \frac{1}{\mu}$. HENCE

$$\mu^2 = 0 \rightarrow (\xi \rightarrow \infty, \; T \rightarrow T_0)$$

* NOTE: SOME MINOR MISTAKES. THE INTEGRAL GIVES $\frac{\pi}{r} e$ IT'S $\frac{d^3k}{(2\pi)^3}$ AND $\int d\varphi = 2\pi$, SO $G_0(r) = \frac{1}{4\pi r} e^{-\mu r}$.

EVEN IN GAUSSIAN THEORY WE HAVE A PHASE TRANSITION.

WE COULD HAVE ARGUED FROM THE BEGINNING THAT, BY DIMENSIONAL ANALYSIS,

$$[\mu^2] = \frac{1}{x^2} \qquad \rightarrow \qquad \mu^2 = \frac{1}{\xi^2}$$

- GENERAL CASE

$$G_0(r) = \int d^dk \, \frac{e^{-ikr}}{k^2+\mu^2} = \frac{1}{\mu^2} \int d^dk \, \frac{e^{-ikr}}{\left(\frac{k}{\mu}\right)^2 + 1}$$

$$= \mu^{d-2} \underbrace{\int d^du \, \frac{e^{-iu(r\cdot\mu)}}{u^2+1}}_{\equiv f(r\cdot\mu)} \qquad \frac{k}{\mu} \equiv u$$

WHERE WE INTRODUCED A SCALING FUNCTION. WE FOUND

$$G_0(r) = \mu^{d-2} f(r \cdot \mu)$$

WHICH MEANS

$$G_0(r) = \frac{1}{r^{d-2}} (\mu \cdot r)^{d-2} f(r \cdot \mu) = \frac{1}{r^{d-2}} g(\mu \cdot r) = \frac{1}{r^{d-2}} g\left(\frac{r}{\xi}\right)$$

SO IN GENERAL, IN THE GAUSSIAN THEORY,

$$\underline{G_0(r) = \frac{1}{r^{d-2}} g\left(\frac{r}{\xi}\right)} \qquad \xi = \frac{1}{\mu}$$

WE'LL FIND THAT, WHEN WE ADD INTERACTION,

$$G(r) = \frac{1}{r^{d-2+\eta}} g\left(\frac{r}{\xi}\right)$$

$\eta \downarrow$ ANOMALOUS DIMENSION

NOTE: i.e. SOMETHING THAT WE CAN'T JUST RETRIEVE FROM DIMENSIONAL ANALYSIS

- THE CUTOFF

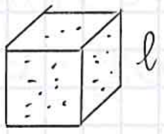
BEING MORE EXPLICIT,

$$G_0(r) = \int_{K_{MIN}}^{K_{MAX}} d^d K \, e^{-iKr} \, G_0(K)$$

$$K_{MIN} = \frac{1}{L} \xrightarrow[L \to \infty]{} 0$$

L: SYSTEM SIZE

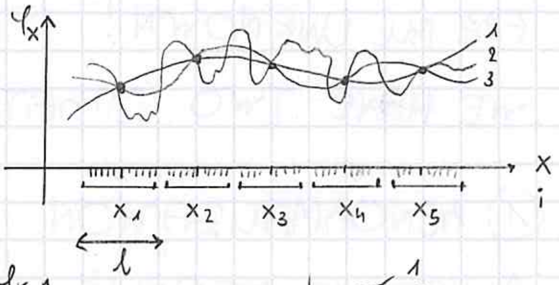
CAN WE SET $K_{MAX} = \infty$? NO!

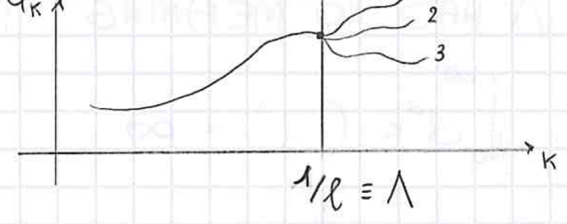


$$\varphi(x) = \frac{1}{v} \sum_{i \in v} \sigma_i = \frac{1}{\ell^d} \sum_{i \in v} \sigma_i$$

BY DEFINITION, WE HAVE NO INFORMATION FROM $\varphi(x)$ ABOUT SCALES $< O(\ell)$.

IN THE GRAPH, I KNOW NOTHING OF THE LINE INTERPOLATING AMONG THE BIG DOTS. IT TURNS OUT THIS IGNORANCE IS EASIER TO REPRESENT IN FOURIER SPACE: NO INFORMATION BEYOND $K_{MAX} = \frac{1}{\ell}$ ($\leftrightarrow$ WITHIN $\Delta x < \ell$).





WE THEN DEFINE THE CUTOFF

$$\Lambda = \frac{1}{l} = k_{MAX}$$

THIS IS THE ADVANTAGE OF SFT ON QFT: THE CUTOFF HAS A CLEAR PHYSICAL INTERPRETATION. IN QFT YOU'RE FORCED TO INTRODUCE IT BECAUSE SOME INTEGRALS DIVERGE AND NEED TO BE REGULARIZED.

* HOW LARGE IS Λ?

WE DON'T KNOW! BUT WE SAID

$$l \simeq m \cdot a \qquad\qquad \Lambda \sim \frac{1}{ma}$$

WHERE m IS AN INTEGER AND IT CAN DEPEND ON THE SIZE OF YOUR SYSTEM (m IS NOT 1, BUT $m \sim 100$ IS OK).

SINCE a IS THE LATTICE SPACING, Λ IS A BIG NUMBER, BUT IT'S FINITE.

* LANDAU - GINZBURG

$$\{\alpha^2, \mu^2, \lambda; \Lambda\}$$

BUT HOW CAN WE MAKE PREDICTIONS IF WE DON'T KNOW Λ? ACTUALLY, WE DON'T EVEN KNOW THE OTHER GUYS, AND

$$\{\alpha^2(\Lambda), \mu^2(\Lambda), \lambda(\Lambda); \Lambda\}$$

ARE ALL UNKNOWN!

WE HAVE TWO STRATEGIES:

① RENORMALIZATION (QFT)

Λ HAS NO MEANING: IT JUST SAVES US FROM

$$\int_0^\infty d^dk\,(\dots) = \infty \qquad \rightarrow \qquad \int_0^\Lambda d^dk\,(\dots) < \infty$$

BUT THEN Λ HAS TO BE ELIMINATED (AND YOU CAN'T JUST SEND $\Lambda \to \infty$).

IN ORDER TO DO SO, YOU MEASURE PHYSICAL QUANTITIES, SO THAT

$$\{\alpha^2(\Lambda), \mu^2(\Lambda), \lambda(\Lambda); \Lambda\} \to \{a^2, m^2, g\} \quad \text{(AND NO CUTOFF)}$$

WE'LL GO THROUGH IT, BUT THE IDEA IS

$$G(k;\Lambda) - G(\bar{k};\Lambda) = A(k;\bar{k})$$

WE'LL DO IT FOR 3 REASONS:

a) GENERAL CULTURE

b) WE WILL DERIVE USEFUL DIAGRAMS

c) AT THE END OF THE DAY, WE'LL GET THE SAME RESULT IN 2 WAYS

② RENORMALIZATION GROUP (RG, IN SFT)

WHY ELIMINATE Λ? THE ONLY PROBLEM IS THAT IT'S ARBITRARY

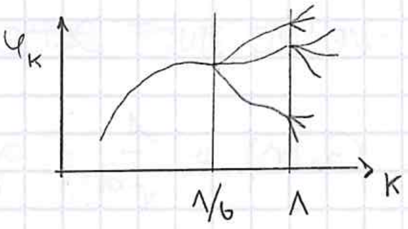
MAYBE YOU KNOW THE FUNCTIONS

$$\{\alpha^2(\Lambda), \mu^2(\Lambda), \lambda(\Lambda); \Lambda\}$$

AND MAYBE IT'S ENOUGH. WILSON SAYS WE SHOULD GET THE SAME PHYSICS IF WE USE Λ OR Λ/b, WHENCE

$$\{\alpha^2(\Lambda/b), \mu^2(\Lambda/b), \lambda(\Lambda/b); \Lambda/b\}$$

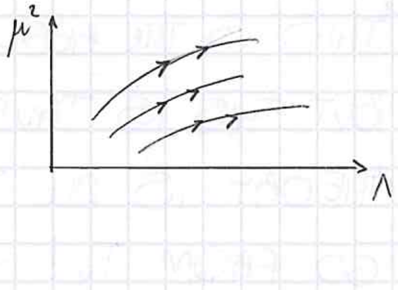
i.e. WE FIND A DIFFERENT SET OF CONSTANTS.



WE CAN EXTRACT INFORMATION FROM THE FLOW OF THE PARAMETERS.

WE'LL NEED THE SAME DIAGRAMS, BUT USING MOMENTUM SHELL YOU NEVER FIND ACTUAL INFINITIES:

$$\int^{\Lambda}, \int^{\Lambda/b}$$



· PUSHING Λ OUT OF THE GAUSSIAN THEORY

$$G_0(r) = \int^{\Lambda} d^d k \, \frac{e^{-ikr}}{k^2+\mu^2} \qquad\qquad \frac{k}{\mu} \equiv u$$

$$= \mu^{d-2} \int^{\Lambda/\mu} d^d u \, \frac{1}{u^2+1} \, e^{-iu(\mu \cdot r)} \qquad\qquad \mu \sim \frac{1}{\xi}$$

$$= \frac{1}{\xi^{d-2}} \int^{\Lambda\xi} d^d u \, \frac{e^{-iu(r/\xi)}}{u^2+1}$$

$$= \frac{1}{\xi^{d-2}} f\left(\frac{r}{\xi} ; \Lambda\xi\right) = \frac{1}{r^{d-2}} \left(\frac{r}{\xi}\right)^{d-2} f\left(\frac{r}{\xi} ; \Lambda\xi\right)$$

HENCE

$$G_0(r) = \frac{1}{r^{d-2}} g\left(\frac{r}{\xi} ; \Lambda\xi\right) \qquad\qquad g\left(\frac{r}{\xi} ; \infty\right) \text{ SMOOTH}$$

FOR INSTANCE, IN $d=3$

$$g \sim e^{-r/\xi}$$

IF $\Lambda\xi \to \infty$ AND ξ IS LARGE, i.e.

$$\xi >> \frac{1}{\Lambda} \sim l \qquad\qquad \begin{array}{l} \xi \sim \text{LONG CORRELATION} \\ l \sim \text{DETAILS} \end{array}$$

WHICH IS TRUE FOR $T \to T_0^+$: THEN YOU CAN USE THE PROCEDURE WE'VE JUST DESCRIBED AND

$$G_0(r) \simeq \frac{1}{r^{d-2}} g\left(\frac{r}{\xi} ; \infty\right)$$

THIS IS THE REAL JUSTIFICATION OF COARSE GRAINING. BUT THIS IS TRUE IN GAUSSIAN THEORY AND WE SAID THIS THEORY IS ALL THE MORE JUSTIFIED THE FURTHER WE GO FROM T_0 ! WE'LL HAVE TO RESORT TO THE INTERACTING THEORY TO SOLVE THIS.

NOTICE AT $T = T_0$, $\xi \to \infty \Rightarrow \Lambda\xi \to \infty$, WHOEVER Λ IS.

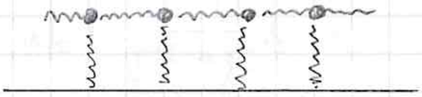
· LESSON 16.04.19

· SOLUTION OF THE EXERCISE: REALIZATION OF THE GAUSSIAN MODEL

$$H = \int d^d x \; \alpha^2 (\nabla \varphi)^2 + \mu^2 \varphi^2$$

THE GRADIENT PART IS COUPLED OSCILLATORS,

$$H_0 = \sum_i \alpha^2 (u_{i+1} - u_i)^2$$



WHICH IS TRANSLATIONALLY INVARIANT UNDER

$$u_i' \to u_i + \Delta$$

BUT THEN EACH OF THEM IS COUPLED TO SOME SUBSTRATE,

$$H = H_0 + \underbrace{\mu^2 \sum_i u_i^2}_{} \rightarrow \text{PINNING POTENTIAL}$$

AND THIS BREAKS TRANSLATIONALLY INVARIANCE, WHICH IS A CONTINUOUS SYMMETRY: BY GOLDSTONE THEOREM, IF $\mu = 0$ WE GET A "ZERO MODE".

· SUSCEPTIBILITY

$$\chi = \beta \int d^d r \; G(r) \qquad G(r) = \langle \delta\varphi(x) \delta\varphi(x+r) \rangle$$

WE'VE ALREADY INTRODUCED

$$\langle \delta\varphi(k_1) \delta\varphi(k_2) \rangle = \delta^{(d)}(k_1 + k_2) \, G(k_1)$$

(PROPAGATOR IN FOURIER SPACE), SO SINCE

$$G(k) = \int d^d r \; e^{ikr} \, G(r)$$

WE GET

$$\underline{\chi = \beta \, G(k=0)}$$

IF $\chi \to \infty$, $G(k)$ HAS A POLE IN $k = 0$.

- FISHER RELATION (γ, ν, η)

$$\chi = \beta \int d^d r \, G(r)$$

$$= \beta \int d^d r \, \frac{f(r/\xi)}{r^{d-2+\eta}} \qquad u \equiv \frac{r}{\xi}$$

$$= \underbrace{\beta}_{\sim \beta_c} \frac{\xi^d}{\xi^{d-2+\eta}} \int du \, \frac{f(u)}{u^{d-2+\eta}} \sim \xi^{2-\eta}$$

IF WE SAY

$$\chi \sim (T-T_c)^{-\gamma} \qquad \xi \sim (T-T_c)^{-\nu}$$

WE GET

$$(T-T_c)^{-\gamma} \sim (T-T_c)^{-\nu(2-\eta)} \quad \Rightarrow \quad \underline{\gamma = \nu(2-\eta)}$$

SINCE THEN

$$G(k) = \int d^d r \, \frac{f(r/\xi)}{r^{d-2+\eta}} e^{ikr} \qquad kr \equiv y$$

$$= \frac{k^{d-2+\eta}}{k^d} \int d^d y \, \frac{f(y/k\xi)}{y^{d-2+\eta}} e^{iy}$$

$$= \frac{1}{k^{2-\eta}} F(k \cdot \xi)$$

WE HAVE IN GENERAL

$$\underline{G(k) = \frac{1}{k^{2-\eta}} F(k\xi)} \qquad \underline{G(r) = \frac{1}{r^{d-2+\eta}} f\left(\frac{r}{\xi}\right)}$$

WITH $(2-\eta) = \gamma/\nu$.

- CRITICAL EXPONENTS (GAUSSIAN THEORY)

$$G(k)\Big|_{\xi=\infty} = \frac{F(k\xi)}{k^{2-\eta}}\Big|_{\xi=\infty} \sim \frac{1}{k^{2-\eta}} \qquad \text{(EXPONENT } \eta \text{)}$$

SO

$$G_0(k) = \frac{1}{k^2+\mu^2} = \frac{1}{k^2+\frac{1}{\xi^2}} \xrightarrow[\xi=\infty]{} \frac{1}{k^2} \quad \Rightarrow \quad \eta = 0$$

THIS HAS NO COUNTERPART IN LANDAU THEORY.

NOTE: $(\nabla\varphi)^2$ IS MISSING IN LANDAU THEORY, SO $G_L(x) = \delta(x)$. IT SAYS NOTHING ABOUT η.

* EXPONENT ν:

$$\xi \sim \frac{1}{(T-T_0)^\nu}$$

$$\mu^2 = \frac{1}{\xi^2}, \quad \mu^2 = \frac{T-T_0}{T_0} \quad \Rightarrow \quad \xi = \frac{1}{\mu} \sim \frac{1}{(T-T_0)^{1/2}}$$

SO $\nu = \frac{1}{2}$: THIS HAS NO LANDAU COUNTERPART, BUT T_0 IS LANDAU CRITICAL TEMPERATURE.

* EXPONENT γ:

$$\chi = (T-T_0)^{-\gamma}$$

$$\gamma/\nu = 2-\eta \quad \text{WITH } \nu = \frac{1}{2}, \eta = 0$$

$$\chi = \beta G(k=0)$$

$$G_0(k=0) = \left.\frac{1}{k^2+\mu^2}\right|_{k=0} = \frac{1}{\mu^2}$$

HENCE

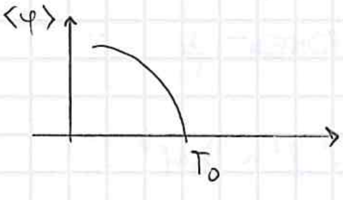
$$\chi \sim \frac{1}{\mu^2} \sim (T-T_0)^1 \quad \Rightarrow \quad \gamma = 1$$

* WHAT ABOUT δ AND β?

$\langle\varphi\rangle = (T-T_0)^\beta$ $\quad T \leq T_0$

$\langle\varphi\rangle \sim h^{1/\delta}$ $\quad T \sim T_0$

SO IN THIS CASE WE HAVE NO PREDICTIONS.



NOTE: I GUESS $\langle\varphi\rangle = 0$ IN GAUSSIAN THEORY.

TO RECAP,

	LANDAU	GAUSS	DIMENSIONAL ANALYSIS
η	/	0	/
ν	/	$1/2$	$1/2$
γ	1	1	1
β	$1/2$	/	$1/2$
δ	3	/	3

WHICH IS STRANGE, BECAUSE THEY ALWAYS SAY GAUSSIAN AND LANDAU ARE THE SAME!

- <u>CRITICAL EXPONENTS FROM DIMENSIONAL ANALYSIS</u>

$$P(\varphi) = \exp\left\{ -\int d^d x \, (\underline{\nabla}\varphi)^2 + \mu^2\varphi^2 + \lambda\varphi^4 \right\}$$

★ EXPONENT η

$$1 \sim x^{d-2}\,\varphi^2(x) \quad \rightarrow \quad \varphi^2(x) \sim \frac{1}{x^{d-2}}$$

$$\langle \varphi(x)\varphi(y) \rangle \sim \frac{1}{r^{d-2}} \quad \rightarrow \quad \eta = 0$$

★ EXPONENT ν

$$(\underline{\nabla}\varphi)^2 \sim \mu^2\varphi^2 \quad \rightarrow \quad \frac{1}{x^2}\varphi^2 \sim \mu^2\varphi^2$$

$$\xi \sim \frac{1}{\mu} \sim (T-T_0)^{-1/2}$$

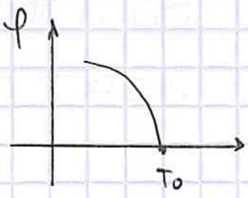
NOTE: $\mu \sim x^{-1} \sim \xi^{-1}$.

IN FACT, ξ IS THE ONLY RELEVANT LENGTH SCALE: SO $\nu = \frac{1}{2}$.

★ EXPONENT γ

$\gamma = 1$ (BY FISHER'S RELATION)

★ EXPONENT β



$$\mu^2\varphi^2 \sim \lambda\varphi^4 \quad \rightarrow \quad \varphi^2 \sim \frac{\mu^2}{\lambda}$$

WE ASSUME λ IS NOT SINGULAR $(\lambda \neq 0, \lambda \neq \infty)$, SO

$$\varphi \sim \mu \sim (T-T_0)^{1/2} \quad \rightarrow \quad \beta = 1/2$$

* EXPONENT δ

$$\lambda \varphi^4 \sim h\varphi \quad \rightarrow \quad \varphi^3 \sim \frac{h}{\lambda}$$

AGAIN, SINCE λ IS NOT SINGULAR,

$$\varphi \sim h^{1/3}$$

* BY USING DIMENSIONAL ANALYSIS, WE GOT THE SAME RESULTS AS THE OTHER 2 THEORIES (INCLUDING THEIR OVERLAP). BY LOOKING AT THE FULL H_{LG}, WE SEE THAT

LANDAU : CUTS THE GRADIENT, $(\underline{\nabla}\varphi)^2$

GAUSSIAN: CUTS THE NON-LINEARITY, $\lambda\varphi^4$

SO WHY ARE THEY BOTH CALLED "FREE THEORY" (UNRENORMALIZED)

THE MESSAGE IS: IT'S THE INTERPLAY BETWEEN THESE TWO TERMS THAT GIVES THE INTERACTION; NONE OF THEM ALONE CAN CHANGE THE EXPONENTS GIVEN BY DIMENSIONAL ANALYSIS. WE NEED BOTH TO CHANGE THE SCALING DIMENSION OF THE FIELD.

- A PARADOX: INTERACTION VS CORRELATION IN GAUSSIAN THEORY

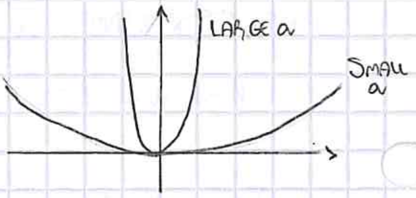
FORGET ABOUT FIELD THEORY AND TAKE

$$H = \frac{1}{2} \sum_{ij} u_i A_{ij} u_j \qquad A \text{ POSITIVE DEF.}$$

$$Z = \int \mathcal{D}u \, e^{-\frac{1}{2}\sum_{ij} u_i A_{ij} u_j} \qquad (\beta \text{ INTO } A)$$

$$C_{ij} \underset{\substack{\uparrow \\ \text{GAUSSIAN}, \ = \langle \delta u_i \delta u_j \rangle}}{=} \langle u_i u_j \rangle = (A^{-1})_{ij} \quad \rightarrow \quad \text{CORRELATION} \overset{?}{\sim} (\text{INTERACTION})^{-1}$$

* 1 DEGREE OF FREEDOM



$$H = \frac{1}{2} a u^2$$

THERE'S NO INTERACTION; a IS THE SELF-INTERACTION, OR A POTENTIAL. WE CAN WRITE

$$\langle u^2 \rangle = \langle u \cdot u \rangle = \frac{1}{a}$$

BUT THIS IS NOT A CORRELATION: IT'S A FLUCTUATION. IT MAKES SENSE THAT IT'S INVERSELY PROPORTIONAL TO a. IT REMINDS OF

$$\chi \sim \frac{1}{g''}$$

* 2 d.o.f.

$$H = \frac{1}{2} \underline{u} A \underline{u}$$

SELF-INTERACTION

MUTUAL INTERACTION

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

IF

$b < 0$: $H \sim - u_1 u_2$ FERROMAGNET / IMITATION

$b > 0$: $H \sim + u_1 u_2$ ANTIFERROMAGNET

BUT IN BOTH CASES WE NEED $a > 0$ FOR H TO BE WELL DEFINED.

$$\langle u_i u_j \rangle = (A^{-1})_{ij} \qquad A^{-1} = \frac{1}{(a^2 - b^2)} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$$

SO FOR INSTANCE

$$\langle u_1 u_2 \rangle = \frac{-b}{(a^2 - b^2)}$$

SO FOR

$b < 0$: POSITIVE CORRELATION

$b > 0$: NEGATIVE CORRELATION

NOTE: IN ORDER FOR A TO BE POSITIVE DEFINED, WE REQUIRE

$$\lambda_1 + \lambda_2 = \mathrm{Tr} A = 2a > 0$$
$$\lambda_1 \lambda_2 = \det A = a^2 - b^2 \geq 0$$

SO IF IT'S TRUE THAT, FOR FIXED a, $\langle u_1 u_2 \rangle$ DECREASES AS $|b|$ GROWS, IN FACT b CAN'T GROW INDEFINITELY AS $|b| \leq a$.

· GAUSSIAN CRITICAL DYNAMICS

z: DYNAMICAL CRITICAL EXPONENT

RECALL THE OVERDAMPED LANGEVIN EQUATION,

$$\eta \frac{\partial}{\partial t} \varphi(x,t) = - \frac{\delta H}{\delta \varphi} + \hat{\xi} \qquad \langle \hat{\xi}(x,t)\hat{\xi}(x',t')\rangle = 2\eta T \delta(t-t')\delta(x-x'$$

LET'S INTRODUCE A KINETIC COEFFICIENT Γ, FOR CONSISTENCY WITH HALPERIN-HOENBERG (RMP '77):

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{\eta}\frac{\delta H}{\delta \varphi} + \frac{1}{\eta}\hat{\xi} \qquad \Gamma \equiv \frac{T}{\eta} \quad , \quad \frac{1}{\eta} = \Gamma \beta$$

WHENCE

$$\frac{\partial \varphi}{\partial t} = -\Gamma \frac{\delta(\beta H)}{\delta \varphi} + \frac{1}{\eta}\hat{\xi}$$

$$\equiv -\Gamma \frac{\delta}{\delta \varphi}\mathcal{H} + \xi$$

SO THE CORRELATOR OF THE NOISE IS

$$\langle \xi \xi \rangle = \frac{1}{\eta^2}\langle \hat{\xi}\hat{\xi}\rangle = \frac{1}{\eta^2} 2\eta T \delta(\tilde{t})\delta(\tilde{x}) = 2\Gamma \delta(t-t')\delta(x-x')$$

NOW LET'S USE

$$\mathcal{H} = \frac{1}{2}\int d^d x \left[(\underline{\nabla}\varphi)^2 + \mu^2 \varphi^2 \right] \qquad (\underline{\nabla}\varphi)^2 \to -\varphi \Delta \varphi$$

$$\frac{\partial \varphi}{\partial t} = -\Gamma\left(-\Delta + \mu^2\right)\varphi(x,t) + \xi(x,t)$$

LET'S CALCULATE THE GREEN FUNCTION:

$$\left[\frac{\partial}{\partial t} + \Gamma\left(-\Delta + \mu^2\right)\right] G_0(x-x', t-t') = \delta^{(d)}(x-x')\delta(t-t')$$

WHERE x IS AN ARGUMENT (AND NOT ONLY A d.o.f., AS IN OUR PREVIOUS DISCUSSION). GOING TO FOURIER SPACE,

$$\left[-i\omega + \Gamma(k^2 + \mu^2)\right] G_0(k,\omega) = 1$$

SO WE FOUND THE GAUSSIAN (FREE) DYNAMICAL PROPAGATOR

$$G_0(k,w) = \frac{1}{-iw + \Gamma(k^2+\mu^2)}$$

ADDING A SOURCE $J(x,t)$ AND THE NOISE ξ, WE GET

$$\langle \varphi(x,t) \rangle = \int d^dx'\, dt'\, G_0(x-x', t-t')\, J(x',t')$$

WHICH IS LINKED TO THE RESPONSE FUNCTION

$$R = \frac{\delta \langle \varphi(x_1,t_1) \rangle}{\delta J(x_2,t_2)} = G_0(x_1-x_2, t_1-t_2)$$

REMARK:

IF WE SEND

$$\mathcal{H} \to \mathcal{H} - h\varphi \qquad \Rightarrow \qquad \Gamma\frac{\delta \mathcal{H}}{\delta \varphi} \to \Gamma\frac{\delta \mathcal{H}}{\delta \varphi} - \Gamma h$$

SO THE SOURCE IS ACTUALLY

$$\frac{\partial \varphi}{\partial t} = -\Gamma\frac{\delta \mathcal{H}}{\delta \varphi} + \Gamma h + \xi \qquad \Rightarrow \qquad J = \Gamma h$$

AND SOMETIMES PEOPLE DEFINE

$$\tilde{R} = \frac{\delta \langle \varphi \rangle}{\delta h} = \Gamma\, G_0(\dots)$$

BACK TO US, WE REWRITE

$$G_0^{-1}(k,w) = -iw + \Gamma\, G_0^{-1}(k) \qquad G_0(k) \equiv \frac{1}{k^2+\mu^2}$$

SO THAT IN THE STATIC LIMIT WE GET THE STATIC PROPAGATOR:

$$G_0^{-1}(k, w=0) = \Gamma \cdot G_0^{-1}(k)$$

LET'S CALCULATE THE CORRELATION:

$$\langle \varphi(x_1,t_1)\varphi(x_2,t_2) \rangle = \int dx_1'\, dx_2'\, dt_1'\, dt_2'\, G_0(x_1-x_1', t_1-t_1') \cdot$$

$$\cdot\, G_0(x_2-x_2', t_2-t_2') \underbrace{\langle \xi(x_1',t_1')\xi(x_2',t_2') \rangle}_{} \sim 2\Gamma\,\delta(x)\delta(t)$$

WHICH GIVES

$$\langle \varphi(x_1, t_1)\varphi(x_2, t_2)\rangle = 2\Gamma \int dx' dt' G_0(x_1 - x', t_1 - t') G_0(x_2 - x', t_2 - t')$$

$$= 2\Gamma \int dx' dt' \int dk\, dq \int d\omega\, d\hat{\omega}\, e^{ik(x_1 - x')} e^{iq(x_2 - x')} e^{-i\omega(t_1 - t')} e^{-i\hat{\omega}(t_2 - t')} G_0(k,\omega) G_0(q,$$

SINCE

$$\int dx' \sim \delta(k+q) \quad , \quad \int dt' \sim \delta(\omega + \hat{\omega})$$

WE GET

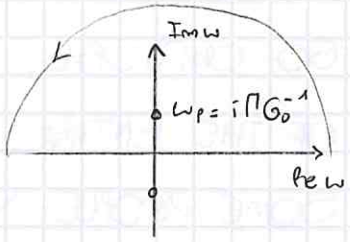
$$C_0(x_1 - x_2, t_1 - t_2) = 2\Gamma \int dk\, d\omega\, e^{ik(x_1 - x_2)} e^{-i\omega(t_1 - t_2)} G_0(k, \omega) G_0(-k, -\omega)$$

WHENCE

$$\underline{C_0(k, \omega) = 2\Gamma\, G_0(k, \omega)\, G_0(-k, -\omega)}$$

USING CAUCHY INTEGRALS (2 POLES → NOT CAUSAL),

$$C(k, t) = 2\Gamma \int d\omega\, e^{i\omega t} \frac{1}{-i\omega + \Gamma G_0^{-1}(k)} \frac{1}{i\omega + \Gamma G_0^{-1}(k)}$$



$$= 2\Gamma \frac{2\pi i}{2i\Gamma G_0^{-1}(k)} e^{-\frac{\Gamma}{G_0(k)} t}$$

$$= 2\pi\, G_0(k)\, e^{-\frac{\Gamma}{G_0(k)} t}$$

SO THE GAUSSIAN RELAXATION TIME IS

$$\tau_k = \frac{G_0(k)}{\Gamma} = \frac{1}{\Gamma(k^2 + \mu^2)}$$

USING DIMENSIONAL ANALYSIS,

$$\frac{\partial\varphi}{\partial t} = -\Gamma \frac{\delta H}{\delta\varphi} = -\Gamma(-\Delta + \mu^2)\varphi$$

$$\frac{\varphi}{t} = \Gamma l^{-2} \varphi \quad \longrightarrow \quad \Gamma \sim \frac{l^2}{t} \quad , \quad \tau \sim t \frac{\cancel{l^2}}{\cancel{l^2}}$$

WHICH IS OK.

LET'S OBSERVE

$$\tau_k = \frac{1}{\Gamma(k^2+\mu^2)}$$

FOR FIXED μ,

$\tau_k \uparrow \quad k \downarrow$

SO LOW K MODES (MORE COLLECTIVE) ARE SLOWER!

THE SLOWEST OF ALL ($k=0$ MODE) HAS

$$\tau_{k=0} = \tau = \frac{1}{\Gamma \mu^2} = \frac{1}{\Gamma}\xi^2 = \frac{1}{\Gamma}\chi$$

NOTE: YES, DARLING, IN THE GAUSSIAN THEORY IT'S $\xi = \frac{1}{\mu}$ FOR ANY d. ($G(r) \sim \frac{1}{r}e^{-\mu r}$ IN $d=3$)

SO AS

$$\chi \to \infty \qquad \Rightarrow \qquad \tau \to \infty$$

THIS IS KNOWN AS CRITICAL SLOWING DOWN: IT'S A GENERAL FEATURE OF ALL COLLECTIVE THEORIES.

NOTICE THIS IS ONLY TRUE AT $k=0$: $k \neq 0$ MODES NEVER GO CRITICAL. IF $k \neq 0$ YOU'RE NOT LOOKING AT THE FLUCTUATIONS OF THE ENTIRE SYSTEM, EVEN IF IT'S INFINITE.

SOME PEOPLE WRITE

$$G_0(k) = \frac{1}{k^2+\mu^2} \equiv \chi_0(k)$$

NOTE: WE'LL SEE THAT K GIVES THE SIZE OF THE WINDOW THROUGH WHICH YOU OBSERVE THE SYSTEM.

SO OUR χ IS ACTUALLY

$$\chi = \chi(k=0)$$

DYNAMICAL CRITICAL EXPONENT

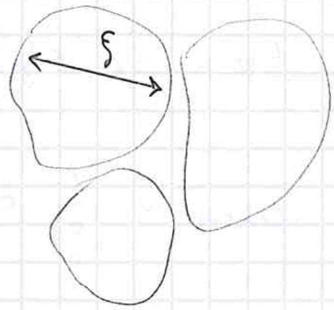
$$\underset{\text{AT } k=0}{\underset{\downarrow}{\tau}} \sim \xi^z \qquad z=2 \text{ (GAUSSIAN)}$$

THIS AGAIN IS SHEER DIMENSIONAL ANALYSIS:

$$(-i\omega + k^2 + \mu^2)\varphi = \dots \qquad \frac{\partial \varphi}{\partial t} \sim \nabla^2 \varphi$$

$$\omega \sim k^2 \sim \mu^2 \qquad \tau \sim \frac{1}{\mu^2} \sim \xi^2$$

GIVEN A SYSTEM, TO GO ERGODIC YOU HAVE TO FLIP THE DOMAINS MANY TIMES: THIS TAKES A LOT OF TIME IF THE DOMAINS GET BIG,



$\tau_{exp} \to \infty$

$\xi \to \infty$.

· DYNAMICAL SCALING HYPOTHESIS

AGAIN, IT WAS AN HYPOTHESIS BEFORE BEING PROVEN BY R.G.

★ STEP 1 : STATICS $(T \simeq T_c,\ \xi \gg a)$

$$G^{(S)}(r;T) = \frac{1}{k^{\gamma/\nu}} f(r \cdot \xi)$$ STATIC SCALING HP $\left(\frac{\gamma}{\nu} = 2-\eta\right)$

ALL EVENTUAL PARAMETERS ARE CONCENTRATED INTO ξ, WHICH ALONE GOVERNS $G^{(S)}$ IN A STRONGLY CORRELATED SYSTEM:

$$G(r; \alpha_1, \alpha_2, .., \alpha_p) = \frac{1}{r^{\gamma/\nu}} f(r \cdot \xi)$$

$$\xi = \xi(\alpha_1, \alpha_2, ..., \alpha_p)$$

NOTE: $G^{(S)}(k;T) = \frac{1}{k^{2-\eta}} f(k\xi)$.

★ STEP 2 : DYNAMICAL CORRELATION FUNCTION

$$G(k,\omega;T) = G^{(S)}(k)\, h\left(\frac{\omega}{\omega_k}; k \cdot \xi\right)$$

WHERE ω_k IS A CHARACTERISTIC FREQUENCY.

★ STEP 3: CHARACTERISTIC FREQUENCY

$$\omega_k = k^2 g(k \cdot \xi)$$

IN A NUTSHELL, EVERYTHING IS EITHER HOMOGENEOUS, OR A FUNCTION OF $k\xi$. ξ RULES THE DYNAMICS TOO!

LET'S TAKE THE CASE $k=0$:

NOTE: THE FUNCTION $g(k\xi)$ MUST DO SOMETHING AT $k=0$ IN ORDER FOR ω_k NOT TO BE ZERO.

$$\omega_k = \frac{1}{\xi^z}(k\xi)^z g(k\xi) = \frac{1}{\xi^z}\hat{g}(k\cdot\xi)$$

$$\omega_{k=0} \sim \frac{1}{\xi^z}\hat{g}(0) \quad \Rightarrow \quad \tau_{k=0} = \frac{1}{\omega_{k=0}} \sim \xi^z$$

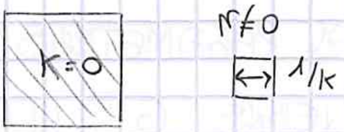
WE CAN EVEN COMBINE THE LAST TWO STEPS TO WRITE

$$G(k,\omega) = G^{(s)}(k)\, h\left(\frac{\omega}{k^z g(k\xi)}; k\xi\right)$$

SO THE EXTERNAL WORLD IS IN THE PRODUCT $k\xi$.

K GIVES THE SIZE OF THE WINDOW THROUGH WHICH YOU WATCH THE SYSTEM, SO

$$k\xi \sim \frac{\xi}{l}$$



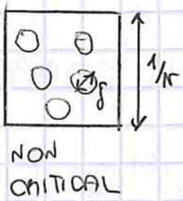
IS THE SIZE OF THAT WINDOW IN UNITS OF ξ. IF

$$k\xi \ll 1 \qquad \frac{1}{k} \gg \xi$$

WE HAVE THE HYDRODYNAMIC LIMIT (OR LANDAU LIMIT). YOU START TO SEE FLUCTUATIONS IF

$$k\xi \gg 1$$

$$\frac{1}{k} \ll \xi$$

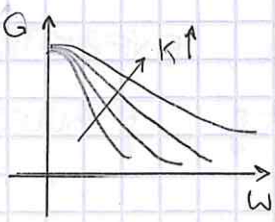


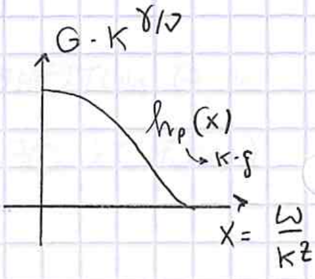
SO THE REAL WAY TO CHECK IF THE SYSTEM IS CRITICAL IS NOT TO LOOK AT ξ ALONE, BUT TO COMPARE IT TO $1/k$.

* IF WE FIX $k\cdot\xi = \rho$ AND SEND $\xi\to\infty$, $k\to 0$,

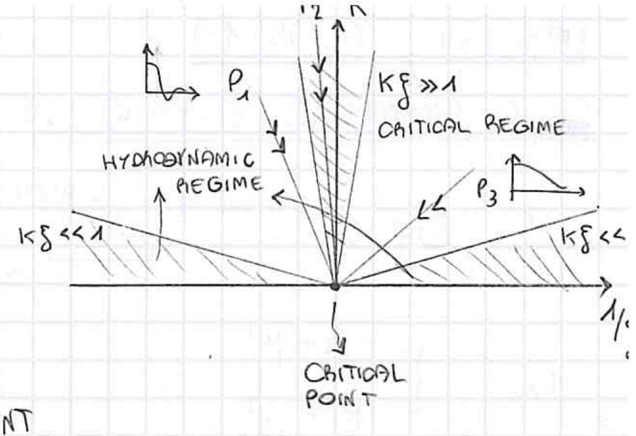
$$G(k,\omega) = \frac{1}{k^{\gamma/\nu}} h_\rho\left(\frac{\omega}{k^z}\right)$$

WHERE h DEPENDS PARAMETRICALLY ON ρ. THE CURVES COLLAPSE ON ONE ANOTHER IF OPPORTUNELY RESCALED.





WE CONCLUDE WITH A GRAPH BY HALPERIN-HOENBERG '69.



K IS THE OBSERVATIONAL WINDOW, $1/\xi$ IS GIVEN BY T (THE CONTROL PARAMETER). WE CAN DISTINGUISH 3 REGIONS AND A SINGLE CRITICAL POINT.

THE ONLY THING THAT CHANGES BY APPROACHING THE CRITICAL POINT FROM DIFFERENT REGION IS THE SHAPE OF THE DYNAMICAL CORRELATION FUNCTION, BUT ALONG THE LINES NOTHING CHANGES.

· LESSON 26.04.19

DIAGRAMMATIC EXPANSION

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\varphi G_0^{-1}\varphi} \, \underbrace{e^{-\lambda\varphi^4}} \hookrightarrow$$ EXPAND THIS (WE'LL NEGLECT THE TECHNICALITIES)

$$G_0^{-1} = \begin{cases} -\Delta + \mu^2 \\ k^2 + \mu^2 \end{cases}$$

$\varphi G_0^{-1}\varphi$ IS SYMBOLIC

NOTE: RECALL WE DERIVED, FOR THE GAUSSIAN THEORY,

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\int dk\, dq\, \varphi(k)\delta(k+q)\varphi(q) \cdot (k^2+\mu^2) + \int dk \, \varphi(k) j}$$

$$\langle \varphi(q)\varphi(k) \rangle = \delta^{(d)}(q+k)\, G_0(k)$$

IT'S GOOD FOR:

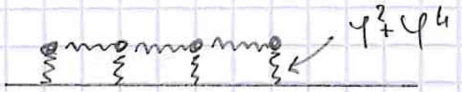
- RENORMALIZATION

  $\mu^2 \to m^2$ (PARTICLE PHYSICS)

- RENORMALIZATION GROUP

  $\Lambda \to \Lambda/b$

- REAL ANHARMONIC CHAIN



THE REAL G WILL BE

$$G = \langle \varphi\varphi \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\varphi G_0^{-1}\varphi} \varphi\varphi(1 - \lambda\varphi\varphi\varphi\varphi + \dots)$$

BUT WE ALSO HAVE TO EXPAND

$$\mathcal{Z} = \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\varphi G_0^{-1}\varphi}(1 - \lambda\varphi\varphi\varphi\varphi + \dots)$$

$$= \mathcal{Z}_0 \left(1 - \lambda \int \frac{\mathcal{D}\varphi}{\mathcal{Z}_0} e^{-\frac{1}{2}\varphi G_0^{-1}\varphi} \varphi\varphi\varphi\varphi + \dots\right)$$

WE ONLY NEED TO DO GAUSSIAN AVERAGES.

· WICK THEOREM (SCALAR)

$$P(\varphi) = \frac{1}{\mathcal{Z}_0} e^{-\frac{1}{2}\varphi G_0^{-1}\varphi}$$

$$\mathcal{Z}(j) = \int \mathcal{D}\varphi \, e^{-\frac{1}{2}\varphi G_0^{-1}\varphi + j\varphi} = \mathcal{Z}_0 \, e^{\frac{1}{2} j G_0 j}$$

THEN WE EASILY GET

NOTE: IT IS ONLY THE m-TH TER
THAT SURVIVES. HERE j IS A
SCALAR, i.e. $\frac{1}{2}G_0 j^2$.

$$\langle \varphi^{2m} \rangle_0 = \frac{\partial^{2m}}{\partial j^{2m}} \frac{Z(j)}{Z_0}\Big|_{j=0} = \frac{\partial^{2m}}{\partial j^{2m}} e^{\frac{1}{2} j G_0 j}\Big|_{j=0}$$

$$= \frac{\partial^{2m}}{\partial j^{2m}} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{2^k} (j G_0 j)^k \Big|_{j=0} = \frac{1}{m!} \frac{1}{2^m} G_0^m (2m)!$$

$$= G_0^m \frac{2m(2m-2)(2m-4)\dots(2m-1)(2m-3)\dots}{2^m m(m-1)(m-2)\dots}$$

THE EVEN PART BECOMES

$$(2m-2)(2m-4)\dots = 2^{m-1}(m-1)(m-2)\dots$$

SO THAT

$$\underline{\langle \varphi^{2m} \rangle_0 = G_0^m (2m-1)!! = \left(\langle \varphi\varphi \rangle_0\right)^m (2m-1)!!}$$

WHERE THE SYMMETRY FACTOR

$(2m-1)!! \equiv$ # OF WAYS OF CONNECTING THE FIELDS

★ THIS IS TRUE IN THE SCALAR CASE. IF INSTEAD φ CARRIES AN INDEX, e.g.

$$\langle \varphi_{i_1} \dots \varphi_{i_{2m}} \rangle = \frac{\partial^{2m}}{\partial j_{i_1} \dots \partial j_{i_{2m}}} \frac{1}{m!} \frac{1}{2^m} \left(j^T G j\right)^m$$

$$= \langle \varphi_{i_1} \varphi_{i_2} \rangle_0 \langle \varphi_{i_3} \varphi_{i_4} \rangle \dots \langle \varphi_{i_{2m-1}} \varphi_{i_{2m}} \rangle + \text{ALL PERMUTATIONS}$$

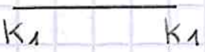
DOING THIS ALGEBRICALLY, ESPECIALLY WHEN

$$\varphi_i \to \varphi(k)$$

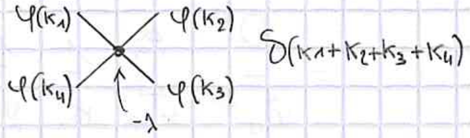
IS HARD: THIS IS WHY WE USE DIAGRAMS. TO EACH PERMUTATION WE ASSOCIATE A DIAGRAM AND A SYMMETRY FACTOR ("OUTSIDE MY PAY RANGE").

- KEEP TRACK OF YOUR δ!

① $\langle \varphi(k_1)\varphi(k_2)\rangle_0 = \delta^{(d)}(k_1+k_2)\,G_0(k_1)$



② $-\lambda \int d^d x\ \varphi^4(x)$



$$= -\lambda \int d^dk_1\, d^dk_2\, d^dk_3\, d^dk_4\ \varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\ \underbrace{\delta^{(d)}(k_1+k_2+k_3+k_4)}$$

$$\hookrightarrow = \int d^d x\ e^{i(k_1+k_2+k_3+k_4)x}$$

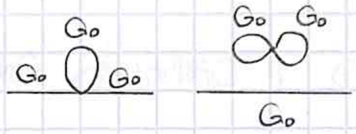
- DIAGRAMS FOR PEDESTRIANS

$$\langle \varphi(k)\varphi(-k)\rangle = \langle \varphi(k)\,\{1+\times+\dots\}\,\varphi(-k)\rangle_0$$

$$= \langle \underset{\uparrow}{-}\ \{1+\times\}\ -\rangle_0 = \underset{\uparrow}{\overline{\qquad G_0 \qquad}} + \langle \overset{\varphi}{-}\ \overset{\varphi\ \ \varphi}{\underset{\varphi\ \ \varphi}{\times}}\ \overset{\varphi}{-}\rangle_0$$

SHORT LINE ≡ FIELD LONG LINE ≡ PROPAGATOR

THE SECOND FACTOR CAN GIVE



EACH ONE TIMES ITS SYMMETRY FACTOR.

★ LET'S CALCULATE ONE FOR REAL.

CALLING p, q THE EXTERNAL MOMENTA,

$$\langle \varphi(q)\varphi(p)\rangle = \frac{1}{2}\int D\varphi\ \varphi(q)\varphi(p)\, e^{-\frac{1}{2}\int dk\, \varphi(k) G_0^{-1}\varphi(k)}\ .$$

$$\cdot \left\{ 1 - \lambda \int dk_1 \dots dk_4\ \varphi(k_1)\dots\varphi(k_4)\ \delta(k_1+\dots+k_4) + O(\lambda^2) \right\}$$

$$= \langle \varphi(q)\varphi(p)\rangle_0 - \lambda \underbrace{\int dk_1\dots dk_4\ \langle \overbrace{\varphi(q)\varphi(k_1)}\overbrace{\varphi(k_2)\varphi(k_3)}\overbrace{\varphi(k_4)\varphi(p)}\rangle_0\, \delta(k_1+\dots+k_4)}_{\equiv I_1(q,p)} + \dots$$

WHERE

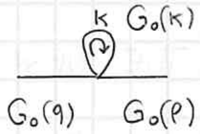
$$I_1(q,p) = \int dk_1 dk_2 dk_3 dk_4\ \underset{\hookrightarrow k_1=-q}{\delta(q+k_1)}\, G_0(q)\, \underset{\hookrightarrow k_3=-k_2}{\delta(k_2+k_3)}\, G_0(k_2)\, \underset{\hookrightarrow k_4=-p}{\delta(k_4+p)}\, G_0(p)\, \delta(k_1+\dots+$$

THEN

NOTE: I SEE HE'S IGNORING THE $(2\pi)^d$, SO I CAN KEEP CONSIDERING THEM TO BE PART OF THE MEASURE d^dq.

$$I_1(q,p) = \int dk_2\, G_0(q)G_0(k_2)G_0(p)\delta(-q+\cancel{k_2}-\cancel{k_2}-p)$$

$$= G_0(q)\left[\int dk\, G_0(k)\right]G_0(p)\,\delta(p+q)$$

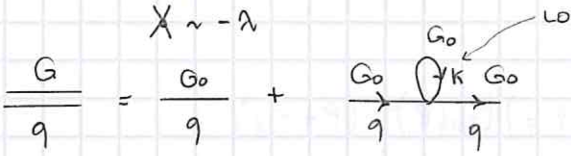


WE FOUND, AT FIRST ORDER,

$$G(q) = G_0(q) - (\ldots)\lambda\, G_0(q)\left(\int d^dk\, G_0(k)\right)G_0(q)$$

MINUS! ↗ ↑ SYMMETRY FACTOR

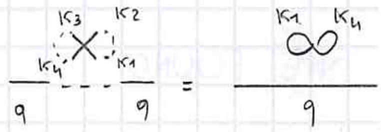
$X \sim -\lambda$



THIS IS THE TADPOLE: THE INTEGRAL IS INDEPENDENT OF THE EXTERNAL MOMENTUM.

- <u>VACUUM FLUCTUATIONS</u>

$$\langle \varphi(q)\underbrace{\varphi(k_1)\varphi(k_2)}\,\underbrace{\varphi(k_3)\varphi(k_4)}\varphi(p)\rangle_0$$



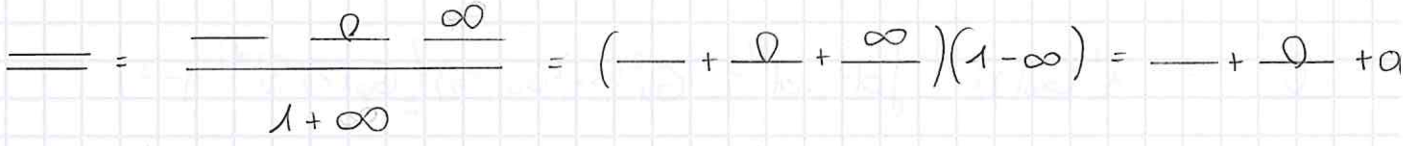
$$= G_0(q)\,\delta(q+p)\int dk_1 dk_2 dk_3 dk_4\; G_0(k_1)\delta(k_1+k_2)G_0(k_4)\delta(k_4+k_3)\delta(k_1+k_2+k_3+k_4)$$

$$= G_0(q)\delta(q+p)\int dk_1 dk_4\, G_0(k_1)G_0(k_4)\,\delta(\cancel{k_1}-\cancel{k_1}-\cancel{k_4}+\cancel{k_4})$$

$$= G_0(q)\delta(q+p)\,\underbrace{\delta^{(d)}(0)}_{V}\int dk_1 dk_2\, G_0(k_1)G_0(k_2) \longrightarrow \infty$$

SO THIS DIAGRAM DIVERGES. BUT WE HAVE TO TAKE INTO ACCOUN

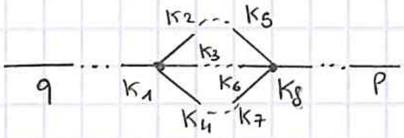
$$\mathcal{Z} = \int \mathcal{D}\varphi\, e^{-\varphi G_0^{-1}\varphi}\left(1-\lambda\varphi\varphi\varphi\varphi\right)\ldots$$



- SATURN DIAGRAM

GOING TO II° ORDER,

ONE OF THE NEW DIAGRAMS WE FIND IS



$$I_2(p,q) = \int dk_1 \dots dk_8\, \delta(k_1+\dots+k_4)\,\delta(k_5+\dots+k_8)\, G_0(q)\,\delta(q+k_1)\,G_0(k_2)\,\cdot$$

$$\cdot\, \delta(k_2+k_5)\,G_0(k_3)\,\delta(k_3+k_6)\,G_0(k_4)\,\delta(k_4+k_7)\,G_0(p)\,\delta(p+k_8)$$

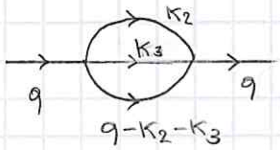
$$= \int dk_2\,dk_3\,dk_4\; G_0(q)\,G_0(k_2)\,G_0(k_3)\,G_0(k_4)\,G_0(p)\,\delta(-q+k_2+k_3+k_4)\,\delta(-k_2-k_3-k_4-p)$$

$$= G_0(q)\,G_0(p)\int dk_2\,dk_3\; G_0(k_2)\,G_0(k_3)\,G_0(q-k_2-k_3)\,\underbrace{\delta(-\cancel{k_2}-\cancel{k_3}-q+\cancel{k_2}+\cancel{k_3}-p)}_{=\,\delta(p+q)}$$

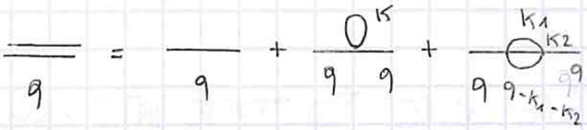
WE FOUND

$$I_2(q) = G_0(q)\left\{\int dk_2\,dk_3\; G_0(k_2)\,G_0(k_3)\,G_0(q-k_2-k_3)\right\} G_0(p)\,\delta(p+q)$$

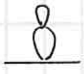
WHICH IS



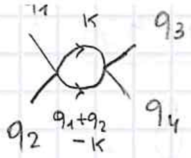
SO AT THIS LEVEL



- EXERCISE

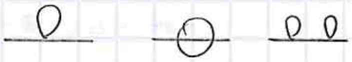


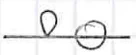
$$= \lambda^2\, G_0(q)\left\{\int d^dk_1\,d^dk_2\; G_0^2(k_1)\,G_0(k_2)\right\} G_0(p)\,\delta^{(d)}(q+p)$$

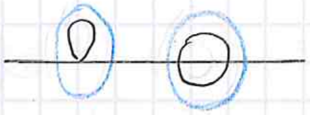
$\langle \varphi(q_1)\varphi(q_2)\varphi(q_3)\varphi(q_4)\rangle =$ 

$$= G_0(q_1)G_0(q_2)G_0(q_3)G_0(q_4)\,\delta(q_1+\ldots+q_4)\int d^d k\, G_0(k)G_0(q_1+q_2-k)$$

- ## ONE-PARTICLE REDUCIBLE DIAGRAMS (1PR)

$O(\lambda^2)$: 

$O(\lambda^3)$: 

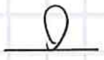
BUT THE JUICY PART IS JUST IN  .

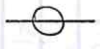
1PR: IFF IT EXISTS AT LEAST ONE "INTERNAL" LINE, SUCH THAT BY CUTTING THAT LINE YOU PRODUCE 2 DISCONNECTED DIAGRAMS

ALL THE OTHER DIAGRAMS ARE CALLED ONE-PARTICLE IRREDUCIBLE, 1PI: IF NOT 1PR.

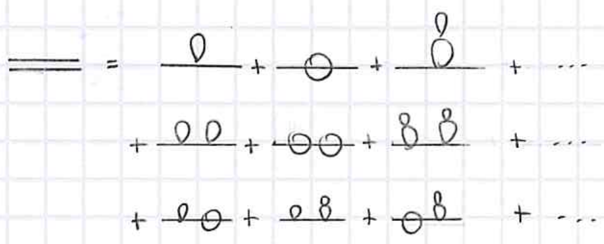
THE 1PI'S ARE THE BUILDING BLOCKS OF THE 1PR'S AND THEREFORE OF THE WHOLE EXPANSION.

BUT EVEN 1PI'S CAN BE SIMPLIFIED, AS THE EXTERNAL LINES ARE ALWAYS THERE:

 =  $\equiv G_0 A G_0$

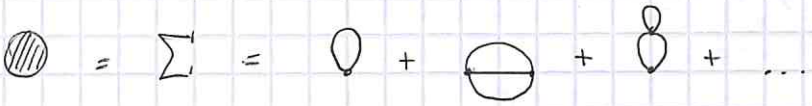
 =  $\equiv G_0 B G_0$

- ## DYSON EQUATION



NOTE: STUDY THE DYSON SPHERE ON WIKIPEDIA.

LET'S INTRODUCE THE SELF-ENERGY :

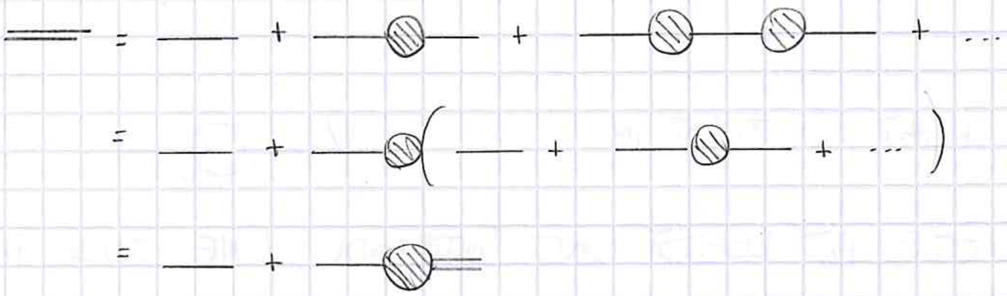


= SUM OF ALL AMPUTATED 1PI DIAGRAMS

THIS IS THE FUN PART (SO TO SPEAK) OF THE DIAGRAMMATIC EXPANSION.

NOTE: IN MY OLD TERMINOLOGY, -◎- = G, ▨ = Σ

NOW



WHERE WE USED THE TELESCOPIC SUM. THIS IS DYSON'S EQUATION:

NOTE: IT'S NOT A TELESCOPIC SUM, IT'S EASIER.

$$\underline{G = G_0 + G_0 \Sigma G} \quad \Rightarrow \quad G = \frac{G_0}{1 - G_0 \Sigma}$$

THIS CAN ALSO BE WRITTEN AS

$$G(q) = \frac{1}{G_0^{-1}(q) - \Sigma(q)} \quad \Rightarrow \quad \underline{G^{-1}(q) = G_0^{-1}(q) - \Sigma(q)}$$

WE CALL VERTEX FUNCTION THE QUANTITY

$$\Gamma \equiv G^{-1}(q)$$

NOTE: IT'S THE USUAL $G = \frac{1}{\Gamma^{(2)}}$.

- RENORMALIZATION OF THE CRITICAL TEMPERATURE

$$G_0^{-1}(q) = q^2 + \mu^2$$

THROUGH DYSON'S EQUATION,

$$G^{-1}(q) = \mu^2 + q^2 - \Sigma(q)$$

WE SAW THAT

NOTE:
$\chi = \int d^d r \, G(r)$

$$\chi = G(q)\big|_{q=0}$$

$$\frac{1}{\chi} = G^{-1}(q=0) = \mu^2 - \Sigma(0)$$

BUT

$$\mu^2 = T - T_0$$

WHERE T_0 IS THE LANDAU (MF) CRITICAL TEMPERATURE.

THE REAL CRITICAL T_c CAN BE DEFINED SUCH THAT

$$@\, T_c \quad \rightarrow \quad \chi = \infty$$

$$\underline{0 = \mu_c^2 - \Sigma(0)}$$

SINCE $\Sigma(0)$ IS IN GENERAL NON-ZERO, SO IS NOT

$$\mu_c^2 = T_c - T_0$$

$$\mu^2 = \mu^2(T) = T - T_0$$

BUT

$$T_c = T_0 + \Sigma(0; T_c)$$

AT $O(\lambda)$, THE SELF-ENERGY IS INDEPENDENT OF THE MOMENTUM:

$$\Sigma(q) = \Sigma(0) = \underset{|}{\bigcirc}$$

$$\Sigma = \underset{|}{\bigcirc} = -\lambda \int d^d k \, G_0(k) = -\lambda \int d^d k \, \frac{1}{k^2 + \mu^2(T)}$$

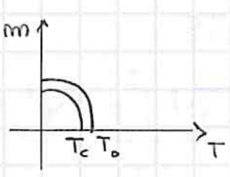
WHENCE

$$T_c = T_0 - \lambda \int d^d k \, \frac{1}{k^2 + \mu_c^2} + O(\lambda^2) = T_0 - \lambda \int d^d k \, \frac{1}{k^2} + O(\lambda^2)$$

$$\uparrow \; \mu_c^2 = \Sigma \sim 0 + O(\lambda)$$

WHICH IS T_0 - (SOMETHING), AS WE'VE ALREADY ARGUED BEFORE:

$$T_c = \underset{\substack{\uparrow \\ \text{MF}}}{T_0} - \lambda \int d^d k \, \frac{1}{k^2} + O(\lambda^2)$$



· LESSON 30.04.19

· LOOP EXPANSION VS λ EXPANSION

EXPANSION AROUND GAUSS VS EXPANSION AROUND LANDAU

$\lambda = 0$

$e^{\lambda\varphi^4} = 1 + \lambda\varphi^4 + \lambda^2 \dots$

$? = 0$

↓ # OF LOOPS

LET'S START FROM HELMOTZ' FREE ENERGY

$$f = -\frac{1}{\beta N}\ln Z = -\frac{1}{\beta N}\ln\int D\varphi\, e^{-\int d^dx\left[\frac{1}{2}(\nabla\varphi)^2+\frac{1}{2}\mu^2\varphi^2+\lambda\varphi^4\right]}$$

LANDAU APPROXIMATION:

$\varphi(x) = \varphi_0$ = CONST. → $\nabla\varphi = 0$

$$f_0 = -\frac{1}{\beta N}\ln e^{-V\left(\frac{1}{2}\mu^2\varphi_0^2+\lambda\varphi_0^4\right)} \qquad \text{(I)}$$

$$= \frac{1}{\beta}\left(\underset{\substack{\downarrow \\ O(\lambda^0)}}{\frac{1}{2}\mu^2\varphi_0^2} + \underset{\substack{\downarrow \\ O(\lambda^1)}}{\lambda\varphi_0^4}\right)$$

0-TH ORDER LANDAU (MF)

THIS MIXES DIFFERENT ORDERS IN λ.

THE PHYSICAL CONDITION UNDER WHICH THIS MAKES SENSE IS

ξ NOT TOO LARGE

BUT WHAT WE DID IN (I) IS ACTUALLY A SADDLE POINT; WHAT IS, THEN, THE ALGEBRAIC CONDITION? LET'S WRITE

$$f = -\frac{1}{\beta N}\ln\int D\varphi\, e^{-\frac{1}{\hbar}\int(\quad)}$$

$$H_{LG} \to \frac{1}{\hbar}H_{LG} = \frac{1}{\hbar}\int d^dx\left((\nabla\varphi)^2+\mu^2\varphi^2+\lambda\varphi^4\right)$$

THEN IF

- $\hbar = 1$: FULL L.G.
- $\hbar = 0$: LANDAU APPROX

WE CAN REWRITE

$$H_{LG} = \int d^dx\, \underbrace{\left(\frac{(\nabla\varphi)^2+\mu^2\varphi^2}{\hbar}\right)}_{= G_0^{-1}} + \underbrace{\frac{\lambda}{\hbar}}_{\hookrightarrow \lambda^{(\hbar)}}\varphi^4$$

WE CAN REPEAT THE WHOLE EXPANSION BY USING

$$G_0^{(\hbar)} = \frac{\hbar}{k^2+\mu^2} \qquad \lambda^{(\hbar)} = \frac{\lambda}{\hbar}$$

RECALL DYSON EQUATION

$$G^{-1} = G_0^{-1} - \Sigma$$

$\Sigma \sim$  $\sim$  $+$  $+ \dots \sim \hbar^{I-V}$

IT'S POSSIBLE TO SHOW THAT

NOTE: WE'RE SAYING WE GAIN A FACTOR $\hbar$ FOR EACH PROPAGATOR, AND WE LOSE ONE FOR EACH VERTEX.

I: # OF LINES

L: # OF LOOPS

V: # OF VERTICES

$\Rightarrow \underline{I - V = L - 1}$

LET'S CHECK IF IT WORKS WITH

	I	V	L
	1	1	1
	3	2	2
	2	2	1
	3	2	2

THIS MEANS

$$\underline{\Sigma \sim (\hbar)^{(\#\text{ of loops})-1}}$$

AN EXPANSION IN $\hbar$ IS ACTUALLY AN EXPANSION IN THE NUMBER OF LOOPS.

THE REAL TECHNICAL PAIN GROWS WITH L: THE NUMBER OF LOOPS MEANS THE NUMBER OF INTEGRALS ONE HAS TO COMPUTE.

NOTE: WHEN WE EXPAND IN λ, THE RADIUS OF CONVERGENCE IS ZERO. IT'S A SIMILAR PROBLEM

THIS JUSTIFIES THE "LOGICAL" WAY WE ORGANIZE THE EXPANSION; IN FACT $\hbar$ IS NOT SMALL ($\hbar = 1$ IN OUR THEORY), BUT IT CAN STILL BE AN ASYMPTOTIC EXPANSION.

· RENORMALIZATION ("A LA" PARTICLE PHYSICS)

$$H_{LG} = \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4!}\lambda\varphi^4 \right\}$$

$$\underbrace{(\mu^2, \lambda, \text{FIELD AMP.}}_{\substack{\text{BARE PARAMETERS} \\ \text{(THEY DEPEND ON } \Lambda)}} ; \Lambda) \longrightarrow (\underbrace{m^2, g}_{\text{PHYSICAL PARAMETERS}} ; \overset{\text{RENORMALIZATION POINT}}{\hat{k}})$$

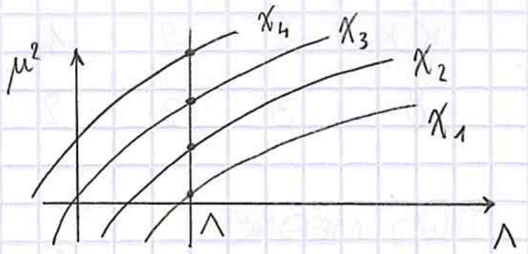
WE HAVE A FAMILY OF MESOSCOPIC THEORIES,

$$\{\Lambda_1 \to \mu^2(\Lambda_1), \lambda(\Lambda_1)\}, \{\Lambda_2 \to \mu^2(\Lambda_2), \lambda(\Lambda_2)\}, \dots$$

BUT THEY ALL HAVE TO CORRESPOND TO THE SAME PHYSICS (MACRO, SMALL K). FOR INSTANCE, TO THE SAME

$\chi \sim G(k=0) \sim$ COLLECTIVE OBJECT, LONG SCALES

THE RENORMALIZATION POINT $\hat{k}$ GIVES THE SCALE AT WHICH I OBSERVE THE PHYSICAL PARAMETERS: $G(k=\hat{k})$.



★ HERE WE PLOTTED $\mu^2(\Lambda)$ FOR VARIOUS TEMPERATURES, EACH ONE GIVING RISE TO A DIFFERENT χ. ONCE WE FIX Λ, WE CAN EXTRACT INFORMATION ON HOW χ CHANGES WITH T.

· MASS RENORMALIZATION

$$G^{-1}(k;\Lambda) = G_0^{-1}(k;\Lambda) - \Sigma(k;\Lambda)$$

$$= \mu^2(\Lambda) + k^2 - \Sigma(k;\Lambda)$$

WE WANT TO:

1) ELIMINATE THE Λ DEPENDENCE.

2) SUBSTITUTE μ^2 WITH SOMETHING PHYSICAL.

1) EVALUATE AT $K=0$ (i.e. WE SET $\hat{K}=0$)

$$G^{-1}(0) = \frac{\beta}{\chi} = \mu^2 - \Sigma(0;\Lambda) \equiv m^2$$

2) INVERT AND WORK OUT "BARE" AS A FUNCTION OF "PHYSICAL"

$$\underbrace{\mu^2(\Lambda)}_{\text{BARE}} = \underbrace{m^2}_{\substack{\text{PHYSICAL} \\ \text{(RENORMALIZED)}}} + \underbrace{\Sigma(0;\Lambda)}_{\text{SELF-ENERGY @ THE RENORMALIZATION POINT } (\hat{K}=0)}$$

IN PARTICLE PHYSICS, μ^2 IS NOT A MEANINGFUL PARAMETER. FOR US, THIS RELATION IS IMPORTANT BECAUSE

$$\mu^2 = \frac{T-T_0}{T_0}$$

3) SUBSTITUTE "BARE" INTO THE ORIGINAL EQUATION

$$\underline{G^{-1}(K) = m^2 + K^2 - \left[\Sigma(K;\Lambda) - \Sigma(0;\Lambda)\right]}$$

THIS IS USEFUL IN THE REGIME

$K \ll \Lambda$, $r \gg l$

FIRST BECAUSE m IS A KNOWN QUANTITY (μ WAS NOT), BUT NOT ONLY THAT. IN GENERAL,

$$\Sigma(K) \sim \int^{\Lambda} d^d q \; F(q, K-q)$$

AND THIS COULD GIVE PROBLEMS IF $q \sim \Lambda$ (ULTRAVIOLET REGION)

IN THE NEW EXPRESSION THERE IS A DIFFERENCE OF QUANTITIES

(UV TROUBLES @ K − UV TROUBLES @ $\hat{K}$)

THE HOPE IS THEY CANCEL OUT, AT LEAST FOR

$K, \hat{K} \ll \Lambda$

- MASS RENORMALIZATION @ 1 LOOP

$$\Sigma_i(k,\Lambda) = \mathcal{Q} = \int^\Lambda d^dq \, \frac{1}{q^2+\mu^2}$$

WHICH IS K - INDEPENDENT ! THEN TRIVIALLY

$$\Delta\Sigma_i(k;\Lambda) \equiv \Sigma_i(k,\Lambda) - \Sigma_i(0,\Lambda) = 0$$

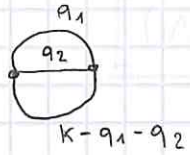
HENCE

$$G^{-1}(k) = m^2 + k^2 + \underbrace{O(\lambda^2)}_{2 \text{ LOOPS}}$$

- MASS RENORMALIZATION @ 2 LOOPS

$$\Sigma \sim 8 \qquad \rightarrow \qquad \Delta\Sigma = 8 - 8 = 0$$

SO THE FIRST INTERESTING BIT IS .

$A(k) =$ 

$$= \int^\Lambda d^dq_1 \int^\Lambda d^dq_2 \, \frac{1}{q_1^2+\mu^2} \cdot \frac{1}{q_2^2+\mu^2} \cdot \frac{1}{(k-q_1-q_2)^2+\mu^2}$$

$$\Delta\Sigma_i = A(k;\Lambda) - A(0;\Lambda) \neq 0$$

YOU CAN PROVE THAT $\Delta\Sigma_i$ DOES NOT DEPEND ON Λ, FOR $k \ll \Lambda$ (YOU CAN LOOK IT UP ON BINNEY P. 233).

NOTE: WHAT YOU DO IS TO PROVE THAT THE INTEGRAND IN $\Delta\Sigma = \Delta A(k)$ GOES LIKE q^{-8}, SO THAT $\Delta\Sigma \sim q^{2d-8}$. IF $d<4$, THIS INTEGRAL CONVERGES; IF $k \ll \Lambda$, WHEN $Q \sim O(\Lambda)$ THE INTEGRAND IS PRACTICALLY ZERO.

IN $d=3$,

$$A(k) \sim \int^\Lambda \frac{q^3 q^3}{q^6} \sim \int^\Lambda \frac{q^6}{q^6} \sim \ln\Lambda$$

FOR $k \ll \Lambda$, $k \ll q$, IT IS POSSIBLE TO SPLIT

$$A(k) = F(k) + \ln\Lambda - F(0) - \ln\Lambda = F(k) - F(0)$$

WHERE $F(k)$ IS A FINITE FUNCTION OF K.

WE FIND $\qquad (\underline{Q} = \underline{q}_1 + \underline{q}_2)$

$$G^{-1}(k) = m^2 + k^2 - \lambda^2 \int_0^{\Lambda\to\infty} d^dq_1 \, d^dq_2 \, \frac{(2\underline{k}\cdot\underline{Q} - k^2)}{(q_1^2+\mu^2)(q_2^2+\mu^2)(Q^2+\mu^2)\left[(k-Q)^2+\mu^2\right]} + O(\lambda^3)$$

WHERE WE COULD SEND $\Lambda \to \infty$ BECAUSE IT'S POSSIBLE TO PROVE THAT

$$[A(k;\Lambda) - A(0;\Lambda)]$$

IS Λ-INSENSITIVE FOR $k \ll \Lambda$. HENCE

$$\mu^2 \Rightarrow m^2$$

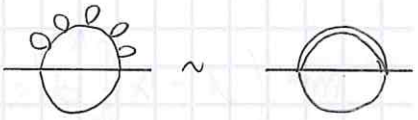
$\Lambda \Rightarrow$  (DISAPPEARED INTO THIN AIR)

★ IF YOU CAN ELIMINATE <u>ALL</u> YOUR Λ-DEPENDENT $\int$ BY SUBSTITUTING THE BARE PARAMETERS WITH THE PHYSICAL ONES, THEN THE THEORY IS SAID TO BE RENORMALIZABLE.

★ IN (II), THERE STILL APPEAR λ (WE'LL TAKE CARE OF IT) AND μ. BUT

$$\mu^2 \sim m^2 + O(\lambda)$$

SO WE MIGHT AS WELL SUBSTITUTE μ WITH m. THIS AMOUNT TO DRESSING UP ALL THE DIAGRAMS WITH BUBBLES, SO WHAT WE GET IS ACTUALLY A MORE ACCURATE PROPAGATOR.



WE FIND

$$G^{-1}(k) = m^2 + k^2 - \lambda^2 \int_0^\infty d^d q_1\, d^d q_2 \frac{(2\underline{k}\cdot\underline{Q} - k^2)}{(q_1^2+m^2)(q_2^2+m^2)(Q^2+m^2)[(k-Q)^2+m^2]^2} + O($$

ONCE WE GET RID OF λ AS WELL ($\lambda \leftrightarrow g$), WE'LL HAVE COMPLETELY FORGOTTEN ABOUT COARSE GRAINING.

- FIRST ATTEMPT TO COMPUTE χ FROM THE DIAGRAMMATIC EXPANSION

(FAILED! i.e. FUCKING WITH YOUR YOUNG MINDS)

$$G^{-1}(k) = \mu^2 + k^2 - \Sigma(k)$$

@ $k=0$: $$\frac{\beta}{\chi} \equiv m^2 = \mu^2 + \lambda \int^\Lambda d^dq \, \frac{1}{q^2+\underline{m^2}}$$ → INSTEAD OF μ^2

NOW @ T_c,

$$m^2 = 0 = \mu_c^2 + \lambda \int^\Lambda d^dq \, \frac{1}{q^2}$$

NOTE: $m^2 = m^2 + 0 = m^2 - \left(\mu_c^2 + \lambda \int^\Lambda d^dq \, \frac{1}{q^2}\right)$.

HENCE

$$m^2 = \mu^2 - \mu_c^2 + \lambda \int^\Lambda d^dq \left(\frac{1}{q^2+m^2} - \frac{1}{q^2} \right)$$

NOTE: NOT SURE IF IT SHOULD BE DIVIDED BY T_0.

$$\mu^2 - \mu_c^2 = T - T_0 - T_c + T_0 \simeq T - T_c$$

$$m^2 = (T-T_c) - \lambda \, m^2 \int^\Lambda d^dq \, \frac{1}{q^2(q^2+m^2)}$$

AND FINALLY

$$m^2 \left(1 + \lambda \int_0^\Lambda d^dq \, \frac{1}{q^2(q^2+m^2)} \right) = (T-T_c) \qquad (\mathrm{III})$$

NOTICE THIS MIGHT DIVERGE FOR $m^2 \to 0$ $(T \to T_c)$:

$$\sim \int_0 \frac{d^dq}{q^4} \qquad \text{IR DIVERGENCE } (d \leq 4)$$

IN ADIMENSIONAL TERMS,

$$\int_0^\Lambda d^dq \, \frac{1}{q^2(q^2+m^2)} \underset{\uparrow \; x = \frac{q}{m}}{=} m^{d-4} \int^{\Lambda/m} d^dx \, \frac{1}{x^2(1+x^2)} = \frac{1}{m^{4-d}} \underbrace{\int^{\Lambda/m} d^dx \, \frac{1}{x^2(1+x^2)}}_{\equiv I_0(\Lambda/m)}$$

WE CALL

$$\underline{\varepsilon \equiv 4 - d}$$

AND REWRITE (III) AS

$$m^2 \left(1 + \frac{\lambda}{m^\varepsilon} I_0(\Lambda/m) \right) = (T-T_c)$$

USING $m^2 \sim \frac{1}{\chi}$ (i.e. $\beta = 1$)

$$\underline{\chi \simeq \left(\frac{1}{T-T_c}\right)\left\{1 + \frac{\lambda}{m^{\varepsilon}} I_0(\Lambda/m)\right\}} \qquad \varepsilon = 4 - d \qquad (IV$$

THIS SEEMS TO BE A GOOD CORRECTION TO MF. IF

$$\frac{\lambda}{m^{\varepsilon}} I_0 \rightarrow 1 \quad \text{(FINITE)} \qquad m \rightarrow 0$$

NO CORRECTIONS: $\gamma = 1$ (MF). IF INSTEAD

$$\frac{\lambda}{m^{\varepsilon}} I_0 \rightarrow \infty \qquad m \rightarrow 0$$

WE GET CORRECTIONS $(\gamma \neq 1)$: THIS HAPPENS IF $d < 4$.

FOR $d > 4$ $(\varepsilon < 0)$, INDEED

$$\frac{1}{m^{\varepsilon}} I_0 = \int_0^{\Lambda} d^d q \, \frac{1}{q^2(q^2+m^2)} \xrightarrow[m^2 \to 0]{} \int_0^{\Lambda} d^d q \, \frac{1}{q^4} < \infty$$

i.e. NO IR DIVERGENCE.

FOR $d < 4$ $(\varepsilon > 0)$,

NOTE: LATER WE'LL STUDY THE CASE $d = 4$. I THINK DOWN HERE WE ASSUME $2 < d < 4$ (CONVERGENCE IN $x =$

$$\frac{1}{m^{\varepsilon}} I_0 = \frac{1}{m^{\varepsilon}} \int^{\Lambda/m} d^d x \, \frac{1}{x^2(1+x^2)} \underset{m^2 \to 0}{\sim} \frac{1}{m^{\varepsilon}} \underbrace{\int^{\infty} d^d x \, \frac{1}{x^4}}_{< \infty} \sim \frac{1}{m^{\varepsilon}} \rightarrow \infty$$

IF $d > 4$, (IV) REDUCES TO MF. BUT IF $d < 4$, CAN WE ACTUALL
EXTRACT INFORMATION FROM (IV)?

NOTE: ON THE ONE HAND, $\frac{1}{m^{\varepsilon}} I$ SHOULD DIVERGE IF WE WANT TO
ANY CORRECTION. ON THE OTHER EN
IF IT DIVERGES IT DOESN'T SEEM
TO MAKE SENSE TO EXPAND AT AL

$$\chi \sim \left(\frac{1}{T-T_c}\right) \cdot \Omega$$

BEST CASE SCENARIO:

1) Ω COMES FROM AN EXPANSION WHOSE 0-TH ORDER SHOULD BE 1,

$$\Omega \sim 1 + y + y^2 \ldots$$

2) Ω SHOULD CORRECT THE DIVERGENCE,

$$\Omega \sim \left(\frac{1}{T-T_c}\right)^{\delta} \qquad \rightarrow \qquad \chi \sim \left(\frac{1}{T-T_c}\right)^{1+\delta}$$

- HOMEWORK

TRY TO SOLVE (IV) RECURSIVELY:

$$\chi = \frac{1}{t}\left(1+\lambda\,\chi^{\varepsilon/2}\right) = \dots = \frac{1}{t}\left(1+\infty+\infty\cdot\infty+\dots\right)$$

YOU'LL END UP WITH A SERIES WHERE EACH TERM IS INFINITELY LARGER THAN THE PREVIOUS.

WE'RE NOT GOING TO GET RID OF THIS FRUSTRATION TODAY.

- THE FUNNY ROLE OF ε

$$\chi \sim \left(\frac{1}{T-T_c}\right)^{\gamma} \quad \rightarrow \quad (T-T_c) \sim \frac{1}{\chi^{1/\gamma}}$$

$$\frac{1}{\chi} \sim m^2 \quad \rightarrow \quad (T-T_c) \sim (m^2)^{1/\gamma}$$

TAKE logs :

$$\ln(T-T_c) \sim \frac{1}{\gamma}\ln(m^2) \quad \rightarrow \quad \underline{\frac{1}{\gamma} = \frac{\partial \ln(T-T_c)}{\partial \ln(m^2)}} \qquad (V)$$

WHICH IS USEFUL TO MAKE CONTACT WITH (IV), WHICH READS

$$(T-T_c) = m^2\left(1+\frac{\lambda}{m^\varepsilon} I_0\right)$$

$$\ln(T-T_c) = \ln(m^2) + \ln\left(1+\frac{\lambda}{m^\varepsilon} I_0\right)$$

AND NOW EXPAND! (WTF? WELL, ASSUME THAT IT'S SMALL...)

$$\ln(T-T_c) = \ln(m^2) + \frac{\lambda}{m^\varepsilon} I_0$$

USING (V),

$$\frac{1}{\gamma} = 1+\lambda I_0 \frac{\partial}{\partial \ln(m^2)} m^{-\varepsilon} = 1+\lambda I_0 \frac{\partial}{\partial \ln(m^2)} e^{-\frac{\varepsilon}{2}\ln m^2}$$

$$= 1 - \frac{1}{2}\lambda I_0 \frac{\varepsilon}{m^\varepsilon}$$

INVERTING AT THIS ORDER,

$$\gamma = 1 + \frac{1}{2} \lambda I_0 \frac{\varepsilon}{m^\varepsilon}$$

THIS SHOWS THAT:

1) YES, WE'RE STUPID.

2) AT $\varepsilon = 0$ ($d = 4$, UPPER CRITICAL DIMENSION) → $\gamma = 1$

3) IS <u>MAYBE</u> ε THE ACTUAL PARAMETER OF THE EXPANSION?

NOTE: IF $d<4$, $\varepsilon > 0$ AND THE CORRECTIC
TO γ DIVERGES AS $m \to 0$.

T.B.H., THERE IS A WAY TO SOLVE THIS WITHOUT R.G., BUT A MORE SATISFACTORY WAY IS TO USE MOMENTUM SHELL, WHERE

$$\int_0^\Lambda \to \int_{\Lambda/b}^\Lambda$$

- <u>THE GINZBURG CRITERION</u>

HOW CLOSE CAN WE GO TO T_c BEFORE MF LANDAU CRASHES?

$$\chi = \left(\frac{1}{T-T_c}\right)\left\{1 + \frac{\lambda}{m^\varepsilon} I_0\right\}$$

$d<4$

NOTE: AGAIN, $I_0\left(\frac{\Lambda}{m}\right)$ IS WELL-BEHAVED,
i.e. $I_0(\infty)$ EXISTS.

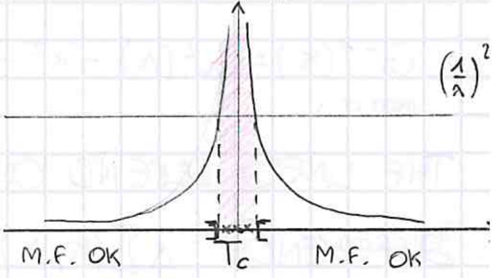
IF

$$\frac{\lambda}{m^\varepsilon} \ll 1 \quad \to \quad \text{MF OK!}$$

SINCE $m^2 \sim \frac{1}{\chi}$, WE'RE SAYING

$$\chi^{\varepsilon/2} \ll \frac{1}{\lambda}$$

$$\chi \sim \frac{1}{T-T_c}$$



HENCE (GINZBURG CRITERION)

$$\underline{\frac{1}{T-T_c} \ll \left(\frac{1}{\lambda}\right)^{2/\varepsilon}}$$

- LESSON 03.05.19

- RENORMALIZATION: GENERAL PROCEDURE

1) IDENTIFY THE BARE PARAMETERS YOU WANT TO ELIMINATE

e.g. MASS: $\mu^2 = \mu^2(\Lambda)$

2) IDENTIFY A PHYSICAL PARAMETER (e.g. CORRELATION FUNCTION) WHICH CONTAINS THE BARE PARAMETER

e.g. $G^{-1}(k) = \mu^2(\Lambda) + k^2 - \Sigma(k,\Lambda) \leftarrow$ PAR.

3) MANIPULATE THE QUANTITY IN SUCH A WAY THAT THE LOWEST ORDER OF IT IS EQUAL TO THE BARE PARAMETER

e.g. $G^{-1}(0) = m^2 = \frac{\beta}{\chi} = \mu^2(\Lambda) - \Sigma(0,\Lambda)$ @ $k = \hat{k} = 0$

4) INVERT. WORK OUT THE BARE AS A FUNCTION OF THE RENORMALIZED PARAMETER

e.g. $\mu^2(\Lambda) = m^2 + \Sigma(0,\Lambda)$

5) INSERT THE BARE PARAMETER (AS A FUNCTION OF THE RENORMALIZED ONE) INTO THE ORIGINAL PHYSICAL QUANTITY.

e.g. $G^{-1}(k) = m^2 + k^2 - [\Sigma(k,\Lambda) - \Sigma(0,\Lambda)]$

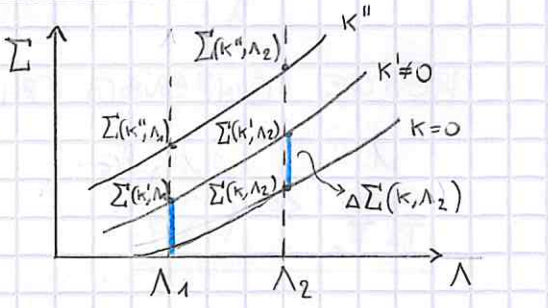
6) CHECK THAT THE NEW Λ-DEPENDENT PARAMETER IS IN FACT Λ-INDEPENDENT, FOR $k \ll \Lambda$

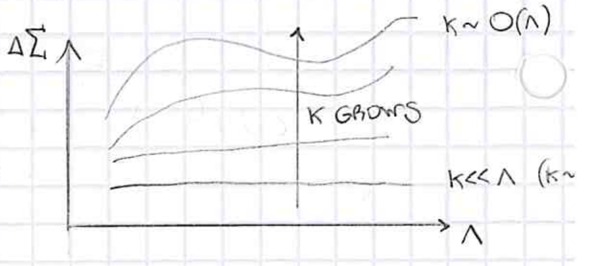
e.g. $[\Sigma(k,\Lambda) - \Sigma(0,\Lambda)]$, Λ-INDEPENDENT

- GEOMETRICAL INTERPRETATION OF RENORMALIZATION

$$\underbrace{G^{-1}(k)}_{\text{INDEP. OF } \Lambda} = \mu^2(\Lambda) + k^2 - \underbrace{\Sigma(k,\Lambda)}_{\text{STRONGLY DEP. ON } \Lambda}$$

THE LINES DEPEND ON Λ, BUT THE DIFFERENCE $\Delta\Sigma(k,\Lambda)$ DOES NOT: THIS WAY THE CONTOURS NEVER INTERCEPT, IT'S A CONTINUITY ARGUMENT. THIS IS TRUE FOR SMALL k $(\Delta\Sigma(k=0,\Lambda) = 0)$, BUT THINGS DRAMATICALLY CHANGE IF k GROWS!





· FIELD RENORMALIZATION

RESTORE THE α^2 IN FRONT OF $(\nabla\varphi)^2$:

$$\alpha^2(\nabla\varphi)^2 \rightarrow \alpha^2 k^2 \varphi^2$$

NOTE: WE HAD INCORPORATED α INTO φ A FEW LECTURES AGO.

THE AMPLITUDE OF THE FIELD MAY DEPEND ON Λ, $\alpha = \alpha(\Lambda)$. IN FACT, WE DEFINED $\varphi(x)$ BY AVERAGING OVER A VOLUME $v \sim \frac{1}{\Lambda^3}$:

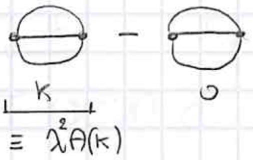
$$\varphi(x) = \frac{1}{l^3}\sum_{i\in x}\sigma_i = \Lambda^3 \sum_{i\in x}^{\Lambda}\sigma_i$$

SO THIS IS NOT SURPRISING.

*NOTE: THIS IS POINT (3) IN THE GENERA PROCEDURE FOR RENORMALIZATION.

THE PROPAGATOR BECOMES

$$G^{-1}(k) = m^2 + \alpha^2 k^2 - \Delta\Sigma(k)$$

$\Delta\Sigma =$


TO ELIMINATE α, WE TAKE *

$$\frac{\partial G^{-1}}{\partial(k^2)} = \alpha^2 - \frac{\partial(\Delta\Sigma)}{\partial(k^2)} = \alpha^2 - \lambda^2\frac{\partial(\Delta A)}{\partial(k^2)}$$

WHERE WE WROTE

$$\Delta\Sigma = \lambda^2 \Delta A(k) \qquad \Delta A = A(k) - A(0)$$

AND CHOSE $\hat{k}$ ARBITRARILY. DEFINE

$$a^2 \equiv \left.\frac{\partial G^{-1}}{\partial(k^2)}\right|_{\hat{k}}$$

SO THAT

$$a^2 = \alpha^2 - \lambda^2\left.\frac{\partial A(k)}{\partial(k^2)}\right|_{\hat{k}} \Rightarrow \alpha^2 = a^2 + \lambda^2\left.\frac{\partial A(k)}{\partial(k^2)}\right|_{\hat{k}}$$

AND WE PLUG IT IN

$$G^{-1}(k) = m^2 + \alpha^2 k^2 - \lambda^2 \Delta A(k)$$

$$= m^2 + a^2 k^2 + \lambda^2\left.\frac{\partial A}{\partial(k^2)}\right|_{\hat{k}} \cdot k^2 - \lambda^2 \Delta A(k)$$

$$= m^2 + a^2 k^2 \left\{1 + \underbrace{\frac{\lambda^2}{a^2}\left(\left.\frac{\partial A}{\partial(k^2)}\right|_{\hat{k}} - \frac{1}{k^2}\Delta A(k)\right)}_{\equiv B(k,\Lambda)}\right\}$$

IT'S POSSIBLE TO PROVE THAT, FOR $k \ll \Lambda$ AND $d < 5$,

IT IS REALLY

$B(k,\Lambda) = B(k)$

NOTE: P. 237 BINNEY. IT PROVES THAT $B(k,\Lambda) \sim g^{(2d-10)}$.

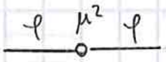
AND MOREOVER WE CAN SAFELY SUBSTITUTE

$$\int \frac{1}{m^2+\alpha^2k^2} \longrightarrow \int \frac{1}{m^2+a^2k^2}$$

· COUPLING CONSTANT RENORMALIZATION

$\lambda = \lambda(\Lambda)$

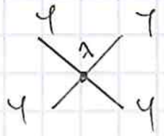
FOR THE MASS, WE USED

$\mu^2\varphi^2$ 

2 FIELDS

THIS IS ALSO TRUE FOR $G = \langle\varphi\varphi\rangle$, A 2-LEGGED OBJECT.

FOR THE COUPLING CONSTANT WE HAVE 4 FIELDS:

$\lambda\varphi^4 \sim$ 

SO WE EXPECT THE RIGHT QUANTITY TO USE IS SOMETHING LIKE

$\langle\varphi\varphi\varphi\varphi\rangle \overset{?}{=} G^{(4)} \overset{?}{=} \lambda + \dots$

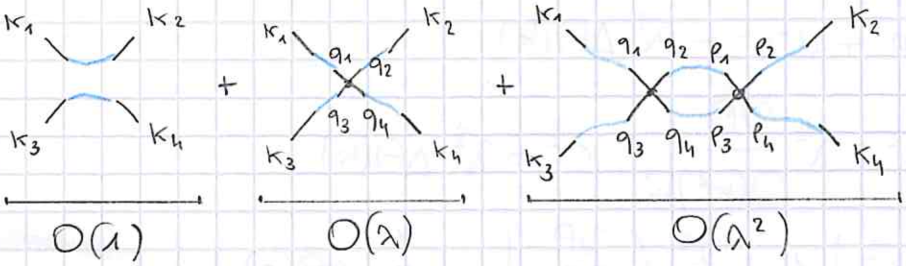
· FOUR POINT CORRELATION FUNCTION

$G^{(4)}(k_1,k_2,k_3,k_4) = \langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\rangle$

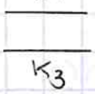
(WE ACTUALLY EXPECT IT TO DEPEND ON 3 K'S ONLY).

WE CAN EXPAND

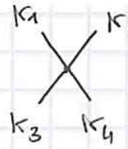
$\sim 1 + \times + \times\times + \dots$

$G^{(4)} =$ 

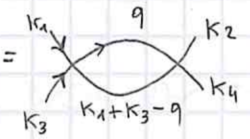
WE FIND

$O(1) =$ 

$= G_0(k_1)\delta(k_1+k_2)G_0(k_3)\delta(k_3+k_4)$

$\mathcal{O}(\lambda) =$ 

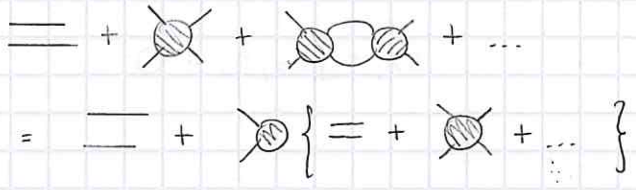
$= -\lambda G_0(k_1)G_0(k_2)G_0(k_3)G_0(k_4)\cdot\delta(k_1+k_2+k_3+k_4)$

$\mathcal{O}(\lambda^2) =$ 

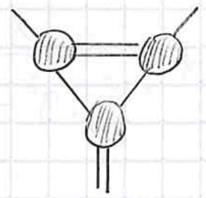
$= \lambda^2 G_0(k_1)G_0(k_2)G_0(k_3)G_0(k_4)\int d^dq\; G_0(q)G_0(k_1+k_3-q)\,\delta(k_1+\ldots+k$

WE FOUND

THIS TIME WE DON'T HAVE SOMETHING LIKE A DYSON EQUATION:



YOU CAN CHECK THAT DIAGRAMS LIKE THIS WOULDN'T BE INCLUDED.



WE DEFINE INSTEAD

$\Gamma^{(4)}(k_1,k_2,k_3,k_4)$ = (−) SUM OF ALL 4-POINTS AMPUTATED 1PI DIAGRAMS

= VERTEX FUNCTION OF ORDER 4

$= -\left(\cdot + \ominus + \ldots \right)$

$= \lambda - \mathcal{O}(\lambda^2)$

THE SAME HOLDS FOR ANY m-TH ORDER VERTEX FUNCTION.

- NOTE ON VERTEX FUNCTIONS

TWO POINT CASE:

Σ = + SUM OF ALL 2-POINT AMPUTATED 1PI DIAGRAMS

$= \bigcirc + \ominus + \ldots$ = SELF-ENERGY

$G^{-1} = G_0^{-1} - \Sigma \qquad\qquad (G)^{-1} = \Gamma^{(2)}$

$\Gamma^{(2)} = \Gamma^{(2)}_0 - \Sigma = \Gamma^{(2)}_0$ − SUM OF ALL 2-POINTS AMPUTATED 1PI DIAGRAMS

THE REASON WHY THEY'RE DEFINED WITH A MINUS SIGN IS THAT THEY CAN BE EVALUATED BY DIFFERENTIATING GIBB'S FREE

ENERGY RATHER THAN HELMOTZ' $(g \leftrightarrow f)$:

$$G^{(m)} \to \frac{\partial f}{\partial j} \qquad\qquad \Gamma^{(m)} \to \frac{\partial g}{\partial \varphi}$$

★ BACK TO OUR 4-POINTS VERTEX FUNCTION, THAT WE NOW EVALUATE:

$$\Gamma^{(4)}(k_1, \dots, k_4) = -\Big(\cdot +$$  $$+ \dots \Big)$$

$$= \lambda - \lambda^2 \int d^d q \, \frac{1}{a^2 q^2 + m^2} \cdot \frac{1}{a^2 (k_1 + k_3 - q)^2 + m^2}$$

DEFINE

$$g \equiv \Gamma^{(4)}(k_1, \dots, k_4) \Big|_{\hat{k} = 0} \qquad\qquad k_1, k_2, k_3, k_4 = 0$$

THAT IS

$$g = \lambda - \lambda^2 \int d^d q \, \frac{1}{(a^2 q^2 + m^2)^2}$$

INVERTING,

$$\lambda = g + \lambda^2 \int d^d q \, \frac{1}{(a^2 q^2 + m^2)^2}$$

AND, WITH SOME PRECAUTIONS, WE MAY SUBSTITUTE λ^2 WITH g^2:

$$\lambda = g + g^2 \int d^d q \, \frac{1}{(a^2 q^2 + m^2)^2}$$

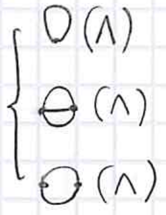
$$\Gamma^{(4)}(k_1, \dots, k_4) = g + g^2 \int d^d q \, \frac{1}{a^2 q^2 + m^2} - \overset{g^2}{\overset{\uparrow}{\lambda^2}} \int d^d q \, \frac{1}{(a^2 q^2 + m^2)\left[a^2 (k_1 + k_3 - q)^2 + m^2\right]}$$

$$= g + g^2 \left\{ \int^{\Lambda} d^d q \, \frac{1}{a^2 q^2 + m^2} - \int^{\Lambda} d^d q \, \frac{1}{(a^2 q^2 + m^2)\left[a^2 (k_1 + k_3 - q)^2 + m^2\right]} \right\}$$

ONCE WE'VE CHECKED THAT THIS DIFFERENCE IS Λ- INDEPENDENT FOR $k \ll \Lambda$, $d < 6$, WE CAN SEND $\Lambda \to \infty$ IN THE UPPER LIMIT OF THE INTEGRALS. WE OBTAIN

$$\Gamma^{(4)}(k_1, \dots, k_4) = g + g^2 \left\{ \int_0^{\infty} d^d q \, \frac{1}{a^2 q^2 + m^2} - \int_0^{\infty} \frac{d^d q}{(a^2 q^2 + m^2)\left[a^2 (k_1 + k_3 - q)^2 + m^2\right]} \right\}$$

- ## RENORMALIZABILITY



Λ - DEPENDENT

$\mu^2(\Lambda)$

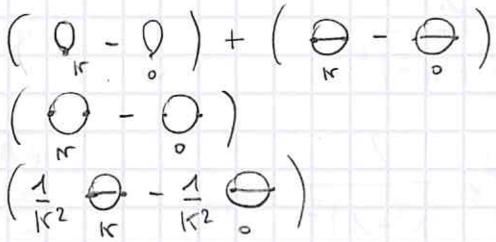
$\alpha^2(\Lambda)$

$\lambda(\Lambda)$

$\mu^2 \to m^2$

$\lambda \to g$

$\alpha^2 \to a^2$



HENCE THE THEORY IS RENORMALIZABLE:

$$\begin{tikzcd}
\{\mu^2(\Lambda_1), \alpha^2(\Lambda_1), \lambda(\Lambda_1); \Lambda_1\} \arrow[dr] & \\
\{\mu^2(\Lambda_2), \alpha^2(\Lambda_2), \lambda(\Lambda_2); \Lambda_2\} \arrow[r] & \{m^2, a^2, g\} \\
\{\mu^2(\Lambda_3), \alpha^2(\Lambda_3), \lambda(\Lambda_3); \Lambda_3\} \arrow[ur] &
\end{tikzcd}$$

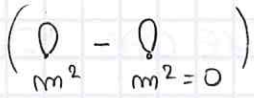
- ## FUCKING WITH YOUR YOUNG MINDS: PART 2

$$\chi = \left(\frac{1}{T-T_c}\right)\left\{1 + \frac{\lambda}{m^\varepsilon} S_0 I_0\right\} \qquad (I)$$

WHERE S_0 IS THE SYMMETRY FACTOR OF , AND

$\varepsilon = 4 - d; \quad d < 4, \; \varepsilon > 0$

$$I_0 = \int_0^\infty d^d x \, \frac{1}{x^2(1+x^2)}$$



$$m^2 = \frac{\beta}{\chi}$$

WE ALSO DERIVED

$$\frac{1}{\gamma} = \frac{\partial \ln(T-T_c)}{\partial \ln(m^2)}$$

AND WE USED IT ON (I) TO GET

$$\gamma = 1 + \underbrace{\frac{1}{2}\varepsilon \frac{\lambda}{m^\varepsilon} S_0 I_0}_{} \ll 1 \qquad (II)$$

(THIS IS THE ASSUMPTION WE HAD TO MAKE, $\ln\left(1+\frac{\lambda}{m^\varepsilon}\right) \simeq \frac{\lambda}{m^\varepsilon}$).

BUT IN FACT THAT QUANTITY IS BY NO MEANS SMALL:

$$\hat{\lambda} = \frac{\lambda}{m^{\varepsilon}} \xrightarrow[\substack{T \to T_c \\ m^2 \to 0}]{} \infty$$

SETTING $\alpha^2 = 1$ (IT CANCELS OUT ANYWAY, BELIEVE ME),

$$g = \lambda - \lambda^2 S_1 \int d^d q \, \frac{1}{(q^2+m^2)^2}$$

$$= \lambda - \lambda^2 S_1 \frac{1}{m^{\varepsilon}} \underbrace{\int d^d x \, \frac{1}{(x^2+1)^2}}_{} \equiv I_1$$

$x = \frac{q}{m}$

S_1

NOTE: WHY ARE WE DOING IT AGAIN? FIRST WE WERE INTERESTED IN MAKING $G^{(2)}$ AND $\Gamma^{(4)}$ FINITE; NOW WE WANT TO EXTRACT INFORMATION ABOUT THE CRITICAL EXPONENTS.

WE FOUND

$$\underline{g = \lambda - \lambda^2 S_1 I_1 \cdot \frac{1}{m^{\varepsilon}}}$$

DEFINE THE EFFECTIVE COUPLING CONSTANTS

$$\hat{\lambda} \equiv \frac{\lambda}{m^{\varepsilon}}$$ (BARE)

$$\hat{g} \equiv \frac{g}{m^{\varepsilon}}$$ (RENORMALIZED)

SO THAT

$$\hat{g} = \hat{\lambda} - \hat{\lambda}^2 S_1 I_1$$

THIS WAY WE CAN REWRITE (II) AS

$$\underline{\gamma = 1 + \frac{1}{2} \varepsilon \hat{g} S_0 I_0}$$ (@ ORDER $\hat{g}$)

BUT WHAT DIFFERENCE DOES IT MAKE? IF $\hat{\lambda}$ DIVERGES, SO DOES $\hat{g}$.

LET'S DO SOME MAGIC.

· β - FUNCTION

$$\beta \equiv \frac{\partial \hat{g}}{\partial \ln(m^2)}$$

IT MEASURES THE CHANGE OF $\hat{g}$ WITH THE TEMPERATURE (m^2).

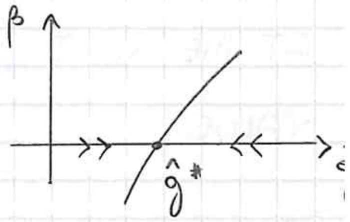
FOR $T \to T_c^+$,

$m^2 \downarrow$, $\delta \ln(m^2) < 0$

MOREOVER,

$$\delta \hat{g} = \beta \cdot \delta \ln(m^2)$$

- IF $\beta < 0$, $\to \delta\hat{g} > 0$ $\quad \hat{g} \uparrow \quad m^2 \downarrow, \quad T \to T_c$
- IF $\beta > 0$, $\to \delta\hat{g} < 0$ $\quad \hat{g} \downarrow \quad m^2 \downarrow, \quad T \to T_c$



IMAGINE THAT FOR SOME REASON β LOOKS LIKE THAT IN THE GRAPH: THEN $\hat{g}^*$ IS AN ATTRACTIVE FIXED POINT FOR $\hat{g}(T)$ AS $T \to T_c$. CAN IT BE SO?

NOTE: ONE IS LED TO THINK THAT $\hat{g}$ GROWS, SO β SHOULDN'T CHANGE ITS SIGN.

$$\hat{\lambda} = \frac{\lambda}{m^\varepsilon}$$

$$\hat{g} = \hat{\lambda} - \hat{\lambda}^2 S_1 I_1$$

$$\frac{\partial \hat{\lambda}}{\partial \ln(m^2)} = \lambda \frac{\partial}{\partial \ln(m^2)} e^{-\frac{\varepsilon}{2}\ln m^2} = -\frac{\varepsilon}{2}\frac{\lambda}{m^\varepsilon} = -\frac{\varepsilon}{2}\hat{\lambda}$$

$$\frac{\partial \hat{\lambda}^2}{\partial \ln(m^2)} = -\varepsilon \hat{\lambda}^2$$

NOTE: YOU MIGHT AS WELL USE THE CHAIN RULE.

$$\beta = \frac{\partial \hat{g}}{\partial \ln(m^2)} = -\frac{\varepsilon}{2}\hat{\lambda} + \varepsilon \hat{\lambda}^2 S_1 I_1$$

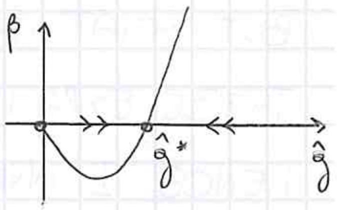
IF THESE WERE REAL EXPANSIONS (i.e. SMALL TERMS), I COULD INVERT

$$\hat{\lambda} = \hat{g} + \hat{\lambda}^2 S_1 I_1 = \hat{g} + \hat{g}^2 S_1 I_1$$

$$\beta = -\frac{\varepsilon}{2}\left(\hat{g} + \hat{g}^2 S_1 I_1\right) + \varepsilon\left(\hat{g} + \hat{g}^2 S_1 I_1\right)^2 S_1 I_1$$

$$= -\frac{\varepsilon}{2}\hat{g} - \frac{\varepsilon}{2}\hat{g}^2 S_1 I_1 + \varepsilon \hat{g}^2 S_1 I_1 + O(\hat{g}^3)$$

$$\underline{\beta(\hat{g}) = -\frac{\varepsilon}{2}\hat{g} + \frac{\varepsilon}{2}\hat{g}^2 S_1 I_1}$$



SO A FIXED POINT REALLY EXISTS (ACTUALLY, $\hat{g}=0$ IS ANOTHER ONE, BUT IT'S REPULSIVE). WE CAN CALCULATE IT:

$$\beta(\hat{g}^*) = 0 \qquad \Rightarrow \qquad \hat{g}^* = \frac{1}{S_1 I_1} = 0.067$$

SO IT'S EVEN SMALL! WE CAN USE IT TO EVALUATE

$$\gamma = 1 + \frac{1}{2}\varepsilon \hat{g} S_0 I_0 \xrightarrow[T \to T_c]{} 1 + \frac{1}{2}\varepsilon \hat{g}^* S_0 I_0 = 1 + \frac{1}{2}\varepsilon \frac{S_0 I_0}{S_1 I_1} = \frac{3}{2} \qquad (d=3, \varepsilon=$$

AT THE NEXT ORDER, THIS GETS MUCH CLOSER TO THE EXPERIMENTAL VALUE.

☆ BUT WHAT THE FUCK DID WE DO?? WE DID 2 OUTRAGEOUS THINGS:

1) $\gamma = 1 + \frac{1}{2}\varepsilon \hat{\lambda} S_0 I_0$ $\qquad \hat{\lambda} = \frac{\lambda}{m^\varepsilon} \xrightarrow[m^2 \to 0]{} \infty$

WE SUBSTITUTED $\hat{\lambda}$ WITH $\hat{g}$, BUT $\qquad \hat{g} \xrightarrow[m^2 \to 0]{} \hat{g}^* < \infty$.

GOT IT?

$\hat{\lambda} \to \infty$, $\hat{g} \to 0.06$

2) $\hat{g}^* = 0.06 < \infty$

$\hat{g} = \hat{\lambda} - \hat{\lambda}^2 S_1 I_1 \qquad \hat{\lambda} \to \infty$

THIS IS A PLAUSIBLE EXPLANATION: ALL EXPANSIONS MAKE SENSE FOR $\hat{\lambda} = \frac{\lambda}{m^\varepsilon} \ll 1$, WHICH MEANS FAR FROM T_c. HENCE

$\gamma = 1 + \frac{1}{2}\varepsilon \hat{\lambda} S_0 I_0 \qquad \hat{\lambda} \ll 1 \qquad$ OK!

$\hat{g} = \hat{\lambda} - \hat{\lambda}^2 S_1 I_1 \qquad \hat{\lambda} \ll 1 \qquad$ OK!

$\to \gamma = 1 + \frac{1}{2}\varepsilon \hat{g} S_0 I_0 \qquad \hat{g} \ll 1$, BECAUSE $\hat{\lambda} \ll 1$.

BUT WHAT HAPPENS FOR $T \to T_c$? CLEARLY, WE CAN'T USE THESE EXPANSIONS, BUT $\hat{g}$ MAY STILL HAVE A FINITE LIMIT. HENCE I INVENT THE β-FUNCTION. I ALREADY HAVE THE

FUNCTION $\hat{g}$, BUT I MAKE UP A DIFFERENTIAL EQUATION FOR $\hat{g}$ TO EMANCIPATE IT FROM $\hat{\lambda}$. THE FUNCTION

$$\beta = \frac{\partial \hat{g}}{\partial \ln(m^2)}$$

MAKES PERFECT SENSE OFF T_c. HERE I ARRIVE AT

$$\beta = -\frac{\varepsilon}{2}\hat{g} + \frac{\varepsilon}{2}\hat{g}^2 S_1 I_1 \equiv \beta(\hat{g})$$

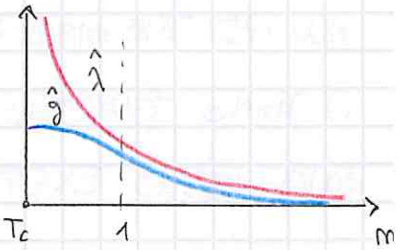
NOTE: IN TAKING THIS LIMIT, WE HAVE TO HOPE NOTHING BAD HAPPE TO THESE FEW TERMS (WHICH SHOULD NOT, AS IT'S JUST A POWER SERIES).

NOW I CAN TAKE THE LIMIT $T \to T_c$.

OBVIOUSLY WE ONLY KNOW THE FIRST FEW TERMS OF THE EXPANSIO

$$\beta(\hat{g}) = -\frac{\varepsilon}{2}\hat{g} + \frac{\varepsilon}{2}\hat{g}^2 S_1 I_1 + O(\hat{g}^3)$$

AND THIS WOULD BE A PROBLEM IF $\hat{g}^*$ WERE BIG. BUT, FOR SOME MIRACLE, WE GOT $\hat{g}^* \simeq 0.06$.



FOR $m^2 >> 1$, $\hat{\lambda} \sim \hat{g} << 1$.

FOR $m^2 << 1$, $\hat{\lambda} \to \infty$, $\hat{g} \to 0.06$.

THINGS MUST SCREW UP SOMEWHERE AROUND $m^2 \sim O(1)$.

★ ANOTHER WAY TO SEE THIS IS TO CONSIDER THE SERIES

$$\hat{g}(m^2) = \hat{\lambda}(m^2) - \hat{\lambda}^2(m^2) + \hat{\lambda}^3(m^2) \quad \ldots$$

IT'S OK FOR $\hat{\lambda} << 1$; BUT HOW CAN I EXTRACT INFORMATION ABOUT $\hat{g}(m^2 = 0)$, ONLY USING THE REGIME $m^2 >> 1$?

WE TRY TO BUILD

$$\frac{\partial \hat{g}}{\partial m^2} = \frac{\partial \hat{g}}{\partial \hat{\lambda}} \cdot \frac{\partial \hat{\lambda}}{\partial m^2}$$

$$\hookrightarrow = 1 - 2\hat{\lambda} + 3\hat{\lambda}^2 + \ldots = 1 - 2(\hat{g} + \hat{g}^2 + \ldots) + 3(\hat{g} + \hat{g}^2 + \ldots)^2$$

$$\frac{\partial \hat{\lambda}}{\partial m^2} = ?$$

BUT $\hat{\lambda} = \frac{\lambda}{m^\varepsilon}$, SO IT CAN ONLY BE A POWER OF m.

SINCE

$$\hat{\lambda} = \lambda (m^2)^{-\frac{\varepsilon}{2}}$$

$$\underbrace{m^2 \frac{\partial \hat{\lambda}}{\partial m^2}}_{\frac{\partial \hat{\lambda}}{\partial \ln(m^2)}} = -\frac{\varepsilon}{2} \hat{\lambda} = -\frac{\varepsilon}{2} (\hat{g} + \hat{g}^2 + \dots)$$

BUT I WAS LOOKING FOR EXPONENTS! IT'S MORE USEFUL TO DERIVE BY LOGS :

$$\frac{\partial \hat{g}}{\partial \ln(m^2)} = \frac{\partial \hat{g}}{\partial \hat{\lambda}} \cdot \frac{\partial \hat{\lambda}}{\partial \ln m^2} = -\frac{\varepsilon}{2} \hat{\lambda} \frac{\partial \hat{g}}{\partial \hat{\lambda}} = -\frac{\varepsilon}{2} \hat{\lambda} (\hat{\lambda} - \hat{\lambda}^2 \dots)$$

$$= -\frac{\varepsilon}{2} (\hat{g} + \hat{g}^2 + \dots)$$

(THERE'S NO PHYSICS HERE : JUST MATHS, AND ALSO FUCKED UP MATHS. YOU GET THE IMPRESSION THAT YOU WOULD NEVER BE ABLE TO DO SUCH A THING YOURSELF. ACTUALLY, WE WILL SEE THAT MOMENTUM SHELL IS MUCH LESS EXOTIC).

· LESSON 07.05.19

THE RENORMALIZATION GROUP, PART 1: INTRODUCTION

· SCALE INVARIANCE

$$G(r) = \frac{f(r/\xi)}{r^{d-2+\eta}} = \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

IN THE GAUSSIAN CASE, $\eta = 0$.

@ T_c, $\xi = \infty$ AND $G(r)$ IS A POWER LAW (SCALE-FREE)

$$G(r; T_c) = \frac{1}{r^{d-2+\eta}}$$

WE RESCALE SPACE BY

$$r \longrightarrow br$$

↳ FUNNY ARROW! WE EVALUATE $G(br)$ INSTEAD OF $G(r)$

AND FIND

$$G(br; T_c) = \frac{1}{b^{d-2+\eta}} \cdot \frac{1}{r^{d-2+\eta}} = \frac{1}{b^{d-2+\eta}} G(r; T_c)$$

THIS IS THE MEANING OF SCALE INVARIANCE: THE SHAPE DOESN'T CHANGE.

WHAT DOES IT MEAN IN TERMS OF FIELDS? AT $T = T_c$,

$$G(r) = \langle \varphi(x)\varphi(y) \rangle \qquad |x-y| = r$$

$$G(br; T_c) = \langle \varphi(bx)\varphi(by) \rangle = \frac{1}{b^{d-2+\eta}} \langle \varphi(x)\varphi(y) \rangle$$

LET'S THEN DEFINE THE SCALING DIMENSION OF φ AS d_φ:

$$\langle \varphi(bx)\varphi(by) \rangle = \frac{1}{b^{2d_\varphi}} \langle \varphi(x)\varphi(y) \rangle \qquad d_\varphi \equiv \frac{1}{2}(d-2+\eta)$$

$$\varphi(bx) = \frac{1}{b^{d_\varphi}} \varphi(x) \qquad \eta: \text{ ANOMALOUS DIMENSION}$$

USING DIMENSIONAL ANALYSIS WE WOULD GET

$$\varphi \sim \frac{1}{x^{\frac{d-2}{2}}} \qquad \varphi \to \frac{1}{b^{d_\varphi}} \varphi \quad \text{IF} \quad x \to bx$$

★ BUT WHAT IS bx? br IS FINE, BECAUSE r IS THE DISTANCE, BUT x IS A COORDINATE.

A POSSIBLE INTERPRETATION OF bx IS A CHANGE OF UNITS.

BUT IF IT WERE A SIMPLE EQUIVALENCE, THEN WE WOULD GET

$$d_\varphi = \frac{1}{2}(d-2)$$

WHICH IS NOT THE CASE. WHAT IS GOING ON?

ANOTHER WAY TO REGARD bx IS A CHANGE OF VARIABLES.

STARTING FROM

$$\langle \varphi(bx)\varphi(by)\rangle = \frac{1}{b^{2d_\varphi}}\langle\varphi(x)\varphi(y)\rangle$$

WE DEFINE A NEW FIELD

$$\underline{\frac{1}{b^{d_\varphi}}\varphi(x) \equiv \varphi_b(x)}$$

SO THAT

$$\langle \varphi(bx)\varphi(by)\rangle = \langle\varphi_b(x)\varphi_b(y)\rangle$$

$$\underline{G(br;\varphi,T_c) = G(r;\varphi_b,T_c)}$$

WHAT IS NOW THE RELATION BETWEEN $P(\varphi)$ AND $P(\varphi_b)$?

HOW DOES ONE FLOW INTO THE OTHER?

★ LET'S GO OFF T_c:

$T \neq T_c \rightarrow \xi \neq \infty$ $\qquad G(r) = \frac{e^{-r/\xi}}{r^{d-2+\eta}}$

RESCALING SPACE,

$$G(br;T) = \frac{e^{-br/\xi}}{r^{2d_\varphi}}\frac{1}{b^{2d_\varphi}} \neq \frac{1}{b^{2d_\varphi}}G(r;T)$$

$$\underline{G(br;\xi) = \frac{1}{b^{2d_\varphi}}G(r;\xi/b)} \qquad b>1 \rightarrow \frac{\xi}{b}<\xi \qquad \text{(V.I.R.)}$$

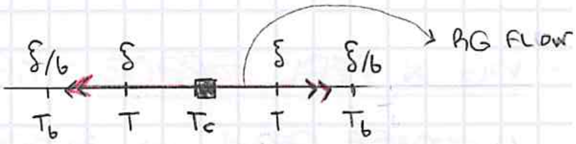
THIS IS A VERY IMPORTANT RELATION: OBSERVING THE SYSTEM AT A LARGER DISTANCE br IS EQUIVALENT TO OBSERVING r,

PROVIDED WE TAKE A SMALLER CORRELATION LENGTH:

$r \to br \quad , \quad \xi = \frac{\xi}{b} \quad , \quad \varphi \to \varphi/_{b^{d_\varphi}}$ $\quad T \to T_b$

HOWEVER, $\xi = \xi(T)$: TO CHANGE ξ, WE ACT ON THE TUNING PARAMETER T. WHEN

$\xi \downarrow \qquad \xi/b, \; b>1$



YOU'RE GETTING FARTHER AWAY FROM T_c; HENCE T HAS TO BE RESCALED AS WELL:

$$\underline{G(br;T) = \frac{1}{b^{2d_\varphi}} G(r;T_b)}$$ (V.S.R

WHICH IS A VERY SEXY RELATION: IT DESCRIBES THE FLOW OF THE PARAMETERS UNDER RESCALING.

★ THE INGREDIENTS SO FAR ARE:

1) RESCALING SPACE / CHANGING SCALE / CHANGING UNITS IS "LIKE" CHANGING THE TEMPERATURE.
2) FLOW OF THE PARAMETERS UNDER RESCALING.
3) T_c IS A REPULSIVE FIXED POINT OF THE FLOW ALONG THE TUNING PARAMETER.

BUT IF THESE WERE ALL, THEN WE WOULDN'T GET ANY ANOMALOUS EXPONENT: ALL THIS SETUP ONLY PRODUCES CRITICAL EXPONENTS EQUAL TO THOSE OF THE GAUSSIAN CASE (i.e. DIMENSIONAL ANALYSIS).

WHAT IS MISSING? THE COARSE-GRAINING!

WHENEVER WE "RESCALE SPACE", WE'RE ACTUALLY LOSING INFORMATION ("DEFOCUSING"). IF I CHANGE UNITS FROM

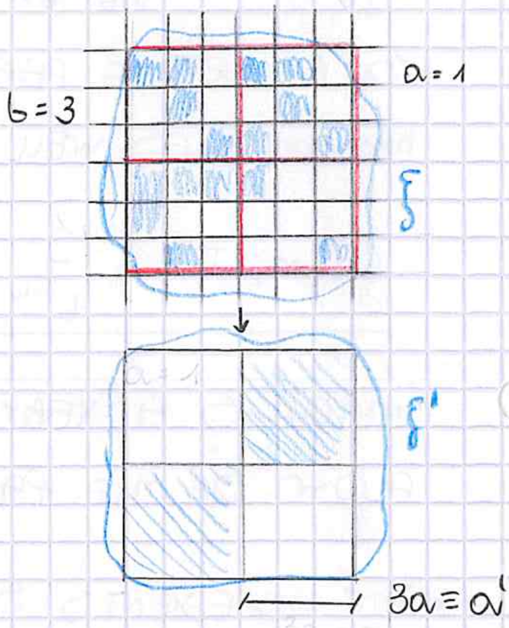
1 cm → 1 LY

IT'S NOT AUTOMATIC THAT I'LL LOSE INFORMATION (THE THEORY DOESN'T SEEM TO LOSE IT). I DO WHEN I

COARSE GRAIN : IT MEANS LOSING INFORMATION BY INTEGRATING OVER THE SHORT WAVELENGTH DETAILS OF THE THEORY. ONLY THEN DO WE RESCALE. THIS CHANGES THE DIMENSIONS.

- RG IN REAL SPACE : KADANOFF BLOCKING PROCEDURE

1) COARSE GRAIN → BLOCKING



REFERENCE : ISING MODEL 2d (BLUE = UP).

YOU INTEGRATE BY SOME RULE (e.g. MAJORITY RULE). IF AT FIRST

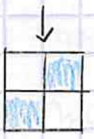
$\xi = 6a$

AFTER THE TRANSFORMATION WE FIND

$\xi' = 2a'$

BECAUSE WE HAVEN'T RESCALED YET.

2) RESCALE IN ORDER TO COMPARE TWO SYSTEMS WITH THE SAME CUTOFF (i.e. LATTICE SPACING).



$a' = 3a \quad \rightarrow \quad \xi' = \xi/3 \quad (\xi \rightarrow \xi/6)$

b = 3 IS BOTH THE COARSE GRAINING AND THE RESCALING FACTOR.

BUT WE ALREADY STEPPED FROM

$$P(\sigma_i) = e^{-H(\sigma_i)} \qquad\qquad P(\varphi) \sim e^{-H(\varphi)}$$
$$H = H(\sigma_i ; J, h) \quad \longrightarrow \quad H = H(\varphi ; \alpha^2, \mu^2, \lambda)$$

THIS WAS REALLY A FIRST COARSE GRAINING STEP.

AND THIS IS THE NONTRIVIAL ONE, WHICH CORRECTS THE CRITICAL EXPONENTS; THE DIAGRAMMATIC WORK IS HERE.

NOTE : AN EXAMPLE IS LOOKING AT THE SAME OBJECT FROM AN INCREASING DISTANCE, TAKING INTO ACCOUNT THE FINITE RESOLUTION OF THE HUMAN EYE. THE FARTHER I AM, THE LESS I SEE.

· RGT AND FIXED POINTS

K: SET OF PARAMETERS $\begin{cases} \alpha^2, \mu^2, \lambda \,/\, y, \mu^2, \lambda \\ \beta J, \beta h \end{cases}$

THIS TRANSFORMS INTO

$$K_b = R_b[K]$$

K_b: NEW SET OF PARAMETERS

K: INITIAL PARAMETERS

R_b: RG TRANSFORMATION $\begin{cases} \text{1) COARSE GRAINING} \\ \text{2) RESCALING} \end{cases}$, $b > 1$

A FIXED POINT IS s.t.

$$K^* = R_b[K^*]$$

THE CRITICAL POINT ($\xi = \infty$) MUST BE AN UNSTABLE FIXED POINT.

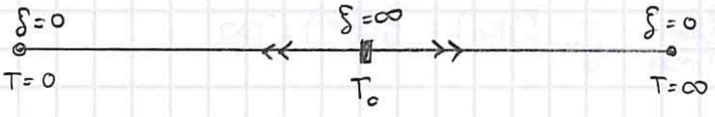
$\xi = \xi(K)$

BUT WE KNOW FOR SURE (IT'S REALLY THE SPACE!) THAT

$\xi \to \xi(Kb) = \xi(K)/b$

HENCE, IF WE ARE AT A FIXED POINT,

$$\xi(K^*) = \frac{\xi(K^*)}{b}$$



WHICH ADMITS AS SOLUTIONS

1) $\xi(K^*) = 0$ TRIVIAL FIXED POINT

$T \neq T_c$. IT IS ATTRACTIVE, $\xi \downarrow \ b \uparrow$

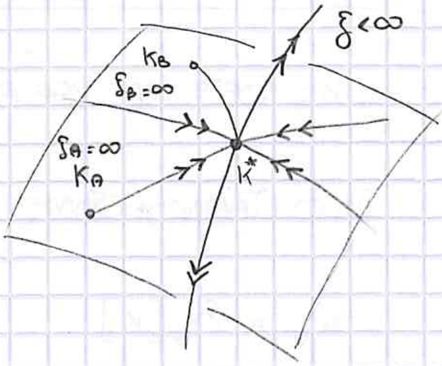
2) $\xi(K^*) = \infty$ CRITICAL FIXED POINT

$T = T_c$. IT IS REPULSIVE, $\xi \downarrow \ b \uparrow$

NOTICE THE OPPOSITE IS NOT TRUE: IF $\xi = \infty$ IT DOESN'T NECESSARILY MEAN THAT WE ARE AT A FIXED POINT.

- CRITICAL MANIFOLD

IT'S THE MANIFOLD SPANNED BY ALL STABLE DIRECTIONS (EIGENSTATES) OF R_b.



- EXERCISE

PROVE THAT $\xi = \infty$ $\forall K \in$ CRITICAL MANIFOLD.

LET $K_A \in CM$. THEN APPLYING THE RG TRANSFORMATION

$$K_A' = R_b[K_A]$$

$$\xi(K_A') = \frac{\xi(K_A)}{b}$$

BY ITERATING, WE GET

$$\xi(K_m) = \frac{1}{b^m}\xi(K_A)$$

BUT WE KNOW, BY DEFINITION, THAT

$$\lim_{m\to\infty} K_m = K^*$$

$$\lim_{m\to\infty} \xi(K_m) = \xi(K^*) = \infty$$

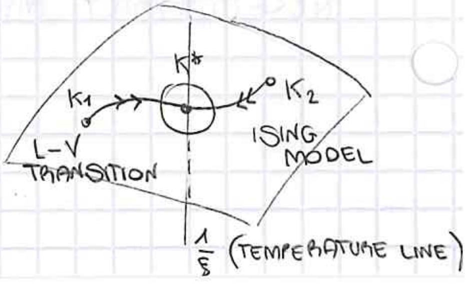
HENCE

$$\lim_{m\to\infty} \frac{1}{b^m}\xi(K_A) = \xi(K^*) = \infty$$

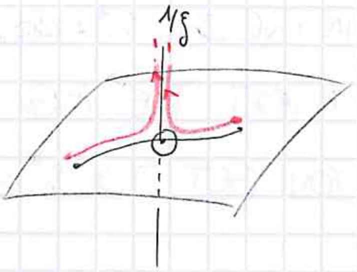
ONLY POSSIBLE IF $\xi(K_A) = \infty$.

NOTICE K_A AND K_B ARE BOTH AT THE CRITICAL POINT, BUT THEY ARE TWO THEORIES WITH DIFFERENT PARAMETERS. HOWEVER, AFTER SOME APPLICATIONS OF R_b THEY END UP IN THE SAME FIXED POINT. THIS IS UNIVERSALITY!

STABLE PARAMETERS ARE CALLED IRRELEVANT (UNSTABLE → RELEVANT). THE FIXED POINT RULES THE LONG SCALE (DISTANCE) PHYSICS OF BOTH K_1 AND K_2.



NOTE IF YOU'RE NOT EXACTLY AT CRITICALITY, RG TAKES YOU CLOSE TO K^* BEFORE RUNNING AWAY.



THIS IS WHY YOU COULD EVEN EXTRACT INFORMATION BY RUNNING THE FLOW BACKWARD.

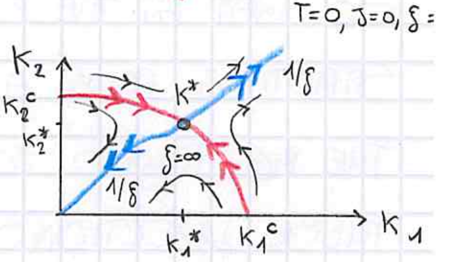
- EXAMPLE: NN vs NNN ISING MODEL

NEAREST NEIGHBOR vs NEXT NEAREST NEIGHBOR

$$\beta H_1 = -K_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

$$\beta H_2 = -K_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j$$

THE RED LINE IS THE CRITICAL MANIFOLD, WHILE THE BLUE ONE IS THE UNSTABLE DIRECTION.



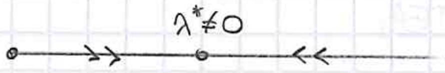
DOES THIS MEAN THAT THE REAL CRITICAL POINT OF ISING IS K_1^*? NOT AT ALL (H_1 DOESN'T KNOW ABOUT K_2). BUT THE CRITICAL EXPONENTS ARE THE SAME AS K_1^*, SO IF YOU APPLY RG YOU GET TO A POINT WHERE THE CRITICAL EXPONENTS ARE THE SAME IN THE TWO MODELS.

- A CRUCIAL IRRELEVANT/STABLE PARAMETER: THE COUPLING CONSTAN

LANDAU GINZBURG, λ, @ $T=T_c$ $\xi=\infty$

Ⓐ


GAUSSIAN (FREE) $d \geq 4$

Ⓑ


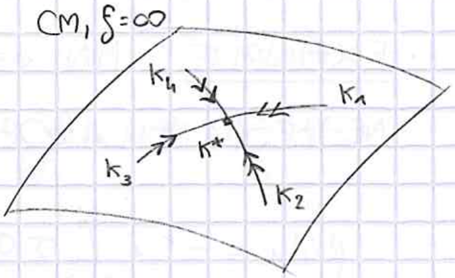
INTERACTING $d < 4$

FOR $d \geq 4$, THE RG TELLS YOU THAT YOU COULD AS WELL NOT USE λ FROM THE START. NOT SO IF $d<4$; BUT IS λ^* SMALL? IF SO, YOU CAN EXPAND; IF NOT, YOU HAVE A PROBLEM.

IN QCD, $\lambda^* = \infty$ (ASIMPTOTICALLY FREE IN THE UV); IF YOU INVERT THE RG FLOW (i.e. LOOK AT SHORT DISTANCES), YOU GET TO $\lambda^* = 0$.

· LESSON 10.05.19

· WHAT IS THE RG GOOD FOR?



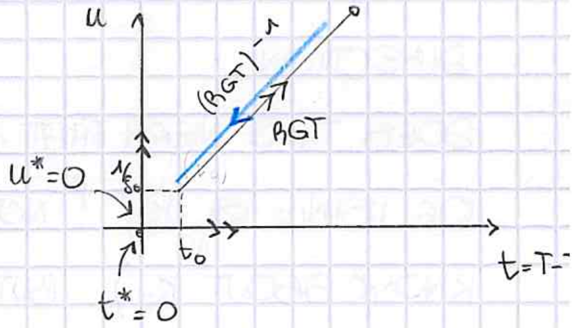
CRITICAL EXPONENTS, LIKE

$$\xi \sim \frac{1}{(T-T_c)^{\nu}} \qquad \chi \sim \frac{1}{(T-T_c)^{\gamma}}$$

MAY BE FOUND USING RG. DEFINE THE SCALING VARIABLES

$$u_\xi = 1/\xi \qquad u_\chi = \frac{1}{\chi}$$

SO THAT AT THE FIXED POINT T_c THEY GO TO ZERO.



STARTING FROM AN INITIAL POINT (t_0, ξ_0), THE RG TAKES YOU AWAY. BOTH THESE DIRECTIONS ARE UNSTABLE, i.e. RELEVANT. BY PLAYING THE RGT BACKWARD, YOU CAN OBTAIN

$$u_\xi \sim t^{\nu}$$

· β - FUNCTIONS

$$u_b = R_b[u_0]$$

SINCE WE WANT TO PERFORM INFINITESIMAL TRANSFORMATIONS, LET'S INTRODUCE THE SMALL PARAMETER

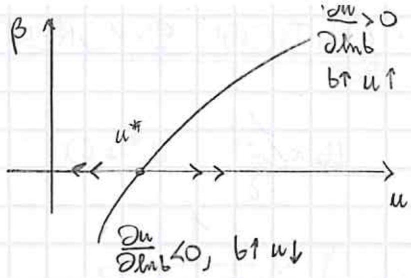
$$\ln b \equiv x \simeq 0 \qquad b \simeq 1$$

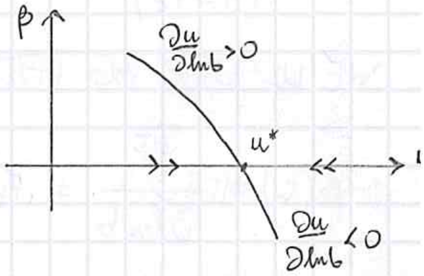
$$u_x = u_0 + x \frac{\partial u}{\partial x} \quad \rightarrow \quad \beta(u) = \frac{\partial u}{\partial \ln b}$$

$\beta(u^*) = 0 \quad \rightarrow \quad u^*$ FIXED POINT OF RGT

WE MAY DISTINGUISH, DEPENDING ON THE SHAPE OF $\beta(u)$,

- UNSTABLE CASE (REPULSIVE, RELEVANT)
- STABLE CASE (ATTRACTIVE, IRRELEVANT)





DEFINE THE SLOPE Y_u, FOR $u \simeq u^*$ (HERE $u^* = 0$),

$$\beta(u) = Y_u \cdot u \qquad \Rightarrow \qquad u_b = b^{Y_u}\, u_0$$

Y_u IS CALLED THE <u>SCALING DIMENSION</u> OF u:

$$u_b = e^{Y_u \ln b}\, u_0 \qquad \beta = \frac{\partial u_b}{\partial \ln b} = Y_u \cdot u_b$$

NOTE: ACTUALLY $\beta = \beta(u_0)$,

$$\beta(u_0) = \frac{\partial}{\partial \ln b} u_b(u_0)$$

$$Y_u = \frac{\partial \beta}{\partial u_0} = \frac{\partial}{\partial u_0}\left(\frac{\partial u_b(u_0)}{\partial \ln b}\right)$$

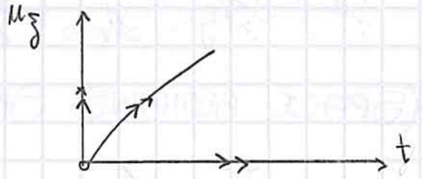
- <u>SCALING DIMENSIONS AND CRITICAL EXPONENTS</u>

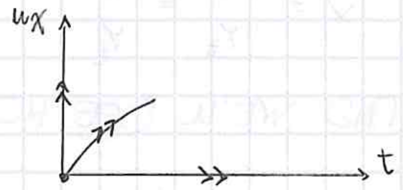
FOCUS ON RELEVANT VARIABLES:

$$u_\xi = \frac{1}{\xi} \quad \rightarrow \quad u^*_\xi = 0$$

$$u_\chi = \frac{1}{\chi} \quad \rightarrow \quad u^*_\chi = 0$$

$$t = T - T_c \quad \rightarrow \quad t^* = 0$$





THEIR β-FUNCTIONS ARE

$$\beta(u) = \frac{\partial u}{\partial \ln b} = Y_u \cdot u \qquad Y_u > 0 \quad \text{UNSTABLE}$$

$$\beta(t) = \frac{\partial t}{\partial \ln b} = Y_t \cdot t \qquad Y_t > 0 \quad \text{UNSTABLE}$$

CRITICAL EXPONENTS ARE RATIOS OF RG SCALING DIMENSIONS:

$$\begin{cases} \frac{1}{u}\frac{\partial u}{\partial \ln b} = Y_u \\ \frac{1}{t}\frac{\partial t}{\partial \ln b} = Y_t \end{cases} \Rightarrow \quad \frac{\partial \ln u}{\partial \ln t} = \frac{Y_u}{Y_t}$$

$$u \sim t^{Y_u/Y_t} \qquad \frac{Y_u}{Y_t} : \text{CRITICAL EXPONENT!}$$

- CRITICAL EXPONENT OF ξ: ν

$$u = \frac{1}{\xi}\ , \quad u^* = 0 \qquad\qquad t = T - T_c\ , \quad t^* = 0$$

$$\xi \sim \frac{1}{(T-T_c)^{\nu}} \quad \rightarrow \quad u \sim t^{\nu}$$

WE'LL USE RG (AT 1-LOOP LEVEL) TO CALCULATE

$$\beta(t) = \frac{\partial t}{\partial \ln b} = y_t \cdot t$$

WHILE WE CAN ALREADY OBTAIN

$$\beta(u) = \frac{\partial u}{\partial \ln b} = y_u \cdot u$$

IN FACT,

$$y_u = y_{1/\xi} = \xi_b \frac{\partial}{\partial \ln b}\left(\frac{1}{\xi_b}\right) = \xi_b \frac{\partial}{\partial \ln b}\left(\frac{b}{\xi}\right) = \xi_b \frac{b}{\xi} = 1$$

(SPACE REMAINS SPACE UNDER A RGT!). HENCE

$$\nu = \frac{y_u}{y_t} = \frac{1}{y_t}$$

NOTE: RECALL y_u IS A SCALING DIMENSION BECAUSE $u_b = b^{y_u} u_0$

AND WE'LL SEE HOW TO CALCULATE y_t (WITH MUCH PAIN).

- FUNNY ARGUMENT!

$$\xi_b = \frac{\xi_0}{b} \qquad\qquad u_b = b^{y_u} u_0$$

AFTER l ITERATIONS,

$$\xi_{l+1} = \frac{1}{b}\xi_l = \frac{\xi_0}{b^l} \qquad\qquad u_{l+1} = b^{y_u} u_l = \overset{\text{SMALL}}{u_0}(b^{y_u})^l$$

AND AT SOME POINT

$$\xi_l \sim O(1)$$

WE STOP RG FLOW AT $\xi_{l+1} \sim 1$:

$$1 = \frac{\xi_0}{b^{l_{STOP}}}$$

STOP CONDITION: $b^{l_{STOP}} \sim \overset{\text{LARGE}}{\xi_0}$

$$1 \equiv \overset{\to \text{NOT INFINITE}}{u_{l+1}} = u_0 (b_{l_{STOP}})^{y_u} = u_0 \xi_0^{y_u} \quad \Rightarrow \quad u_0 \sim \frac{1}{\xi_0}^{y_u}$$

DROPPING THE SUFFIX FROM u_0 (IT HOLDS FOR ANY u),

$$u \sim \frac{1}{\xi^{y_u}} \qquad \Rightarrow \qquad \xi \sim \frac{1}{u^{1/y_u}}$$

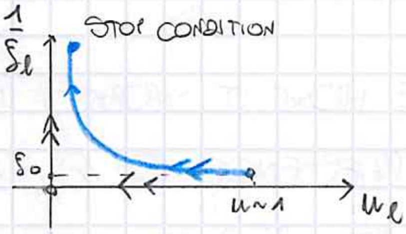
CHOOSING $u \equiv t$, WE RECOVER

$$\nu = \frac{1}{y_t} \qquad \text{NOTE: } \xi \sim t^{-\nu}.$$

*THIS CONSTRUCTION ONLY HOLDS FOR RELEVANT PARAMETERS.

IMAGINE WE REPEAT IT FOR A STABLE / IRRELEVANT ONE; WITH THE SAME STOP CONDITION

$$b^{l} \sim \xi_0$$



APPLIED ON $(RGT)^{-1}$, YOU GET

$$u \sim O(1)$$

INSTEAD OF $u \ll 1$ THAT WE FOUND IN THE UNSTABLE CASE,

$$u \sim \frac{1}{\xi^{y_u}} \to 0$$

IN FACT, IN THE LATTER CASE

RGT: $\xi\downarrow \; u\uparrow$

$(RGT)^{-1}$: $\xi\uparrow \; u\downarrow$

WHILST IN THE FORMER

RGT: $\xi\downarrow \; u\downarrow$

$(RGT)^{-1}$: $\xi\uparrow \; u\uparrow$

HENCE THE CONDITION

$$u \sim \frac{1}{\xi^{y_u}}$$

IS WRONG IN THIS CASE.

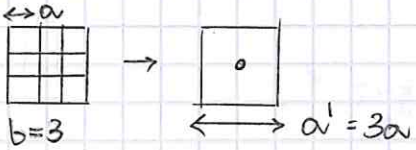
NOTE: i.e. $u_0 \, \xi_0^{y_u} \sim O(1)$ IS NOT TRUE WITH STABLE PARAMETERS, BECAUSE ξ_0 IS BIG AND $u_0 \sim O(1)$ (INSTEAD OF $u_0 \ll 1$ OF THE UNSTABLE CASE).

• MOMENTUM SHELL

RECAP OF REAL SPACE RG

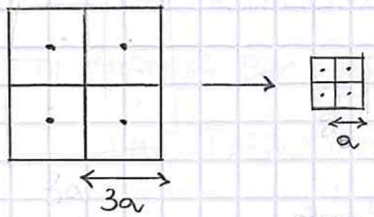
WE HAD TWO STEPS:

1) COARSE-GRAINING (DEFOCUSING/LOSING INFO)



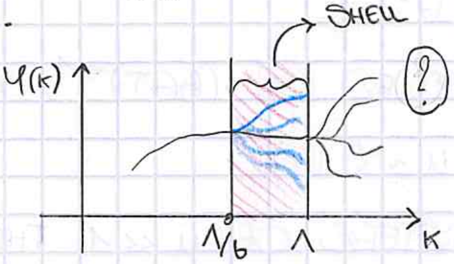
2) RESCALING. CHANGE COORDINATE x SO THAT WE GET THE SAME LATTICE SPACING*.

$x \to \frac{x}{b}$, $ba \to a$



LET'S SEE HOW IT WORKS IN K-SPACE.

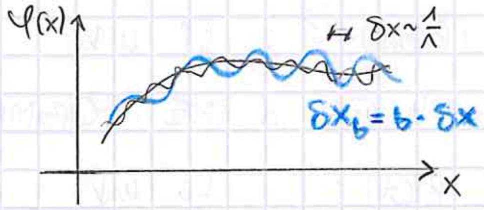
HERE THE PRESENCE OF A CUTOFF MEANS THAT $\varphi(k)$ IS UNKNOWN BEYOND Λ, AND THIS REFLECTS OUR IGNORANCE WITHIN $\delta x \sim \frac{1}{\Lambda}$ IN REAL SPACE ($\varphi(x)$).



RG STEP ① MEANS FURTHER COARSE GRAINING!

IN K-SPACE, THIS SIMPLY MEANS MOVING THE CUTOFF AND INTEGRATING OVER THE SHELL.

IN REAL SPACE, IT PRODUCES SMOOTHER FIELDS.



RG STEP ②: RESCALE K s.t. $\frac{\Lambda}{b} \to \Lambda$. THIS WAY WE HAVE THE SAME CUTOFF.

*NOTE: as before, AND HERE, THE SAME CUTOFF AS BEFORE.

• GAUSSIAN CASE

$$H = \int \underset{\substack{\downarrow \\ \text{POSITION}}}{d^d x} \left[(\nabla \varphi)^2 + \mu^2 \varphi^2 \right] = \int^{\Lambda} \underset{\substack{\downarrow \\ 1/\text{DISTANCE}}}{d^d k} \, \varphi(k) \left[k^2 + \mu^2 \right] \varphi(-k)$$

NOTICE YOU COARSE GRAIN USING DISTANCE, NOT POSITION.

HOW WOULD WE FURTHER COARSE GRAIN IN POSITION SPACE?

IN K-SPACE IT'S STRAIGHTFORWARD. RECALL

$$P(\varphi) = \frac{e^{-\int^\Lambda d^dk\, \varphi(k)(k^2+\mu^2)\varphi(-k)}}{\int \mathcal{D}\varphi\, e^{-\int^\Lambda d^dk\, \varphi(k)(k^2+\mu^2)\varphi(-k)}}$$

MORALLY, $\varphi(k)$ IS A VECTOR:

$$\varphi(k) = \{\varphi_1, \dots, \varphi_{\Lambda/b}, \varphi_{\Lambda/b+1}, \dots, \varphi_\Lambda\}$$

$$P(\varphi) = P(\underbrace{\varphi_1, \dots, \varphi_{\Lambda/b}}_{\varphi^<}, \underbrace{\varphi_{\Lambda/b+1}, \dots, \varphi_\Lambda}_{\varphi^>})$$

→ WE MARGINALIZE $P(\varphi)$ BY INTEGRATING OVER THESE GUYS (ON-SHELL FIELDS)

SO WE LOOK FOR THE DISTRIBUTION OF THE OFF-SHELL FIELDS BY CALCULATING

$$\underline{P(\varphi_1, \dots \varphi_{\Lambda/b}) = \int d\varphi_{\Lambda/b+1} \dots d\varphi_\Lambda\, P(\varphi, \dots, \varphi_{\Lambda/b}, \dots, \varphi_\Lambda)} \qquad (I)$$

WHERE IS THE ADVANTAGE? WE ONLY COMPUTE INTEGRALS BETWEEN

$$\int_{\Lambda/b}^{\Lambda} \rightarrow \text{FINITE!}$$

NOTICE EQUATION (I) IS WHAT WE PRACTICALLY MEAN BY COARSE-GRAINING. WE CAN REWRITE IT, FOR SIMPLICITY, AS

$$P(\varphi^<) = \int \mathcal{D}\varphi^>\, \frac{1}{Z} e^{-\int^\Lambda d^dk\, \varphi(k)(k^2+\mu^2)\varphi(-k)}$$

SPECIFICALLY,

$$\int_0^\Lambda d^dk\, \varphi(k)(k^2+\mu^2)\varphi(-k)$$

$$= \int_0^{\Lambda/b} d^dk\, \varphi^<(k)(k^2+\mu^2)\varphi^<(-k) + \overbrace{\int_{\Lambda/b}^{\Lambda} d^dk\, \varphi^>(k)(k^2+\mu^2)\varphi^>(-k)}^{\text{SHELL INTEGRAL}}$$

NOTICE THERE ARE NO MIXED TERMS LIKE

$$\varphi^>(k)\varphi^<(k)$$

(DECOUPLING OF LARGE/SMALL MOMENTA).
UV / IR

THIS WOULD HAPPEN IN AN INTERACTING THEORY, LIKE $\lambda\varphi^4$:

$$\int d^dx\, \varphi^4(x) \to \int_0^\Lambda d^dk_1 \int_0^\Lambda d^dk_2 \int_0^\Lambda d^dk_3 \int_0^\Lambda d^dk_4\; \varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)$$

$$= \left(\int_0^{\Lambda/b} + \int_{\Lambda/b}^{\Lambda}\right)\left(\int_0^{\Lambda/b} + \int_{\Lambda/b}^{\Lambda}\right)(\dots)(\dots)$$

WE GET, IN THE GAUSSIAN CASE,

$$P(\varphi^<) = \int D\varphi^> \left\{ e^{-\int_0^{\Lambda/b} d^dk\, \varphi^<(\dots)\varphi^<}\, e^{-\int_{\Lambda/b}^{\Lambda} d^dk\, \varphi^>(\dots)\varphi^>} \right\} \cdot \frac{1}{Z}$$

$$Z = \int D\tilde\varphi^< e^{-\int_0^{\Lambda/b} d^dk\, \tilde\varphi^<(\dots)\tilde\varphi^<} \cdot \int D\tilde\varphi^> e^{-\int_{\Lambda/b}^{\Lambda} d^dk\, \tilde\varphi^>(\dots)\tilde\varphi^>} = (Z^<)(Z^>)$$

$$P(\varphi^<) = \frac{1}{Z^<}\, e^{-\int_0^{\Lambda/b} d^dk\, \varphi^<(\dots)\varphi^<} \cdot \frac{\cancel{\int D\varphi^>\, e^{-\int_{\Lambda/b}^{\Lambda} d^dk\, \varphi^>(\dots)\varphi^>}}}{\cancel{\int D\tilde\varphi^>\, e^{-\int_{\Lambda/b}^{\Lambda} d^dk\, \tilde\varphi^>(\dots)\tilde\varphi^>}}}$$

(COURTESY OF GAUSS). IN THE GAUSSIAN CASE, INTEGRATION OVER THE UV DEGREES OF FREEDOM HAS NO IMPACT ON THE IR DEGREES OF FREEDOM.

THIS IS WHY WE GET NO CORRECTIONS TO THE NAIVE DIMENTIONAL ANALYSIS CRITICAL EXPONENTS.

★ STEP 2: RESCALE TO RESTORE THE CUTOFF

RESCALING ↔ CHANGE OF VARIABLE K!

$$k_b \equiv b \cdot k \qquad\qquad d^dk = \frac{1}{b^d}\, d^dk_b$$

$$P(\varphi^<) = \frac{1}{Z^<}\, e^{-\int_0^\Lambda d^dk_b \frac{1}{b^d}\left(\frac{k_b^2}{b^2} + \mu^2\right)\varphi^<\left(\frac{k_b}{b}\right)\varphi^<\left(-\frac{k_b}{b}\right)}$$

BUT

$$\varphi^<\varphi^< \frac{1}{b^d}\left(\frac{k_b^2}{b^2} + \mu^2\right) = \frac{\varphi^<\varphi^<}{b^{d+2}}\left(k_b^2 + b^2\mu^2\right)$$

LET'S THEN INTRODUCE A NEW FIELD

$$\varphi_b(k_b) \equiv \frac{1}{b^{\frac{d+2}{2}}}\, \varphi\left(\frac{k_b}{b}\right)$$

WHAT IS THE PROBABILITY DISTRIBUTION OF $P(\varphi_b)$?

$$P(\varphi_b) D\varphi_b = P(\varphi^<) D\varphi^<$$

NOTE: IT'S EASY BECAUSE IT'S LINEAR
IN GENERAL

$$P(\tilde{\varphi}) = \int D\varphi \, P(\varphi) \, \delta(\tilde{\varphi} - F[\varphi])$$

$$Y = Y(X)$$

IN THIS CASE IT'S EASY:

$$P(x) = \frac{f(x)}{\int dx \, f(x)}$$

$$P(y) = P(x) \frac{dx}{dy} = \frac{f(x^{-1}(y))}{\int dy \frac{dx}{dy} f(x^{-1}(y))} \cdot \frac{dx}{dy}$$

HENCE

$$P(\varphi_b) = \frac{1}{Z(b)} e^{-\int_0^\Lambda d^d k_b \, \varphi_b(k_b)(k_b^2 + b^2\mu^2)\varphi_b(-k_b)}$$

★ THE EFFECT OF THE RGT IS THUS

$$\mu_b^2 = b^2 \mu^2$$

$$\begin{cases} \mu^2_{\ell+1} = b^2 \mu^2_\ell \\ \varphi_{\ell+1} = \frac{1}{b^{\frac{d+2}{2}}} \varphi_\ell \end{cases}$$

(RG FLOW EQUATIONS)

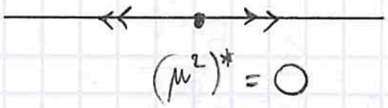
FIXED POINT OF THE MASS:

$\mu^{2*} = 0$ (NONTRIVIAL)

$\mu^2 = T - T_0 \rightarrow T^* = T_0$

HENCE THE GAUSSIAN $T_c = T_0$.

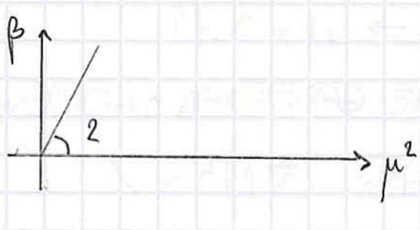
IT'S NONTRIVIAL BECAUSE IT IS CLEARLY UNSTABLE. TO GET ν,



$$\beta(t) = \frac{\partial t}{\partial \ln b}$$

$t = T - T_c \sim \mu^2$

$$\beta(\mu^2) = \frac{\partial \mu^2}{\partial \ln b} = \overset{y_t}{\overset{\uparrow}{2}} \mu^2 \Rightarrow \nu_{GAUSS} = \frac{1}{y_t} = \frac{1}{2}$$



* THE NEW FIELD (NEW THEORY) IS

$$\varphi_b(k) = \frac{1}{b^{\frac{d+2}{2}}} \varphi\left(\frac{k}{b}\right) \qquad (k \sim k_b)$$

↳ OLD FIELD (OLD THEORY)

WHAT HAPPENS TO THE <u>CORRELATION FUNCTIONS</u>?

$$\langle \varphi_b(k_1) \varphi_b(k_2) \rangle = \frac{1}{b^{d+2}} \langle \varphi\left(\frac{k_1}{b}\right) \varphi\left(\frac{k_2}{b}\right) \rangle$$

$$\langle \varphi(k_1) \varphi(k_2) \rangle = \delta^{(d)}(k_1 + k_2)\, G(k_1)$$

$$\delta^{(d)}(k_1 + k_2)\, G_b(k_1) = \frac{1}{b^{d+2}} \delta^{(d)}\left(\frac{k_1 + k_2}{b}\right) G\left(\frac{k_1}{b}\right)$$

$$= \frac{b^d}{b^{d+2}} \delta^{(d)}(k_1 + k_2)\, G\left(\frac{k_1}{b}\right)$$

WHENCE

$$G_b(k) = \frac{1}{b^2} G(k/b)$$

WHERE G_b IS RELATED TO A DIFFERENT TEMPERATURE.

AT $T = T_c$, $\mu^2 = 0$; SINCE IT'S A FIXED POINT,

$$G(k) = \frac{1}{b^2} G(k/b) \qquad\qquad G(k) = \frac{1}{k^{2-\eta}}$$

$$\frac{1}{k^{2-\eta}} = \frac{1}{b^2} \frac{b^{2-\eta}}{k^{2-\eta}}$$

NOTE: $G(k)$ UP HERE IS AN ANSATZ JUSTIFIED BY FISHER'S RELATION, BUT IT CAN BE AVOIDED BY USING EULER'S METHOD (→ FOCUS).

HENCE

$$G(k) = \frac{1}{k^2} \quad \Rightarrow \quad \begin{cases} \eta_G = 0 \\ \nu_G = \frac{1}{2} \end{cases}$$

* WHAT HAPPENS IF WE ADD A VERY SMALL NON-GAUSSIAN TERM?

$$H \to H + \lambda \varphi^4 \qquad\qquad \lambda << 1 \quad (\sim 10^{-5})$$

IN THE GAUSSIAN FRAMEWORK, WE USE DIMENSIONAL ANALYSIS:

$$\int d^d x \, (\nabla \varphi)^2 \sim 1 \quad \Rightarrow \quad \varphi_x^2 \sim \frac{1}{x^{d-2}}$$

SIMILARLY, THE DIMENSION OF λ IS

$$\lambda \int d^d x \, \varphi_x^4 \sim 1 \quad \Rightarrow \quad \lambda \sim \frac{1}{x^d \varphi^4} \sim \frac{1}{x^d} x^{2d-4} \sim x^{d-4} \sim k^{4-d}$$

AND AGAIN WE FOUND

$$\underline{4-d \equiv \varepsilon} \qquad \underline{\lambda \sim k^{\varepsilon}}$$

NOTE: THIS IS IN GENERAL THE WAY WE DETERMINE THE UPPER CRITICAL DIMENSION OF A THEORY.

LET'S APPLY RG:

$$k \rightarrow k_b = b \cdot k \qquad \lambda_b = b^{\varepsilon} \lambda$$

ITERATING,

$$\lambda_{l+1} = b^{\varepsilon} \lambda_l$$

NOTE: THIS IS THE NAIVE (GAUSSIAN) APPROACH. NEXT TI
WE'LL SEE THAT THE FLOW GETS CORRECTED AS

$$\lambda_{l+1} = b^{\varepsilon} \lambda_l (1 - \lambda_l I_2(b)) \quad (1 \text{ LOOP})$$

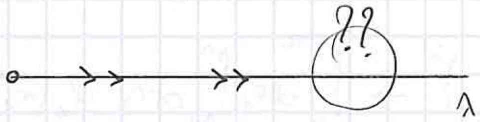
AND WE HAVE A PROBLEM: HOWEVER SMALL λ IS, FOR

$\varepsilon \leq 0$ $(d \geq 4)$

$\lambda^* = 0$ ATTRACTIVE F.P.

$\varepsilon > 0$ $(d < 4)$

$\lambda^* = 0$ UNSTABLE F.P.



THIS IS NOT GOOD: SMALL SCALE PHYSICS CANNOT BE RULED BY $\lambda^* = 0$. WE NEED TO RE-DO THE CALCULATIONS.

- FOCUS: EULER'S METHOD (BINNEY P.286)

LET $f(\underline{x})$ BE HOMOGENEOUS OF DEGREE D IN $x_1 \ldots x_m$, i.e.

$$f(\rho \underline{x}) = \rho^D f(\underline{x})$$

APPLYING $\rho \frac{d}{d\rho}$ ON BOTH SIDES GIVES

$$\rho (\underline{x} \cdot \underline{\nabla}) f(\underline{x}) = D \rho^D f(\underline{x})$$

CHOOSING $\rho = 1$ GIVES EULER'S EQUATION

$$(\underline{x} \cdot \underline{\nabla} - D) f(\underline{x}) = 0$$

IN OUR CASE, AT $T = T_c$ $(\mu^2 = 0)$ THE FIXED-POINT CONDITION READS

$$G(k) = {}^{1}/_{b^2} \, G({}^{k}/_{b})$$

i.e. $G(k)$ IS HOMOGENEOUS OF DEGREE $D = -2$ IN $k = |\underline{k}|$. HENCE, SINCE $\underline{\nabla}_k G(k) = \frac{\partial G}{\partial k} \hat{k}$,

$$(\underline{k} \cdot \underline{\nabla}_k + 2) G(k) = 0$$

$$k \frac{\partial G}{\partial k} = -2 G(k) \quad \Rightarrow \quad G(k) \sim \frac{1}{k^2}$$

· LESSON 14.05.19

WHY THE $\lambda\varphi^4$ CASE IS DIFFERENT FROM THE GAUSSIAN CASE

★ GAUSSIAN :

$$\int \mathcal{D}\varphi(x)\, e^{-\underset{\substack{\downarrow\\ \text{LOCAL IN SPACE}}}{\int dx\, \varphi^2(x)}} = \int \mathcal{D}\varphi(k)\, e^{-\underset{\substack{\downarrow\\ \text{LOCAL IN MOMENTUM}}}{\int dk\, \varphi^2(k)}}$$

HENCE THE d.o.f. SEPARATE:

$$\int_0^\Lambda dk\, \varphi^2(k) = \int_0^{\Lambda/b} dk\, \underset{\substack{\downarrow\\ \text{IR}}}{\varphi_<^2(k)} + \int_{\Lambda/b}^{\Lambda} dk\, \underset{\substack{\downarrow\\ \text{UV}}}{\varphi_>^2(k)}$$

★ NON-GAUSSIAN :

$$\int \mathcal{D}\varphi(x)\, e^{-\lambda\underset{\substack{\downarrow\\ \text{LOCAL IN SPACE}}}{\int dx\, \varphi^4(x)}} = \int \mathcal{D}\varphi(k)\, e^{-\lambda\underset{\substack{\downarrow\\ \text{NON-LOCAL IN MOMENTUM}}}{\int dk_1 dk_2 dk_3 dk_4\, \varphi(1)\varphi(2)\varphi(3)\varphi(4)\delta(\dots)}}$$

$$\left(\int_0^{\Lambda/b} + \int_{\Lambda/b}^{\Lambda}\right)^4 \rightarrow \int_0^{\Lambda/b} dk_1 \int_{\Lambda/b}^{\Lambda} dk_2\, \underset{\substack{\downarrow\\ \text{IR}}}{\varphi_<(k_1)}\, \underset{\substack{\downarrow\\ \text{UV}}}{\varphi_>(k_2)} \cdot \varphi \dots$$

AS WE ARE ABOUT TO SEE.

AGAIN, WE WILL CALL

$$\varphi(k) = \begin{cases} \varphi_<(k) & k < \Lambda/b \\ \varphi_>(k) & k > \Lambda/b \end{cases}$$

COARSE-GRAINING = INTEGRATION OF THE $\varphi_>$

$$P(\varphi^<) = \int \mathcal{D}\varphi^>\, P(\varphi^<, \varphi^>) = \int \mathcal{D}\varphi^>\, \frac{1}{Z} e^{-H(\varphi^<, \varphi^>)} \qquad (\mathrm{I})$$

$$H = \int_0^\Lambda d^d k\, \varphi(k)(\mu^2 + k^2)\varphi(-k) + \lambda S(\varphi)$$

LET'S CALL

$$\Gamma_0 \equiv (\mu^2 + k^2)$$

$$\int_0^\Lambda dk\, \varphi\Gamma_0\varphi = \int_0^{\Lambda/b} dk\, \varphi_<\Gamma_0\varphi_< + \int_{\Lambda/b}^{\Lambda} dk\, \varphi_>\Gamma_0\varphi_>$$

SO THAT

$$P(\varphi^<) = \int D\varphi^> \frac{e^{-\int_0^{\Lambda/b} \varphi^< \Gamma_0 \varphi^< - \int_{\Lambda/b}^{\Lambda} \varphi^> \Gamma_0 \varphi^> - \lambda S(\varphi)}}{\int D\tilde{\varphi}^> D\tilde{\varphi}^< \, e^{-\int_0^{\Lambda/b} \tilde{\varphi}^< \Gamma_0 \tilde{\varphi}^< - \int_{\Lambda/b}^{\Lambda} \tilde{\varphi}^> \Gamma_0 \tilde{\varphi}^> - \lambda S(\tilde{\varphi})}}$$

LET'S INTRODUCE THE ON-SHELL GAUSSIAN PARTITION FUNCTION

$$Z_0^> = \int D\varphi^> e^{-\int_{\Lambda/b}^{\Lambda} \varphi^> \Gamma_0 \varphi^>}$$

$$\langle A \rangle^0_{SHELL} = \int D\varphi^> \frac{1}{Z_0^>} e^{-\int_{\Lambda/b}^{\Lambda} \varphi^> \Gamma_0 \varphi^>} A(\varphi^<, \varphi^>)$$

THIS WAY, MULTIPLICATING AND DIVIDING (I) BY $Z_0^>$,

$$P(\varphi^<) = \frac{e^{-\int_0^{\Lambda/b} \varphi^< \Gamma_0 \varphi^<} \langle e^{-\lambda S(\varphi^>, \varphi^<)} \rangle^0_{SHELL}}{\int D\tilde{\varphi}^< e^{-\int_0^{\Lambda/b} \tilde{\varphi}^< \Gamma_0 \tilde{\varphi}^<} \langle e^{-\lambda S(\tilde{\varphi}^>, \tilde{\varphi}^<)} \rangle^0_{SHELL}} \equiv \frac{e^{-H(\varphi^<)}}{\int D\tilde{\varphi}^< e^{-H(\tilde{\varphi}^<)}}$$

WHERE WE DEFINED

$$e^{-H(\varphi^<)} \equiv e^{-\int_0^{\Lambda/b} d\mathbf{r}\, \varphi^< \Gamma_0 \varphi^<} \langle e^{-\lambda S(\varphi^<, \varphi^>)} \rangle^0_{SHELL}$$

LET'S USE A CUMULANT EXPANSION:

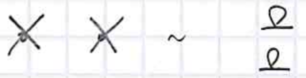
$$\langle e^{-\lambda S} \rangle = e^{\ln \langle e^{-\lambda S} \rangle} = e^{\ln (1 - \lambda \langle S \rangle + \frac{1}{2} \lambda^2 \langle S^2 \rangle + \dots)}$$

NOTE: $\ln \langle e^{tx} \rangle = \mu t + \frac{1}{2} \sigma^2 t^2 + \dots$

$$= e^{-\lambda \langle S \rangle + \frac{1}{2} \lambda^2 \langle S^2 \rangle - \frac{1}{2} \lambda^2 \langle S \rangle^2 + O(S^3)}$$

$$= e^{-\lambda \langle \varphi^4 \rangle^0_{SHELL}} \, e^{\frac{1}{2} \lambda^2 \underbrace{(\langle \varphi^4 \varphi^4 \rangle^0_{SHELL} - \langle \varphi^4 \rangle^0_{SHELL} \langle \varphi^4 \rangle^0_{SHELL})}} \rightarrow \text{CONNECTED 4-POINTS DIAGRAMS}$$

NOTE $\langle \varphi^4 \rangle$ IT'S REALLY AN INTEGRAL (HIGHLY SYMBOLIC). MOREOVER


NOT INCLUDED! (CONNECTED)


INCLUDED

NOTE:

$S(\varphi) = \int d^dx \, \varphi^4(x)$

AND

$\langle \varphi^4 \rangle^0_{SHELL} = \int d^dx \, \langle \varphi^4 \rangle^0_{SHELL}$

IS THE USUAL GAUSSIAN AVERAGE WHERE INTEGRATIONS ARE TAKEN OVER $\int_0^{\Lambda/b}$.

NOW ADD AND SUBTRACT

$$e^{-\lambda \langle \varphi^4 \rangle^0_{SHELL}} = e^{-\lambda \varphi^4_<} e^{-\lambda\left(\langle \varphi^4 \rangle^0_{SHELL} - \varphi^4_<\right)}$$

NOTE: AGAIN,
$\varphi^4_< \equiv \int_0^{\Lambda/b} d^d k_1 .. d^d k_4 \, \varphi(k_1) .. \varphi(k_4) \delta(1+..+$

AND REWRITE

$$H(\varphi^<) = \int_0^{\Lambda/b} dk \, \varphi^< \Gamma_0 \varphi^< + \lambda \int_0^{\Lambda/b} dk_1 dk_2 dk_3 dk_4 \, \varphi_<(1)\varphi_<(2)\varphi_<(3)\varphi_<(4) \delta(1+2+3+4)$$

$$+ \lambda\left(\langle \varphi^4 \rangle^0_{SHELL} - \varphi^4_<\right) - \frac{1}{2}\lambda^2\left(\langle \varphi^4 \varphi^4 \rangle^0_{SHELL} - \langle \varphi^4 \rangle^0_{SHELL} \langle \varphi^4 \rangle^0_{SHELL}\right) \qquad (\text{II})$$

NOW THE REAL CALCULATIONS. WE WANT TO EVALUATE

$$\langle \varphi^4 \rangle^0_{SHELL} = \left\langle \int_0^\Lambda dk_1 \int_0^\Lambda dk_2 \int_0^\Lambda dk_3 \int_0^\Lambda dk_4 \, \varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4) \delta(1+2+3+4) \right\rangle_{SHELL}$$

$$\int_0^\Lambda dk \, \varphi(k) = \int_0^{\Lambda/b} dk \, \varphi^<(k) + \int_{\Lambda/b}^\Lambda dk \, \varphi^>(k)$$

$$\langle \varphi^4 \rangle^0_{SHELL} = \left\langle \left(\int_0^{\Lambda/b} \varphi_1^< + \int_{\Lambda/b}^\Lambda \varphi_1^>\right)\left(\int_0^{\Lambda/b} \varphi_2^< + \int_{\Lambda/b}^\Lambda \varphi_2^>\right)\left(\int_0^{\Lambda/b} \varphi_3^< + \int_{\Lambda/b}^\Lambda \varphi^>_3\right)\left(\int_0^{\Lambda/b} \varphi_4^< + \int_{\Lambda/b}^\Lambda \varphi_4^>\right) \delta(1+..+4) \right\rangle^0_{SH}$$

OVER ↓ φ

$$= \varphi^4_< + \varphi^2_< \langle \varphi^2_> \rangle^0_{SHELL} + \langle \varphi^4_> \rangle^0_{SHELL}$$

*NOTE: i.e. TERMS WITH $\varphi_>$ OR $\varphi^3_>$.
I THINK THERE ARE REALLY 6 TERMS
LIKE $\varphi^2_< \langle \varphi^2_> \rangle^0_{SHELL}$.

ALL THE ODD TERMS* ARE NULL (THEY'RE GAUSSIAN INTEGRALS).

NOTICE THE LAST TERM

$\langle \varphi^4_> \rangle^0_{SHELL}$ = CONST., NOT DEPENDING ON $\varphi_<$

NOTE: $\varphi^4_<$ WILL SIMPLIFY (CHECK EQUATION (II)) WHEN YOU PUT $\langle \varphi^4 \rangle^0_{SHELL}$ BACK INTO $H(\varphi^<)$.

SO THE ONLY INTERESTING TERM IS THE SECOND: IT CONTAINS THE ACTION. LET'S EXPLICITLY WRITE IT :

$$\lambda \varphi^2_< \langle \varphi^2_> \rangle^0_{SHELL} = \lambda \int_0^{\Lambda/b} \varphi^< \varphi^< \int_{\Lambda/b}^\Lambda \langle \varphi^> \varphi^> \rangle^0_{SHELL}$$

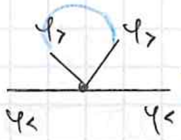
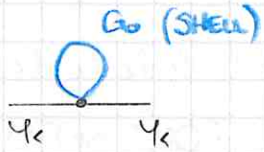
NOTE: THIS LINE IS STILL A BIT SYMBOLIC.

$$= \lambda \int_0^{\Lambda/b} d1 \, d2 \, \varphi^<(1)\varphi^<(2) \cdot \int_{\Lambda/b}^\Lambda d3 \, d4 \, \langle \varphi^<(3)\varphi^<(4) \rangle^0_{SHELL} \, \delta(1+2+3+4)$$

$$= \lambda \int_0^{\Lambda/b} d1 \, d2 \, \varphi^<(1)\varphi^<(2) \int_{\Lambda/b}^\Lambda d3 \, d4 \, G_0(3) \, \delta(3+4) \, \delta(1+2+(3+4))$$

$$= \lambda \int_0^{\Lambda/b} d1 \, d2 \, \varphi^<(1)\varphi^<(2) \delta(1+2) \int_{\Lambda/b}^\Lambda dq \, G_0(q) = \int_0^{\Lambda/b} dk \, \varphi^<(k)\varphi^<(-k) \cdot \lambda \int_{\Lambda/b}^\Lambda dq \, G_0(q)$$

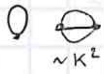
WE FOUND

$\lambda \varphi_<^2 \langle \varphi_>^2 \rangle^0_{\text{SHELL}} \equiv B(\varphi^<) = \Big\langle$  $\Big\rangle_{\text{SHELL}} =$ 

i.e. THE INTEGRATION PRODUCED A BUBBLE ON THE SHELL THAT DOESN'T DEPEND ON K: THUS, IT'S A CORRECTION TO THE MASS IN Γ_0.

MASS CORRECTION

$$\varphi^<(k^2+\mu^2)\varphi^< \xrightarrow[\text{COARSE GRAINING}]{} \varphi^<\left(k^2+\mu^2+\lambda\int_{\Lambda/b}^{\Lambda} d^dq \, \frac{1}{q^2+\mu^2}\right)\varphi^<$$

$G_0^{-1}(k) \xrightarrow[\text{C.G.}]{} G_0^{-1}(k) -$  $\dots = G_0^{-1}(k) - \Sigma_{\text{SHELL}}$

WHICH LOOKS A LOT LIKE DYSON'S EQUATION, BUT WITH NO IR DIVERGENCES, EVEN AT $T=T_c$!

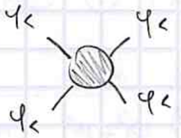
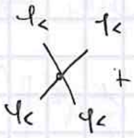
$$G_0^{-1}(k) \xrightarrow[\text{C.G.}]{} \underline{G_b^{-1}(k) = G_0^{-1}(k) - \Sigma_{\text{SHELL}}(k)}$$

BY CHANGING THE THEORY, WE STEPPED FROM

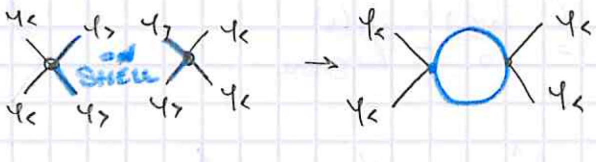
$$\mu^2 \longrightarrow \mu_b^2 = \mu^2 + \lambda\int_{\Lambda/b}^{\Lambda} d^dq \, \frac{1}{q^2+\mu^2}$$

COUPLING CONSTANT CORRECTIONS

AS FOR RENORMALIZATION, WE CHOOSE A TERM CONTAINING λ:

$\lambda \varphi^< \varphi^< \varphi^< \varphi^< \sim$  $\sim$  + (NON TRIVIAL)

AS WE DID FOR RENORMALIZATION, LET'S ISOLATE IT:



$$\langle(\varphi_<+\varphi_>)^4(\varphi_<+\varphi_>)^4\rangle^0_{\text{SHELL}} - \langle(\varphi_<+\varphi_>)^4\rangle^0_{\text{SHELL}}\langle(\varphi_<+\varphi_>)^4\rangle^0_{\text{SHELL}}$$

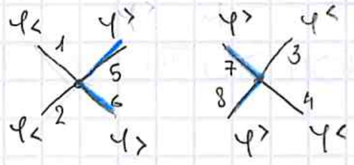
FOCUS ON

$$D \equiv \varphi_<\varphi_<\varphi_<\varphi_< \langle\varphi_>\varphi_>\varphi_>\varphi_>\rangle^0_{\text{SHELL}}$$

OTHER TERMS, LIKE

$$\varphi_< \varphi_< \varphi_< \varphi_< \varphi_< \varphi_< \langle \varphi_> \varphi_> \rangle \sim \varphi_<^6$$

WILL FORTUNATELY TURN OUT TO BE IRRELEVANT. WE'RE LEFT WITH



$$\Theta = \int_0^{\Lambda/b} \varphi_<(1)\varphi_<(2)\varphi_<(3)\varphi_<(4) \int_{\Lambda/b}^{\Lambda} \varphi_>(5)\varphi_>(6)\varphi_>(7)\varphi_>(8) \cdot \delta(1+2+5+6)\delta(7+8+3+4)$$

WE ONLY NEED TO EVALUATE THE SECOND PART AND WE'LL USE WICK'S THEOREM (KEEPING ONLY CONNECTED DIAGRAMS):

$$\Theta = \int_0^{\Lambda/b} \varphi_<\varphi_<\varphi_<\varphi_< \int_{\Lambda/b}^{\Lambda} d5\,d6\,d7\,d8\; G_0(5)\delta(5+7)G_0(6)\delta(6+8)\delta(1+2+5+6)\delta(7+8+3+4)$$

$$= \int_0^{\Lambda/b} \varphi_<\varphi_<\varphi_<\varphi_< \int_{\Lambda/b}^{\Lambda} d5\,d6\; G_0(5)G_0(6)\delta(1+2+5+6)\delta(-5-6+3+4)$$

$$= \int_0^{\Lambda/b} \varphi_<\varphi_<\varphi_<\varphi_< \int_{\Lambda/b}^{\Lambda} d5\; G_0(5)G_0(3+4-5)\delta(1+2+3+4)$$

NOTE:

FISH ON SHELL

WE FOUND

$$\Theta = \int_0^{\Lambda/b} \varphi_<(k_1)\ldots\varphi_<(k_4)\delta(1+2+3+4) \overbrace{\int_{\Lambda/b}^{\Lambda} d^dq\, \frac{1}{q^2+\mu^2}\cdot\frac{1}{(k_3+k_4-q)^2+\mu^2}}$$

WHICH IS THE SAME DIAGRAM WE'D HAVE FOUND WITH STANDARD RENORMALIZATION, BUT IT'S ON SHELL:

$$\lambda\varphi_<\varphi_<\varphi_<\varphi_< \xrightarrow[\text{COARSE GRAINING}]{} \left(\lambda - \frac{1}{2}\lambda^2 \int_{\Lambda/b}^{\Lambda} d^dq\, \frac{1}{q^2+\mu^2}\cdot\frac{1}{(k_3+k_4-q)^2+\mu^2}\right)$$

$\Gamma_0^{(4)} \xrightarrow[C.G.]{} \Gamma_0^{(4)} -$  $= \Gamma_0^{(4)} - \Sigma^{(4)}_{SHELL}$

$$\lambda \xrightarrow{\qquad} \lambda_b = \lambda - \frac{1}{2}\lambda^2 \int_{\Lambda/b}^{\Lambda} d^dq\, \frac{1}{q^2+\mu^2}\cdot\frac{1}{(k_3+k_4-q)^2+\mu^2}$$

WITH NO INFRARED DIVERGENCES.

★ THE NEW HAMILTONIAN LOOKS LIKE*

$$H(\varphi^<) = \int_0^{\Lambda/b} d^dk\, \varphi^<(k)\varphi^<(k)\left[k^2+\mu^2-\overset{\bigcirc}{\Sigma'^{(2)}_{\text{SHELL}}}\right]$$

$$+\int_0^{\Lambda/b} d^dk_1 \dots d^dk_4\, \varphi^<(k_1)\dots\varphi^<(k_4)\left[\lambda-\overset{\bigcirc}{\Sigma'^{(4)}_{\text{SHELL}}}\right] + (\text{BYPRODUCTS})$$

1) IT HAS A DIFFERENT MASS AND COUPLING CONSTANT

2) IT HAS A DIFFERENT CUTOFF: Λ/b (THAT'S WHY WE'LL RESCALE)

*NOTE: CHECK EQUATION (II) AND FOLLOWING
THE FOLLOWING IS TRUE @1 LOOP.

SPECIFICALLY,

$$\mu^2 \longrightarrow \mu_b^2 = \mu^2 - \Sigma'^{(2)}_{\text{SHELL}}(b) \qquad \Sigma'^{(2)}_{\text{SHELL}}(b) = \bigcirc = -\lambda\int_{\Lambda/b}^{\Lambda} d^dq\, \frac{1}{q^2+\mu^2}$$

$$\lambda \longrightarrow \lambda_b = \lambda - \Sigma'^{(4)}_{\text{SHELL}}(b) \qquad \Sigma'^{(4)}_{\text{SHELL}}(b) = \bigcirc = \frac{1}{2}\lambda^2\int_{\Lambda/b}^{\Lambda} d^dq\, \frac{1}{q^2+\mu^2}\cdot\frac{1}{(k_3+k_4-q)^2+}$$

· <u>RENORMALIZATION vs RENORMALIZATION GROUP</u> (1ST STEP)

$$G^{-1} = G_0^{-1} - \Sigma'^{(2)} = k^2+\mu^2-\Sigma'^{(2)} \qquad \Sigma'^{(2)} = \int_0^{\Lambda} \sim \bigcirc + \bigcirc$$

$$m^2 = \mu^2 - \Sigma'^{(2)}(k=0) \qquad \text{PHYSICAL QUANTITY} \sim \frac{1}{\chi}$$

IN RG, WE FIND INSTEAD

$$\mu_b^2 = \mu^2 - \Sigma'^{(2)}_{\text{SHELL}}(k=0) \qquad \Sigma'^{(2)}_{\text{SHELL}} = -\lambda\int_{\Lambda/b}^{\Lambda} d^dq\,\frac{1}{q^2+\mu^2}$$

WHERE μ_b^2 IS ANOTHER BARE PARAMETER AND THE ONLY DIVERGENCE WE SEE IS AT $T=T_c$.

HOWEVER, PRACTICALLY SPEAKING WE CAN USE THE SAME CALCULATIONS AS IN STANDARD RENORMALIZATION.

SIMILARLY,

$$g = \lambda - \Sigma'^{(4)}(\hat{k}) \quad \rightsquigarrow \quad \int_0^{\Lambda}$$

$$\lambda_b = \lambda - \Sigma'^{(4)}_{\text{SHELL}} \quad \rightsquigarrow \quad \int_{\Lambda/b}^{\Lambda}$$

WE USE THE SAME TECHNIQUES, BUT THEY'RE DIFFERENT CONCEPTS.

* NOW LET'S CALL

$$I_1(b) = \int_{\Lambda/b}^{\Lambda} d^d q \; \frac{1}{q^2+\mu^2}$$

$$I_2(b) = \int_{\Lambda/b}^{\Lambda} d^d q \; \frac{1}{q^2+\mu^2} \cdot \frac{1}{(k_{ext}-q)^2+\mu^2}$$

TAKING INTO ACCOUNT THE SYMMETRY FACTORS, WE GET

$$\mu_b^2 = \mu^2 + 3\lambda I_1(b)$$

$$\lambda_b = \lambda - 9\lambda^2 I_2(b)$$

- RG STEP 2: RESCALING

"RESCALE" MOMENTUM: $\Lambda/b \rightarrow \Lambda$

$k_b = b \cdot k$ $\qquad\qquad \Lambda_b = \frac{\Lambda}{b} \cdot b = \Lambda$

(IT'S REALLY ONLY A CHANGE OF VARIABLE). HENCE

$$H(\varphi^<) = \int_0^{\Lambda} d^d k_b \; \frac{1}{b^d} \varphi^<\left(\frac{k_b}{b}\right) \varphi^<\left(-\frac{k_b}{b}\right) \left[\frac{1}{b^2} k_b^2 + \mu^2 + 3\lambda I_1(b) \right] +$$

$$+ \int_0^{\Lambda} d^d k_b^{(1)} \dots d^d k_b^{(4)} \; \frac{1}{b^{4d}} \underset{\substack{\uparrow \\ \text{FROM THE } \delta(\dots)}}{b^d} \varphi^<(1) \dots \varphi^<(4) \left[\lambda - 9\lambda^2 I_2(b) \right] \delta^{(d)}_{(1+2+3+4)}$$

$$= \int_0^{\Lambda} d^d k_b \; \frac{1}{b^{d+2}} \varphi^<\left(\frac{k_b}{b}\right) \varphi^<\left(-\frac{k_b}{b}\right) \left[k_b^2 + \mu^2 b^2 + b^2 3\lambda_1 I_1(b) \right] +$$

$$+ \int_0^{\Lambda} d1 \dots d4 \; \frac{1}{b^{3d}} \varphi_1^< \dots \varphi_4^< \left[\lambda - 9\lambda^2 I_2(b) \right]$$

DEFINE

$$\frac{1}{b^{\frac{d+2}{2}}} \varphi^<\left(\frac{k_b}{b}\right) \equiv \varphi_b(k_b)$$

SAME AS GAUSSIAN, BECAUSE WE'RE WORKING AT 1 LOOP: IF YOU INCLUDED , YOU WOULD FIND ANOMALOUS DIMENSION SCALING.

IT FOLLOWS, FROM OUR DEFINITION, THAT

$$\varphi_<^4 = b^{2d+4}\varphi_b^4$$

HENCE

$$H(\varphi_b) = \int_0^\Lambda d^d k \, \varphi_b \varphi_b \left[k^2 + b^2\left(\mu^2 + 3\lambda I_1(b)\right)\right]$$

$$+ \int_0^\Lambda d^d k_1 \dots d^d k_4 \, \varphi_b(1) \dots \varphi_b(4) \, b^{4-d} \left[\lambda - 9\lambda^2 I_2(b)\right]$$

WHICH IS THE SAME HAMILTONIAN AS BEFORE, BUT WITH A NEW MASS AND COUPLING CONSTANT:

$$\begin{cases} \mu_b^2 = b^2\left(\mu^2 + 3\lambda I_1(b)\right) \\ \lambda_b = b^\varepsilon\left(\lambda - 9\lambda^2 I_2(b)\right) \end{cases} \qquad I_1(b) = \int_{\Lambda/b}^\Lambda \frac{d^d q}{q^2+\mu^2} \qquad I_2(b) = \int_{\Lambda/b}^\Lambda \frac{d^d q}{(q^2+\mu^2)\left((k_{ext}-q)^2+\mu^2\right)}$$

- THE CLASSIC RG FLOW STRUCTURE

$$X_b = b^\Delta \cdot X\left(1 + Y\, I(b)\right)$$

X_b: NEW BARE PARAMETER

b^Δ: NAIVE PHYSICAL DIMENSION OF X (STEP 2)

$I(b)$: ON-SHELL INTEGRALS

Y: COMBINATION OF PARAMETERS & COUPLING CONSTANT (PERTURBATION PART, STEP 1)

$$X_{l+1} = b^\Delta X_l \left(1 + Y_l\, I(b)\right)$$

FIXED POINT:

$$X^* = b^\Delta X^* \left(1 + Y^* \ln b\right)$$

(WE'LL SEE WHY $\ln b$).

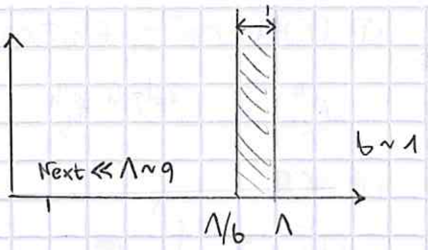
★ HOW COMES λ_b SEEMS TO DEPEND ON k_{ext}?

$$\int dx\, \varphi^4(x) \rightleftarrows \int dk_1 \dots dk_4 \, \varphi\varphi\varphi\varphi \, \delta(1+2+3+4) \underbrace{\lambda(3,4)}$$

NON LOCAL IN SPACE!

BUT

$$\int_{\Lambda/b}^{\Lambda} d^d q \; \frac{1}{(q^2+\mu^2)\left[(k_{ext}-q)^2+\mu^2\right]}$$



IS ON THE SHELL, WHERE q IS LARGE, AND WE'RE INTERESTED IN THE THEORY AT SMALL k (LARGE DISTANCE PHYSICS). WE MIGHT AS WELL CANCEL k_{ext} FROM THE INTEGRAL (WE'LL SEE IT AMOUNTS TO A SMALL CORRECTION IN ε).

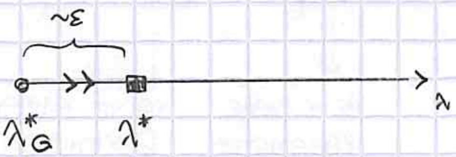
- THE ε - EXPANSION

GO RECURSIVE (RG FLOW EQUATIONS),

$$\begin{cases} \mu^2_{\ell+1} = b^2\left(\mu_\ell^2 + 3\lambda_\ell I_1(b)\right) \\ \lambda_{\ell+1} = b^{\varepsilon}\left(\lambda_\ell - 9\lambda_\ell^2 I_2(b)\right) \end{cases}$$

AT THE FIXED POINT,

$$\lambda^* = b^{\varepsilon}\left(\lambda^* - 9\lambda^{*2} I_2(b)\right)$$



$\lambda^*_G = 0$: GAUSSIAN FIXED POINT, UNSTABLE!

NOTE: I THINK THAT THE ACTUAL WAY TO CHECK THAT λ^*_G IS UNSTABLE IS TO CALCULATE $\beta(\lambda) = \frac{\partial \lambda}{\partial \ln b}$ FOR SMALL λ AND CHECK THAT $\beta > 0$.

SIMPLIFYING, WE GET A NEW ONE AT

$$\frac{1}{b^{\varepsilon}} = 1 - 9\lambda^* I_2(b)$$

$$\lambda^* = \left(1 - \frac{1}{b^{\varepsilon}}\right)\frac{1}{9 I_2(b)} = \left(\frac{b^{\varepsilon}-1}{b^{\varepsilon}}\right)\frac{1}{9 I_2(b)}$$

WHICH IS A NEW FIXED POINT AND IT'S NULL IF $\varepsilon = 0$ $(d=4)$.

FOR SMALL ε,

NOTE: A WAY TO SEE THIS IS $b^{\varepsilon} = e^{\varepsilon \ln b} \simeq 1 + \varepsilon \ln b + O(\varepsilon^2)$

$$\lambda^* = \varepsilon \frac{\ln b}{9 I_2(b)} \sim O(\varepsilon)$$

$\varepsilon \ll 1$

(WILSON-FISHER FIXED POINT).

WE OBSERVE THAT:

1) EVEN AT $d=4$ THE THEORY IS FREE/GAUSSIAN

2) IT SEEMS CONVENIENT TO EXPAND IN ε!

$d \simeq 4, \quad d = 4-\varepsilon$

THIS ALLOWS US TO EVALUATE ALL INTEGRALS @ $d=4$:

$$\lambda^p I(b;\varepsilon) = \underset{\downarrow \atop O(\varepsilon)}{\lambda^p} \left\{ I(b;0) + \varepsilon \Delta(b) \right\}$$

3) THE CORRECTION WE MADE BEFORE IS

$$\int d^d q \, \frac{1}{(k_{ext}-q)^2+\mu^2} \simeq \int d^d q \, \frac{1}{q^2+\mu^2} + O(\varepsilon)$$

i.e. BETTER THE SMALLER IS ε.

· LESSON 21.05.19

· RG RECURSIVE RELATIONS @ 1 LOOP

$$\mu^2_{\ell+1} = \boxed{b^2} \boxed{\left(\mu_\ell^2 + 3\lambda_\ell I_1(b)\right)}$$

$$\lambda_{\ell+1} = \boxed{b^\varepsilon} \boxed{\left(\lambda_\ell - 9\lambda_\ell^2 I_2(b)\right)}$$

↳ FROM STEP 2
RESCALING, NAIVE SCALING DIMENSION (PHYSICAL DIMENSION).

↳ FROM STEP 1
COARSE GRAINING, SHELL INTEGRATION $\int_{\Lambda/b}^{\Lambda}$
CORRECTION TO NAIVE SCALING DIMENSION.

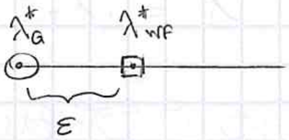
GAUSSIAN FIXED POINT:

$\lambda^*_G = 0$

IT'S UNSTABLE (b^ε, b^2 ARE GREATER THAN 1).

WILSON-FISHER FIXED POINT:

$$\lambda^*_{WF} = \left(\frac{b^\varepsilon-1}{b^\varepsilon}\right) \frac{1}{9 I_2(b)} \sim O(\varepsilon), \quad \varepsilon \to 0$$



THIS SUGGESTS IT'S CONVENIENT TO EXPAND THE THEORY IN ε.

LET'S TAKE

$$I_2(b) = \int_{\Lambda/b}^{\Lambda} d^d q \, \frac{1}{(q^2+\mu^2)^2} \qquad K_{ext} << q$$

AND SET $d=4$. IN FACT

$$\lambda^*_{WF} = \frac{\varepsilon \ln b}{g\, I_2(b,\varepsilon)} = \frac{\varepsilon \ln b}{g\, I_2(b,0)} + O(\varepsilon^2)$$

NOTE:
$$\frac{1}{I_2(b,\varepsilon)} = \frac{1}{I_2(b,0)} \cdot \frac{1}{1+O(\varepsilon)} = \frac{1}{I_2(b,0)}\left(1+O(\varepsilon)\right).$$

$$I_2(b,\varepsilon) = I_2(b,0) + O(\varepsilon)$$

LET'S THEN EVALUATE

$$I_2(b, d=4) = \int_{\Lambda/b}^{\Lambda} d^4 q \, \frac{1}{(q^2+\mu^2)^2} \qquad q \equiv \Lambda x$$

$$= \int_{1/b}^{1} d^4 x \, \frac{1}{\left(x^2 + \frac{\mu^2}{\Lambda^2}\right)^2} \quad \cancel{\Lambda^{4-4}}$$

IF WE TAKE $b \simeq 1$, THE SHELL THICKNESS IS

$$1 - \frac{1}{b} = \frac{b-1}{b} \simeq \ln b$$

NOTE: HE KEEPS DOING THIS, BUT ACTUALLY $1-\frac{1}{b} = 1 - e^{-\ln b} \simeq \ln b$

HENCE* (WE WILL HAVE TO CHECK THAT $\frac{\mu^2}{\Lambda^2} << 1$)

$$I_2(b, d=4) \simeq S_4 \ln b \, \frac{1}{\left(1+\mu^2/\Lambda^2\right)^2} \simeq S_4 \ln b$$

WE'LL SEE ALL SHELL INTEGRALS BOIL OUT AS $\ln b$.

NOW THE EQUATION FOR $\lambda_{\ell+1}$ IS INDEPENDENT OF μ^2:

$$\lambda_{\ell+1} = b^{\varepsilon} \lambda_\ell (1 - g \lambda_\ell S_4 \ln b)$$

$$\cancel{\lambda^*} = b^{\varepsilon} \cancel{\lambda^*} (1 - g\lambda^* S_4 \ln b)$$

$$1 = (1+\varepsilon \ln b)(1 - g\lambda^* S_4 \ln b) \implies \lambda^*_{WF} = \frac{\varepsilon}{g S_4}$$

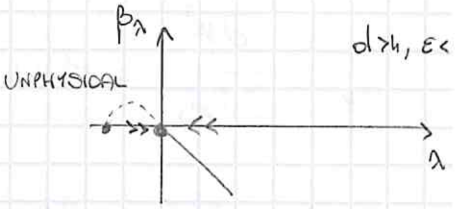
*NOTE: S_4 IS THE SURFACE OF AN HYPERSPHERE IN 4 DIMENSIONS WITH UNIT RADIUS (i.e. $\frac{4}{3}\pi$, I THINK). IT WILL CANCEL OUT ANYWAY.

IS IT STABLE? TO SEE THAT, WE NEED TO COMPUTE THE β-FUNCTION.

· β - FUNCTION OF λ

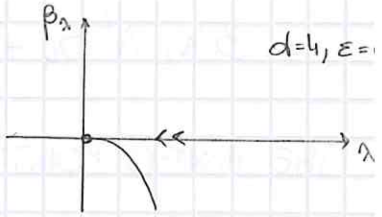
$$\beta_\lambda = \frac{\partial \lambda}{\partial \ln b} = \lambda(\varepsilon - 9 S_4 \lambda)$$

$b \simeq 1$!



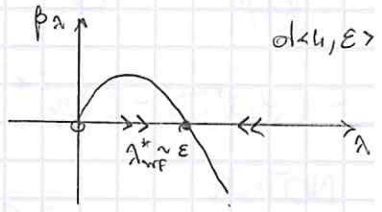
THE FIXED POINT MAY BE FOUND BY REQUIRING

$$\beta(\lambda^*) = 0 \quad \rightarrow \quad \lambda^* = \varepsilon / 9 S_4$$



BY LOOKING AT ITS SHAPE, WE SEE THE THEORY IS IR FREE FOR $d \geq 4$.

HOWEVER, λ^*_{WF} IS STABLE FOR $d<4$.



· RG EQUATION FOR μ^2

$$\mu^2_{l+1} = b^2\left(\mu^2_l + 3\lambda_l I_1(b)\right)$$

$$I_1(b) = \int_{\Lambda/b}^{\Lambda} d^d q \, \frac{1}{q^2+\mu^2} \overset{d=4}{\simeq} \int_{\Lambda/b}^{\Lambda} d^4 q \, \frac{1}{q^2+\mu^2} \qquad q \equiv \Lambda x$$

$$= \int_{1/b}^{1} \Lambda^2 d^4 x \, \frac{1}{x^2 + \mu^2/\Lambda^2} \overset{\text{THIN SHELL}}{\simeq} \Lambda^2 \ln b \, S_4 \frac{1}{(1+\mu^2/\Lambda^2)}$$

KEEP THIS, YOU HAVE Λ^2 IN FRONT

$$\simeq \Lambda^2 \ln b \, S_4 \left(1 - \frac{\mu^2}{\Lambda^2}\right)$$

HENCE

$$\mu^2_{l+1} = b^2\left[\mu^2_l + 3\lambda_l \Lambda^2 S_4 \left(1 - \frac{\mu^2_l}{\Lambda^2}\right)\ln b\right]$$

· β - FUNCTION OF μ^2

$$b^2 \simeq 1 + 2\ln b$$

$$\mu^2_{l+1} = (1+2\ln b)\left[\mu^2_l + 3\lambda_l \Lambda^2 S_4\left(1-\frac{\mu^2_l}{\Lambda^2}\right)\ln b\right]$$

$$= \mu^2_l + 3\lambda_l \Lambda^2 S_4 \left(1 - \frac{\mu^2_l}{\Lambda^2}\right)\ln b + 2\mu^2_l \ln b$$

THIS CAN BE DERIVED TO FIND

$$\beta_{\mu^2} = \frac{\partial \mu^2}{\partial \ln b} = 3\lambda_l \Lambda^2 S_4 \left(1 - \frac{\mu_l^2}{\Lambda^2}\right) + 2\mu_l^2$$

$$= 3\lambda_l \Lambda^2 S_4 - 3\lambda_l S_4 \mu_l^2 + 2\mu_l^2$$

$$= 3\lambda_l \Lambda^2 S_4 + (2 - 3\lambda_l S_4)\mu_l^2$$

MA SE IL FLUSSO DI λ FOSSE DIPESO DA μ IN MODO NON BANALE, COSÌ CHE NON SI POTESSERO METTERE A SISTEMA ??

THE FIXED POINT IS GIVEN BY

$$\beta(\mu^{*2}) = 0$$

$$(3\lambda^* S_4 - 2)\mu^{2*} = 3\lambda^* \Lambda^2 S_4$$

$$\lambda^* = \frac{\varepsilon}{9 S_4}$$

NOTICE

$$3\lambda^* S_4 = \frac{\boxed{3}\,\varepsilon S_4}{\boxed{9}\, S_4} \underbrace{\left(\frac{(2\pi)^4}{(2\pi)^4}\right)}_{\text{FROM F.T.}} = \frac{\varepsilon}{3}$$

SYMMETRY FACTORS

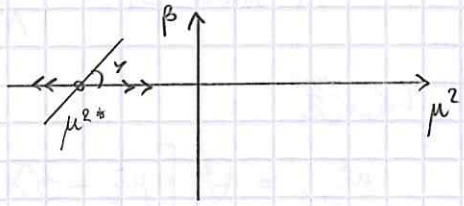
HENCE

$$\underline{\mu^{2*} = -\frac{\varepsilon}{6}\Lambda^2}$$

$$\frac{\mu^{2*}}{\Lambda^2} \ll 1$$

(NOT UNIVERSAL, IT'S NOT AN EXPONENT: $\mu^2 = \frac{T - T_0}{T_0}$).

MOREOVER,



$$\beta(\mu^2) = 3\lambda^* S_4 \Lambda^2 + (2 - 3\lambda^* S_4)\mu^2$$

$$= \frac{\varepsilon}{3}\Lambda^2 + \left(2 - \frac{\varepsilon}{3}\right)\mu^2 \equiv y\left(\mu^2 - \mu^{2*}\right)$$

THE SCALING DIMENSION OF THE MASS (TEMPERATURE) IS THEN

$$\underline{y = \frac{d\beta}{d\mu^2} = 2 - \frac{\varepsilon}{3}}$$

- <u>CRITICAL EXPONENT ν</u>

$$\begin{cases} t \to b^y t \\ \xi \to \frac{1}{b}\xi \end{cases}$$

RECALL

$$\begin{cases} \beta_t = y \cdot t \\ \beta_\xi = -1 \cdot \xi \end{cases} \rightarrow \frac{\partial \ln \xi}{\partial \ln t} = -\frac{1}{y}$$

NOTE: $\frac{1}{t}\frac{\partial t}{\partial \ln b} = \frac{\partial \ln t}{\partial \ln b} = y$.

SO THAT

$$\xi \sim (t)^{-1/y} = \frac{1}{t^{1/y}} \qquad \xi = \frac{1}{(T-T_c)^{\boxed{1/y}}} \rightarrow \nu$$

$$\nu = \frac{1}{2-\varepsilon/3} = \frac{1}{2}\left(1-\frac{\varepsilon}{6}\right)^{-1} = \frac{1}{2}\left(1+\frac{\varepsilon}{6}\right)$$

IF $d=3$, $\varepsilon = 1$ (!!) AND WE FIND

$$\begin{cases} \nu_{GAUSS} = \frac{1}{2} = 0.5 \\ \nu_\varepsilon = 0.58 \\ \nu_{exp} = 0.62 \end{cases}$$

6% ERROR

20% ERROR

OBSERVE

$$\underline{\nu = \frac{1}{2} + \frac{\varepsilon}{12}}$$

THERE ARE:

- NO Λ
- NO S_4, 2π ...
- NO BARE PARAMETERS

THIS IS UNIVERSALITY: CRITICAL EXPONENTS DO NOT DEPEND ON THE DETAILS OF THE THEORY.

- <u>CRITICAL EXPONENT γ</u>

$$\frac{\gamma}{\nu} = 2 - \eta$$

FROM

$$\chi = \beta \int d^d r \, C(r) = \beta \, G(k=0)$$

NOTE:

$$\chi = \beta G(k=0) = \beta \int d^d x \frac{1}{x^{d-2+\eta}} f\left(\frac{x}{\xi}\right)$$

$$\simeq \int^\xi d^d x \frac{1}{x^{d-2+\eta}} \sim \xi^{2-\eta} \sim (T-T_0)^{-\nu(2-\eta)}$$

AT 1 LOOP, $\eta = 0$: φ HAS THE NAIVE SCALING DIMENSION.

$$\gamma = 2\nu = \frac{2}{2-\varepsilon/3} = 1 + \frac{\varepsilon}{6}$$

FOR $d=3$,

$$\begin{cases} \gamma_{GAUSS} = 1 \\ \gamma_{\varepsilon} = 1.17 \\ \gamma_{exp} = 1.24 \end{cases} \quad \Big\uparrow 6\% \quad \Big\uparrow 20\%$$

- <u>NONTRIVIAL FIELD RENORMALIZATION</u>

$$\varphi_b = \frac{1}{b^{\frac{d+2}{2}}}\varphi$$

NAIVE DIM: $G(k) = \frac{1}{k^2}$, $\eta = 0$

NOTE: THIS IS K SPACE. IN REAL SPACE IT WOULD BE $\varphi_b = \frac{1}{b^{(d-2)/2}}\varphi$.

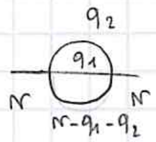
AFTER SHELL INTEGRATION,

$$\int_0^{\Lambda/b} d^dk\, \varphi^< \varphi^< \left[\mu^2(1+X_b) + (..)k^2\right] + \lambda(1+Y_b)\int \ldots$$

$\uparrow$ ℓ $\qquad \downarrow (1+Z_b) \sim O(\varepsilon^2) \qquad \uparrow$ ≻○≺

@ 2 LOOPS,

$$G^{-1} = \ldots$$



$$\int_{\Lambda/b}^{\Lambda} \sim k^2 \quad \rightarrow Z_b$$

AFTER SOME CALCULATIONS, YOU GET

$$Z_b = \eta \ln b \qquad\qquad \eta \sim O(\varepsilon^2)$$

$$(1+Z_b) = 1 + \eta \ln b \simeq b^{\eta}$$

$$\int^{\Lambda/b} d^dk\, \varphi^<\varphi^< \left(b^{\eta}k^2 + \mu^2(1+X_b)\right) + \ldots$$

RESCALING $k_b = b\cdot k$,

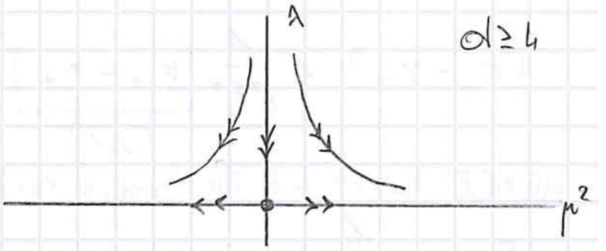
$$= \int^{\Lambda} d^dk_b\, \frac{1}{b^d}\varphi^<\varphi^< \left(\frac{b^{\eta}}{b^2}k_b^2 + \mu^2(1+X_b)\right) + \ldots$$

$$= \int^{\Lambda} d^dk_b\, \frac{1}{b^{d+2-\eta}}\varphi^<\varphi^< \left(k_b^2 + \mu^2 \underset{\downarrow\, O(\varepsilon^2)}{b^{2-\eta}} \underset{\searrow\, O(\varepsilon)}{(1+X_b)}\right) + \ldots$$

DEFINE A FIELD

$$\varphi_b = \frac{1}{b^{\frac{d+2-\eta}{2}}}\varphi_< \quad \rightarrow \quad G(k) = \frac{1}{k^{2-\eta}}$$

- FLOW DIAGRAM

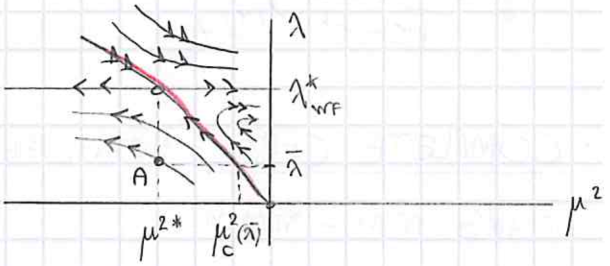


FOR $d \geq 4$, THE FIXED POINT IS

$\lambda^* = 0$

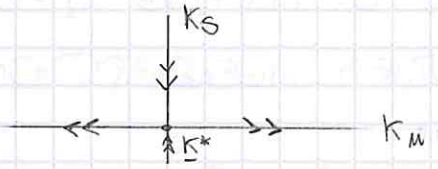
$\mu^{2*} = 0 \quad (T_c = T_0)$

$Y = 2, \nu = \frac{1}{2}, \gamma = 1$

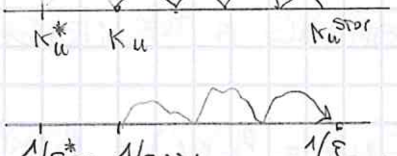


FOR $d < 4$, THE RED LINE IS THE CRITICAL MANIFOLD. BEING THE LATTER THE SET OF ALL CRITICAL POINTS, POINT A IS NOT CRITICAL (EVEN THOUGH IT HAS THE "CRITICAL TEMPERATURE"). IF THE COUPLING CONSTANT IS $\bar{\lambda}$, ITS CRITICAL TEMPERATURE IS $\mu^2_c(\bar{\lambda})$, NOT μ^{2*}.

- WORKING AROUND A CRITICAL POINT



EASY CASE: K_S AND K_u ARE THE STABLE AND UNSTABLE DIRECTIONS, AND THEY'RE ORTOGONAL (DIAGONAL CASE).



NOTE: K'S ARE PARAMETERS, NOT WAVENUME

WE START CLOSE TO THE CRITICAL POINT, AND STOP THE FLOW WHEN

START: $K_u \sim K_u^* \qquad \xi >> 1$

STOP: $K_u = K_u^* + O(1) \qquad \xi_m \sim \frac{1}{\Lambda}$ SMALL

$$\xi_{l+1} = \frac{1}{b}\xi_l \quad \rightarrow \quad \xi_m = \frac{1}{b^m}\cdot\xi \overset{\overset{\text{STOP CONDITION}}{\downarrow}}{=} \frac{1}{\Lambda}$$

THE STOP CONDITION IS

$b^m = \xi \cdot \Lambda$

NOTE: A FEW LECTURES AGO, THE STO CONDITION WAS $\xi_m \sim O(1)$. NOW WE PICKED $\xi_m \sim \frac{1}{\Lambda} \sim a$, THE LATTICE SPACING, WHICH IS REASONABLE.

BY K WE WILL MEAN K_u FROM NOW ON.

RUNNING THE RG FLOW UP UNTIL THE STOP CONDITION*,

$$(K_{l+1} - K_*) = b^y (K_l - K_*)$$

$$(K_m - K_*) = (b^m)^y (K_0 - K_*)$$

$$(K_0 - K_*) = \frac{1}{\Lambda^y \xi^y} \underbrace{(K_m - K_*)}_{O(1)}$$

NOTE: ξ DOWN HERE IS ACTUALLY ξ_0, THE ξ WE START FROM. THEN WE CAN DROP THE "0".

$$b^m \sim \xi \cdot \Lambda$$

(INVERTED, REWIND)

HENCE (DROPPING THE SUFFIX FROM K_0)

$$\xi \sim \frac{1}{(K - K^*)^{\enclose{circle}{1/y}\ \nu}}$$

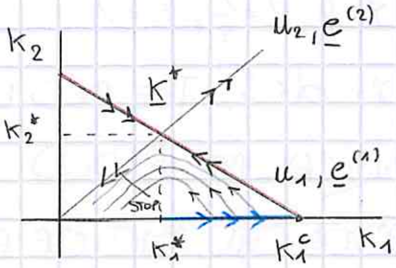
*NOTE: WHAT'S GOING ON? AS AN EXAMPLE,

$u_\chi = 1/\chi$, $u^*_\chi = 0$, $u_{l+1} = b^{y_u} u_l$, β(

$t = T - T_c$, $t^* = 0$, $t_{l+1} = b^{y_t} t_l$, β(

WE ALWAYS CHOOSE AS SCALING VARIABLES QUANTITIES THAT ARE NULL AT THE FIXED POIN

- COMPLETE CASE (NON-DIAGONAL)



ISING mm + mmm

IF YOU SET $K_2 = 0$, THE CRITICAL VALUE OF THE MODEL IS K_1^c. IF YOU RUN A SIMULATION, YOU FIND

$$\xi \sim \frac{1}{(K_1 - K_1^c)^\nu} = \frac{1}{(K_1 - K_1^c)^{1/y_{\text{UNSTABLE}}}}$$

EVEN IF YOU MOVE ALONG THE BLUE LINE, i.e. IF YOU APPROACH K_1^c BY KEEPING $K_2 = 0$! BUT y_{UNSTABLE} IS THE EXPONENT OF THE OTHER EIGENDIRECTION... DOESN'T THIS MAKE YOU ANGRY?

LET $\underline{K}^*$ BE THE FIXED POINT, i.e.

$$\underline{R}(\underline{K}^*) = \underline{K}^*$$

NOTE: DOES IT MEAN WE HAVE TO CHOOSE $\underline{K}$ START IN THE NEIGHBORHOOD OF $\underline{K}^*$? ACTUALLY WE NEED TO BE CLOSE ENOUGH SO THAT WE'R ACTUALLY ATTRACTED BY THE FLOW.

STARTING IN THE NEIGHBORHOOD OF $\underline{K}^*$,

$$\underline{K}_{l+1} = \underline{R}(\underline{K}_l) = \underline{R}(\underline{K}^*) + \left.\frac{\partial R}{\partial K}\right|_{\underline{K}^*} \cdot (\underline{K}_l - \underline{K}^*)$$

NOTE: CLEARLY, IT'S

$$= \underline{R}(\underline{K}^*) + (\underline{\nabla} \otimes \underline{R})\big|_{\underline{K}^*} \cdot (\underline{K}_l - \underline{K}_*)$$

DEFINE THE MATRIX (NON SYMMETRICAL IF THE DIRECTIONS AREN'T ORTHOGONAL

$$\left.\frac{\partial R_a}{\partial K_b}\right|_{K^*} \equiv T_{ab} \qquad \Rightarrow \qquad (\underline{K}_{l+1} - \underline{K}^*) = T(\underline{K}_l - \underline{K}^*)$$

LET'S INTRODUCE THE LEFT EIGENVECTOR OF T,

$$\underline{e}^{(i)} T = \lambda_i \underline{e}^{(i)}$$

DEFINE THE SCALING VARIABLES

$$u_i = \underline{e}^{(i)} \cdot (\underline{K} - \underline{K}^*)$$

THE RGT BECOMES

$$u_i^{l+1} = \underline{e}^{(i)} \cdot (\underline{K}^{l+1} - \underline{K}^*) = \underline{e}^{(i)} T (\underline{K}^{l} - \underline{K}^*) = \lambda_i \, \underline{e}^{(i)} (\underline{K}_l - \underline{K}^*) = \lambda_i \, u_i^l$$

$$\underline{u_i^{l+1} = \lambda_i u_i^l \overset{*}{=} b^{y_i} u_i^l} \quad \rightarrow \quad \begin{cases} y_i > 0 \rightarrow u_i \text{ UNSTABLE } (u_i^* = 0) \\ y_i < 0 \rightarrow u_i \text{ STABLE (IRRELEVANT)} \end{cases}$$

*NOTE: TAKE IT AS THE DEFINITION OF y_i.

ANY $\underline{\Delta K}$ MAY BE WRITTEN AS

$$\underline{\Delta K} = (\underline{K} - \underline{K}^*) = \sum_i u_i \, \underline{e}^{(i)}$$

NOTE:
$$= \sum_i (\underline{e}^{(i)} \cdot \underline{\Delta K}) \, \underline{e}^{(i)}$$

SO THAT ITS EVOLUTION IS GIVEN BY

$$(\underline{K}^{l+1} - \underline{K}^*) = \sum_i b^{y_i} u_i^l \, \underline{e}^{(i)}$$

ITERATING m TIMES,

$$(\underline{K}^{m} - \underline{K}^*) = \sum_i (b^m)^{y_i} u_i \, \underline{e}^{(i)}$$

↳ COORDINATES OF THE STARTING POINT OF RGT

OUR CASE
↓
$$= (b^m)^{y_1} u_1 \underline{e}^{(1)} + (b^m)^{y_2} u_2 \underline{e}^{(2)}$$

ALL THE IRRELEVANT COORDINATES WILL SHRINK TO ZERO.

THE CRITICAL MANIFOLD IS SPANNED BY u_1 ALONE $(u_2 = 0)$.

* HOW DOES ξ DIVERGE AS WE APPROACH THE PHYSICAL CRITICA POINT $(K_1^c, 0)$?

START: $\begin{cases} u_1^{START} \sim O(1) \\ u_2^{START} << 1 \end{cases}$ BECAUSE $\xi >> 1$

STOP: $\begin{cases} u_1^{STOP} << 1 \\ u_2^{STOP} \sim O(1) \end{cases}$

BECAUSE CLOSE TO $u_1^* = 0$

BECAUSE $\xi_{STOP} \sim \frac{1}{\Lambda}$

LET'S DERIVE THEM:

$$u_2^{STOP} = (b^{y_2})^m u_2^{START} \xrightarrow[\text{REWIND}]{} u_2^{START} = \left(\frac{1}{b^m}\right)^{y_2} u_2^{STOP}$$

USING

$$\xi_m = \frac{1}{b^m}\xi \overset{\text{STOP}}{\overset{\downarrow}{=}} \frac{1}{\Lambda} \qquad \rightarrow \qquad \xi \sim b^m$$

NOTE: THERE'S NO NEED TO SPECIFY
$\xi = b^m \Lambda$
JUST REMEMBER DISTANCES ARE MEASURED IN UNITS OF THE LATTICE SPACING.
ALSO RECALL THAT $u_2^{\text{STOP}} \sim O(1)$.

WE FIND

$$u_2^{\text{START}} \sim \frac{1}{\xi^{y_2}}$$

$$\left(\underline{K}^{\text{START}} - \underline{K}^*\right) = u_1^{\text{START}}\, \underline{e}^{(1)} + \frac{1}{\xi^{y_2}}\, \underline{e}^{(2)}$$

LET'S USE IT NOW ON TWO POINTS: $\overset{\text{GENERIC, } \xi \gg 1}{(K_1, 0) \equiv \underline{K}}$ AND $\overset{\xi = \infty}{(K_1^c, 0) \equiv \underline{K}_c}$.
THIS WAY WE CAN GET RID OF $\underline{K}^*$. WE FIND

@ GENERIC, $\quad \underline{K} - \underline{K}^* = u_1\, \underline{e}^{(1)} + \frac{1}{\xi^{y_2}}\, \underline{e}^{(2)}$

@ CRITICAL, $\quad \underline{K}_c - \underline{K}^* = u_1^c\, \underline{e}^{(1)}$

SUBTRACT TO ELIMINATE $\underline{K}^*$:

$$(\underline{K} - \underline{K}_c) = (u_1 - u_1^c)\, \underline{e}^{(1)} + \frac{1}{\xi^{y_2}}\, \underline{e}^{(2)}$$

BOTH POINTS HAVE $K_2 = 0$, SO

NOTE: WE'RE PROJECTING THE EQUATION ONTO $\hat{K}_2$ AND $\hat{K}_1$ INSTEAD OF USING THE EIGENVECTOR BASIS.

$$0 = (u_1 - u_1^c)\, e_2^{(1)} + \frac{1}{\xi^{y_2}}\, e_2^{(2)} \qquad \rightarrow \qquad (u_1 - u_1^c) = -\frac{1}{\xi^{y_2}} \frac{e_2^{(2)}}{e_2^{(1)}}$$

$$(K_1 - K_1^c) = -\frac{1}{\xi^{y_2}} \underbrace{\frac{e_2^{(2)}}{e_1^{(2)}}}_{\text{IT'S TELLING YOU HOW SKEWD THE EIGENVECTOR SYSTEM IS WRT THE ORIGINAL ONE}} e_1^{(1)} + \frac{1}{\xi^{y_2}}\, e_1^{(2)} = \boxed{-}\ \frac{1}{\xi^{y_2}}$$

HENCE

$$\xi \sim \frac{1}{(K_1 - K_1^c)^{1/y_2}}$$

LESSON 24.05.19

SPONTANEOUS SYMMETRY BREAKING

LET'S CONSIDER A COARSE GRAINED "HAMILTONIAN", WHICH IS REALL AN EFFECTIVE ENERGY, e.g. LANDAU GINZBURG $H(\varphi)$. IF

$$H(R\varphi) = H(\varphi)$$

THEN H IS SYMMETRIC UNDER R.

THE GROUND STATE OF L-G IS $\varphi_0(\beta)$, THE MINIMUM OF $H(\varphi)$. IF

1) φ_0 SHARES THE SAME SYMMETRY OF H,

$$R\varphi_0 = \varphi_0$$

THE SYMMETRY IS NOT BROKEN.

2) φ_0 DOES NOT SHARE THE SAME SYMMETRY,

$$R\varphi_0 \neq \varphi_0$$

THEN YOU HAVE SSB.

REMARK

THE SYMMETRY MAY BE BROKEN NON-SPONTANEOUSLY, e.g. BY ADDING A FIELD WHICH MODIFIES THE HAMILTONIAN

$$H \to H - h\varphi = H(h) \qquad \to \qquad H(R\varphi; h) \neq H(\varphi; h)$$

HOWEVER, φ_0 ITSELF IS USUALLY NOT SYMMETRIC AFTER THE INTRODUCTION OF h, SO THAT NO AMBIGUITY ARISES.

REMARK

IF THERE'S SSB, THEN IN PARTICULAR

$$H(R\varphi_0) \equiv H(\varphi_0') = H(\varphi_0)$$

i.e., WE'VE FOUND ANOTHER GROUND STATE. THIS MEANS THERE ARE AT LEAST TWO SUCH STATES (BUT IT CAN BE A CONTINUUM), AND R TAKES YOU FROM ONE TO THE OTHER.

$$\begin{tikzcd}
\varphi_0 \arrow[r, "R"] & \varphi_0' \arrow[d] \\
\varphi_0''' \arrow[u] & \varphi_0'' \arrow[l]
\end{tikzcd}$$

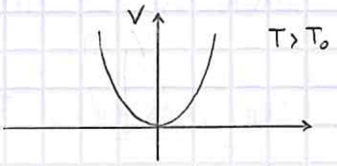
· DISCRETE SSB

ISING (MICROSCOPIC):

$H(\sigma)=H(-\sigma)$ $\qquad$ $R\sigma=-\sigma$

THIS IS OBVIOUSLY SYMMETRIC, BUT THE HAMILTONIAN WE HAVE TO CHECK IS ITS COARSE GRAINED VERSION, L-G $\lambda\varphi^4$, $\varphi\in\mathbb{R}$:

$$H=\int d^dx\,\{(\underline{\nabla}\varphi)^2+\underbrace{\mu^2\varphi^2+\lambda\varphi^4}_{V(\varphi)}\}$$

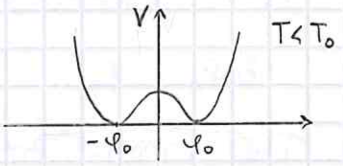


FOR $T>T_0$, $H(-\varphi)=H(\varphi)$, $\varphi_0=0$, $R\varphi_0=\varphi_0$.

IF $T<T_0$, $H(-\varphi)=H(\varphi)$ BUT

$\varphi_0\neq0$, $R\varphi_0\neq\varphi_0$

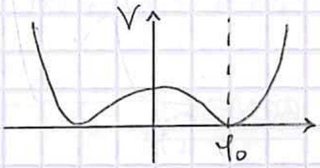
AND R TAKES YOU FROM ONE TO THE OTHER.



TAKING FLUCTUATIONS INTO ACCOUNT ($\psi(x)$), FOR $T<T_0$ WE DEFINE

$$\varphi(x)=\varphi_0+\psi(x)$$

$$V(\psi)=V_0+\underbrace{\frac{\partial^2V}{\partial\varphi^2}(\varphi_0)}_{\equiv\mu_\psi^2\ \text{BARE!}}\cdot\psi^2(x)$$



FOR L-G,

$$V(\varphi)=\mu^2\varphi^2+\lambda\varphi^4$$

$$\dot V(\varphi)=0 \quad\Rightarrow\quad \varphi_0^{\,2}=-\frac{\mu^2}{2\lambda} \qquad (\mu^2<0)$$

$$\mu_\psi^2=-4\mu^2=4|\mu_\varphi^2|$$

AND YOU CAN BUILD A NEW THEORY THAT IS PRETTY MUCH THE SAME AS THE PREVIOUS: μ_ψ^2 IS PROPORTIONAL TO μ_φ^2, AND THE GRADIENT PART IS EXACTLY THE SAME ($\varphi(x)$ AND $\psi(x)$ ONLY DIFFER BY A CONSTANT).

· CONTINUOUS SSB

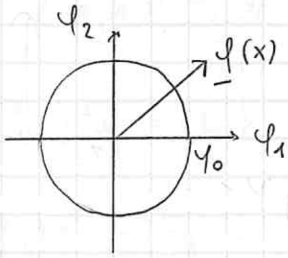
(NAMBU PR 1960, GOLDSTONE Nuovo Cimento 1961)

IN ORDER TO HAVE A CONTINUOUS SYMMETRY, YOU NEED AT LEAST A 2-COMPONENTS FIELD:

$\underline{\varphi}(x)=(\varphi_1(x),\varphi_2(x))$ $\qquad$ $SO(2)$ $(U(1))$

THE L-G HAMILTONIAN BECOMES

$$H = \int d^d x \left\{ \partial_a \varphi^\alpha \partial^a \varphi_\alpha + \mu^2 \varphi_\alpha \varphi^\alpha + \lambda (\varphi_\alpha \varphi^\alpha)^2 \right\}$$



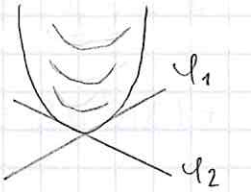
WHICH IS CLEARLY ROTATIONALLY INVARIANT IN

$\alpha = 1, \dots, D$ INTERNAL SPACE $\qquad RR^T = 1$

NOT TO BE CONFUSED WITH

$a = 1, \dots, d$ EXTERNAL SPACE

NOTE:
$$\partial_a \varphi^\alpha \partial^a \varphi_\alpha \equiv \sum_{a=1}^{d} \sum_{\alpha=1}^{D} (\partial_a \varphi^\alpha$$

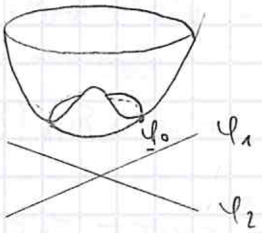


IF $\mu^2 > 0$ $(T > T_c)$,

$\underline{\varphi}_0 = 0, \quad R\underline{\varphi}_0 = \underline{\varphi}_0$

IF $\mu^2 < 0$ $(T < T_c)$,

$V(\underline{\varphi}) = V(\varphi)$ $\qquad$ (BARE POTENTIAL)



ANY $\underline{\varphi}$ S.t.

$$|\underline{\varphi}_0|^2 = -\frac{\mu^2}{2\lambda}$$

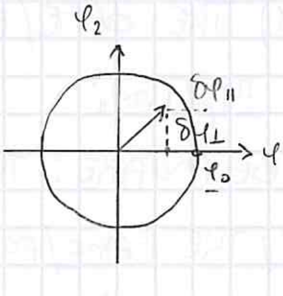
IS A L-G GROUND STATE.

NOTICE THE SYSTEM ALLOWS FOR SMALL ROTATIONS,

$$R_\theta \underline{\varphi}_0 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1^\circ \\ \varphi_2^\circ \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \begin{pmatrix} \varphi_1^\circ \\ \varphi_2^\circ \end{pmatrix} \neq \begin{pmatrix} \varphi_1^\circ \\ \varphi_2^\circ \end{pmatrix}$$

★ REWRITE H AROUND AN ARBITRARY $\underline{\varphi}_0$:

$$\underline{\varphi}_0 = (\varphi_1^\circ, \varphi_2^\circ) \qquad \begin{cases} \varphi_1^\circ = \left(-\frac{\mu^2}{2\lambda}\right)^{1/2} \equiv a \\ \varphi_2^\circ = 0 \end{cases}$$



EVEN BY MERELY CHANGING THE ANGLE, YOU WOULD CREATE $\delta\varphi_\parallel$ AND $\delta\varphi_\perp$. LET'S REWRITE

$$\underline{\varphi} = (\varphi_1^\circ + \delta\varphi_\parallel, \varphi_2^\circ + \delta\varphi_\perp) = (a + \delta\varphi_\parallel, \delta\varphi_\perp)$$

THE GRADIENT BECOMES

$$(\underline{\nabla}\underline{\varphi})^2 \equiv (\underline{\nabla}\varphi_\parallel)^2 + (\underline{\nabla}\varphi_\perp)^2$$

NOTE: IT'S JUST SHORTHA
REFER TO THE PREVIOUS NOT

THE POTENTIAL CAN THEN BE EXPANDED AS

$$V(\underline{\varphi}) = \mu^2(\varphi_1^2+\varphi_2^2) + \lambda(\varphi_1^2+\varphi_2^2)^2$$

$$= \mu^2(a+\delta\varphi_\parallel)^2 + \mu^2\delta\varphi_\perp^2 + \lambda\left[(a+\delta\varphi_\parallel)^2+\delta\varphi_\perp^2\right]^2$$

$$= \mu^2a^2 + \mu^2\delta\varphi_\parallel^2 + 2\mu^2a\delta\varphi_\parallel + \mu^2\delta\varphi_\perp^2 + \lambda\left[a^2+\delta\varphi_\parallel^2+2a\delta\varphi_\parallel+\delta\varphi_\perp^2\right]^2$$

$$= \mu^2a^2 + \mu^2\delta\varphi_\parallel^2 + 2\mu^2a\delta\varphi_\parallel + \mu^2\delta\varphi_\perp^2 + \lambda a^4 + \lambda\delta\varphi_\parallel^2 + \lambda 4a^2\delta\varphi_\parallel^2 + \lambda\delta\varphi_\perp^4 + 2\lambda a^2\delta\varphi_\parallel^2$$

$$+ 4\lambda a^3\delta\varphi_\parallel + 2\lambda a^2\delta\varphi_\perp^2 + 4\lambda a\delta\varphi_\parallel^3 + 2\lambda\delta\varphi_\parallel^2\delta\varphi_\perp^2 + 4\lambda a\delta\varphi_\parallel\delta\varphi_\perp^2$$

$$= \underbrace{\mu^2a^2+\lambda a^4}_{\text{COST.}} + \underbrace{2\mu^2a\delta\varphi_\parallel + 4\lambda a^3\delta\varphi_\parallel}_{\text{LINEAR IN }\delta\varphi_\parallel} + \underline{\mu^2\delta\varphi_\parallel^2 + \mu^2\delta\varphi_\perp^2 + 4\lambda a^2\delta\varphi_\parallel^2 + 2\lambda a^2\delta\varphi_\parallel^2 + 2\lambda a^2\delta\varphi_\perp^2}$$

$$+ \underline{4\lambda a\delta\varphi_\parallel(\delta\varphi_\parallel^2+\delta\varphi_\perp^2) + \lambda(\delta\varphi_\parallel^2+\delta\varphi_\perp^2)^2}$$

WE EXPECT THE LINEAR PARTS TO SIMPLIFY. USING THE FACT THAT

$$a^2 = -\mu^2/2\lambda$$

WE NOTICE THE $\delta\varphi_\perp$ TERM CANCELS AS WELL: THE BARE MASS OF $\delta\varphi_\perp$ IS ZERO! WHAT REMAINS IS

$$H = \int d^dx\ \left[(\underline{\nabla}\delta\varphi_\parallel)^2 + (\underline{\nabla}\delta\varphi_\perp)^2\right] + 2|\mu^2|\delta\varphi_\parallel^2 + 4\lambda a\delta\varphi_\parallel(\delta\varphi_\parallel^2+\delta\varphi_\perp^2)$$

$$+ \lambda(\delta\varphi_\parallel^2+\delta\varphi_\perp^2)^2\Big]$$

WHERE A TRIPLE VERTEX APPEARS. WE NOTICE THAT:

1) THE BARE/FREE (GAUSSIAN) LONGITUDINAL FIELD $\delta\varphi_\parallel$ IS MASSIVE,

$$2|\mu^2|\delta\varphi_\parallel^2 \quad \rightarrow \quad \mu_\parallel^2 = 2|\mu^2|$$

(BE AWARE: THE RENORMALIZED ONE WILL CHANGE).

2) THE BARE/FREE TRANSVERSE FIELD $\delta\varphi_\perp$ IS MASSLESS:

$$\text{NO } \delta\varphi_\perp^2 \quad \rightarrow \quad \mu_\perp^2 = 0$$

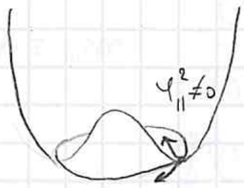
THIS IS TRUE AT ALL RG LEVELS: IT IS JUST TRUE!

· NAMBU - GOLDSTONE THEOREM

WHEN A CONTINUOUS SYMMETRY IS SPONTANEOUSLY BROKEN, MASSLESS FIELDS EMERGE (i.e. ∞ SUSCEPTIBILITIES).

THE MASSLESS FIELDS ARE THE PARAMETERS OF THE SYMMETRY.

THE NUMBER OF MASSLESS FIELDS IS EQUAL TO THE NUMBER OF PARAMETERS.



(SEE RYDER, SECTION 8.2, FOR THE FULL PROOF)

SUCH FIELDS ARE CALLED GOLDSTONE MODES/BOSONS, OR ZERO/MARGINAL MODES.

· GOLDSTONE FROM WARD IDENTITIES

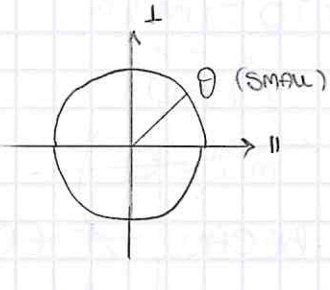
$$f(\underline{h}) = -\frac{1}{N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi}) - \underline{h}\cdot\underline{\varphi}}$$

USING THE SYMMETRY $H(\underline{\varphi}) = H(R\underline{\varphi})$ AND CHANGING VARIABLES TO $R\underline{\varphi}$ (IT'S A ROTATION, SO THE JACOBIAN IS 1),

$$f(\underline{h}) = -\frac{1}{N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(R\underline{\varphi}) - \underline{h}\cdot\underline{\varphi}}$$

NOTE: $R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$$= -\frac{1}{N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi}) - \underline{h}R^T\underline{\varphi}} = f(\underline{h}R^T)$$



WHERE (SO(2))

$\underline{h} = (h_{\parallel}, h_{\perp})$

$$\underline{h}R^T = \begin{cases} h_{\parallel} - \theta h_{\perp} \\ h_{\perp} + \theta h_{\parallel} \end{cases}$$

NOTE: IT'S ACTUALLY $\underline{h}^T R^T$.

TAKING θ SMALL,

$$f(h_{\parallel}, h_{\perp}) = f(h_{\parallel} - \theta h_{\perp}, h_{\perp} + \theta h_{\parallel}) = f(h_{\parallel}, h_{\perp}) - \theta h_{\perp} \frac{\partial f}{\partial h_{\parallel}} + \theta h_{\parallel} \frac{\partial f}{\partial h_{\perp}}$$

THIS GIVES THE WARD IDENTITY

$$\underline{h_{\perp} \frac{\partial f}{\partial h_{\parallel}} = h_{\parallel} \frac{\partial f}{\partial h_{\perp}}} \qquad (I)$$

THE COMPONENT $\chi_{\perp}$ OF THE SUSCEPTIBILITY $\chi_{\perp} \sim \frac{1}{m_{\perp}^2}$ IS WHAT WE'RE INTERESTED IN.

SO WE DERIVE (I) BY $\partial h_\perp$ AND EVALUATE IT AT $h_\perp = 0$:

$$\frac{\partial f}{\partial h_\parallel} + \left(h_\perp \frac{\partial^2 f}{\partial h_\perp \partial h_\parallel} \right)\Bigg|_{h_\perp = 0} = h_\parallel \frac{\partial^2 f}{\partial h_\perp^2} \Rightarrow \frac{\partial f}{\partial h_\parallel} = h_\parallel \cdot \frac{\partial^2 f}{\partial h_\perp^2}$$

BUT

$$\frac{\partial f}{\partial h_\parallel} = m_\parallel \equiv m$$

NOTE: STUPID AS IT MAY SOUND, HERE m IS THE MAGNETIZATION. $m_\perp^2$, A FEW LINES LATER, IS THE TRANSVERSE MASS...

$$\frac{\partial^2 f}{\partial h_\perp^2} = \chi_\perp \Rightarrow \underline{m = h_\parallel \chi_\perp}$$

IF THE SYMMETRY IS UNBROKEN, THIS IS TRIVIAL BECAUSE

$$h_\parallel \to 0, \quad m \to 0$$

BUT IF IT'S BROKEN, THEN

$$h_\parallel \to 0, \quad m \neq 0 \qquad \chi_\perp = \frac{m}{h_\parallel} \xrightarrow[h_\parallel \to 0]{} \infty, \quad m_\perp^2 = 0$$

THIS IS ALL A CONSEQUENCE OF

$$\frac{\partial^2 V}{\partial \varphi_\perp^2} = 0$$

NOTE: WE USED $\chi_\perp \sim \frac{1}{m_\perp^2}$.

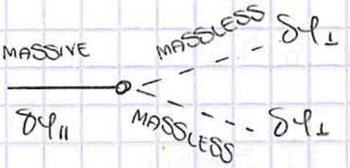
★ TO SUM UP,

$$\mu_\parallel^2 \neq 0$$

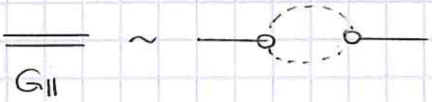
$$m_\perp^2 = 0$$

RECALL WE FOUND A TERM

$$\lambda \, \delta\varphi_\parallel \, \delta\varphi_\perp^2$$



(IN PARTICLE PHYSICS, IT'S A MASSIVE PARTICLE DECAYING INTO ITS GAUGE FIELD). THIS GIVES PROPAGATORS LIKE



- MERMIN-WAGNER THEOREM

$$\begin{cases} G^0_{\parallel} = \dfrac{1}{k^2+\mu^2_{\parallel}} \\ G^0_{\perp} = \dfrac{1}{k^2} = G_{\perp} \end{cases}$$

LET'S STUDY FLUCTUATIONS WHEN WE HAVE SSB ($\underline{\varphi}_0 \neq 0$). IF THEY GET INFINITE, THEN IT WOULDN'T MAKE SENSE TO EXPAND AROUND $\underline{\varphi}_0$: SO WE CHECK IT a posteriori. WE HAVE

$$\langle \delta\varphi^2(x) \rangle = \langle \delta\varphi^2_{\parallel}(x) \rangle + \langle \delta\varphi^2_{\perp}(x) \rangle$$

BUT

$$\langle \delta\varphi^2_{\perp}(x) \rangle = \langle \delta\varphi_{\perp}(x)\delta\varphi_{\perp}(x) \rangle = G_{\perp}(r=0) = \int d^dk \; G_{\perp}(k)$$

$$= \int_0^{\Lambda} d^dk \; \frac{1}{k^2}$$

IF $L=\infty$, WE MAY HAVE AN INFRARED DIVERGENCE. IF L IS FINITE,

$$G_{\perp}(r=0) = \int_{1/L} d^dk \; \frac{1}{k^2}$$

SO IF

$$d=1, \quad \int_{1/L} dk \; \frac{1}{k^2} \sim L \to \infty$$

$$d=2, \quad \int_{1/L} d^2k \; \frac{1}{k^2} \sim \int_{1/L} dk \; \frac{1}{k} \sim \ln L \to \infty$$

$$d \geq 3, \quad \int_{1/L} d^dk \; \frac{1}{k^2} \sim O(1)$$

$\Rightarrow$ IT'S IMPOSSIBLE TO HAVE LONG-RANGE ORDER (SSB) FOR A CONTINUOUS SYMMETRY IN $d \leq 2$.

★ WE OBSERVE THAT

$$G_\perp(k) = \frac{1}{k^2}$$ MASSLESS PROPAGATOR

$$\chi_\perp = G_\perp(k=0) = \infty$$ AT ALL d

$$\chi_\perp \to \infty, \quad \xi_\perp \to \infty$$ AT ALL d

BUT

$$\langle \varphi^2(x) \rangle \to \infty$$ $d \le 2$ ONLY

IN FACT

$$\chi = \int d^d r \, G(r) = G(k=0)$$ (ALWAYS DIVERGENT IF MASSLESS)

$$\langle \delta\varphi^2 \rangle = G(r=0) = \int d^d k \, G(k)$$ (IT CAN CONVERGE)

SO IN $d=3$, $\chi_\perp = \infty$ AND $\langle \delta\varphi^2 \rangle = 1$.

PHYSICALLY, THIS IS DUE TO "SPIN WAVES": YOU DON'T ACTUALLY PAY TO GENERATE AN INFINITE WAVELENGTH SPIN WAVE.

SO, IN A NUTSHELL,

$\chi_T = \infty$ ALWAYS (GOLDSTONE THM)

BUT DEPENDING ON d, $\langle \delta\varphi^2 \rangle$ MAY OR MAY NOT DESTROY THE LONG RANGE ORDER (MERMIN-WAGNER THM),

$$\langle \delta\varphi^2 \rangle = \infty, \quad d \le 2$$

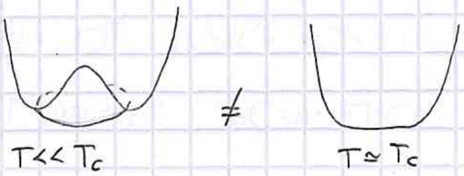
· REMARK

THIS IS NOT BONA FIDE CRITICALITY!

$\chi_\perp = \infty$, $\xi_\perp = \infty$

$G_\perp = \frac{1}{k^2} \to G_\perp = \frac{1}{r^{d-2}}$

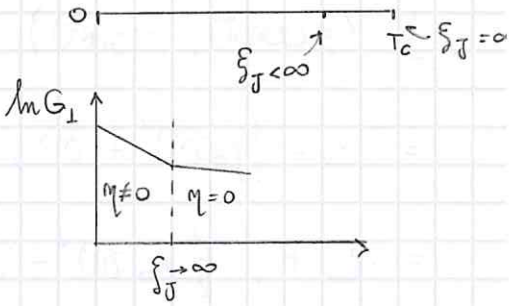
$\langle \delta\varphi^2 \rangle = \infty$, $d \le 2$



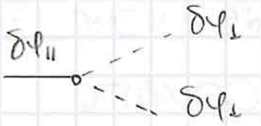
WHAT'S MISSING IS THE SELF-SIMILARITY (THE PRESENCE OF ALL SCALES OF FLUCTUATIONS). IN THE OVERLAP REGION,

$$G_\perp \sim \frac{1}{r^{d-2+\eta}}$$

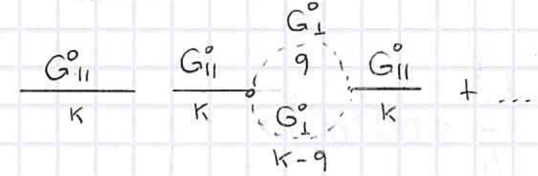
JOSEPHSON DEFINED HIS OWN ξ_J THAT REMAINS FINITE IN THIS CROSSOVER... BUT IT'S ALMOST IMPOSSIBLE TO MEASURE IT IN SIMULATIONS.



- SURPRISE! $\chi_{\parallel} \to \infty$ TOO!



$$G_{\parallel} = \langle \delta\varphi_{\parallel} \delta\varphi_{\parallel} \rangle$$

=  + ...

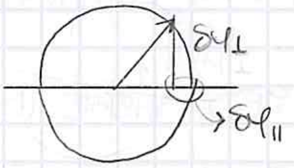
 $= \int d^d q \; G^0_\perp(q) G^0_\perp(k-q) = \int d^d q \; \frac{1}{q^2} \frac{1}{(k-q)^2}$

$$\chi_{\parallel} = G_{\parallel}(k=0) = \frac{1}{\mu^2_{\parallel}} + \frac{1}{(\mu^2_{\parallel})^2} \int d^d q \; \frac{1}{q^4} \xrightarrow[d \leq 4]{} \infty$$

NOTE: IT'S AN INFRARED DIVERGENC

DUE TO THE COUPLING BETWEEN $\perp$ AND $\parallel$ d.o.f.. IT'S ONLY A PLAUSIBILITY ARGUMENT: OTHER TERMS MAY CANCEL THIS DIVERGENCE

★ GEOMETRIC ARGUMENT.



LET'S CHOOSE A DIFFERENT PARAMETRIZATION,

$\underline{\varphi} = (\rho\cos\theta, \rho\sin\theta)$

$\rho(x), \theta(x)$

SSB: $\rho_0 \neq 0$, $\theta_0 = 0$

$$\begin{cases} \rho = \rho_0 + \delta\rho \\ \theta = \delta\theta \end{cases}$$

THIS WAY WE WOULD FIND

$$H = (\nabla \delta\rho)^2 + \rho_0^2 (\nabla \delta\theta)^2 + V(\delta\rho)$$

AND NO MIXED VERTICES!

$m^2_\theta = 0$

$\mu^2_\rho \neq 0 \Rightarrow m^2_\rho \neq 0$

AND THIS IS THE REAL MASSIVE PARTICLE.

FLUCTUATIONS THEN BECOME

NOTE: I THINK WE'RE NEGLECTING $\langle\delta\rho\,\delta\theta\rangle$ BECAUSE IT WOULD VANISH ANYWAY (NO MIXED VERTICES).

$$\underline{\varphi} = (\rho\cos\theta, \rho\sin\theta) = \left((\rho_0+\delta\rho)(1-\tfrac{1}{2}\delta\theta^2), (\rho_0+\delta\rho)\delta\theta\right)$$

$$= \left(\rho_0 - \tfrac{1}{2}\rho_0\delta\theta^2 + \delta\rho,\ \rho_0\delta\theta + \underset{\sim\sim}{\delta\rho\,\delta\theta}\right)$$

$$= \underline{\varphi}_0 + \left(-\tfrac{1}{2}\rho_0\delta\theta^2 + \delta\rho,\ \rho_0\delta\theta\right) = \underline{\varphi}_0 + (\delta\varphi_\parallel, \delta\varphi_\perp)$$

BUT WE KNOW $\delta\rho$ IS MASSIVE AND $\delta\theta$ IS MASSLESS. LET'S THEN COMPUTE

$$G_\parallel = \langle\delta\varphi_\parallel\,\delta\varphi_\parallel\rangle \simeq \underset{\substack{\downarrow \\ \text{SHORT RANGE, } \frac{e^{-r/\xi_\rho}}{r^\alpha}}}{\langle\delta\rho\,\delta\rho\rangle} + \frac{\rho_0^2}{4}\underset{\hookrightarrow \text{ LONG RANGE, GAUSSIAN}}{\langle\delta\theta^2\delta\theta^2\rangle}$$

AND

$$\langle\delta\theta\delta\theta\rangle_k = \frac{1}{k^2} \longrightarrow \langle\delta\theta\delta\theta\rangle = \frac{1}{r^{d-2}} \quad \text{FREE}$$

$$\langle\delta\theta^2\delta\theta^2\rangle \underset{\text{WICK THEOREM}}{=} (\langle\delta\theta\delta\theta\rangle)^2 \sim \frac{1}{r^{2(d-2)}}$$

SO THAT

$$G_\parallel(r) = \langle\delta\varphi_\parallel\delta\varphi_\parallel\rangle \sim \frac{e^{-r/\xi_\rho}}{r^\alpha} + \frac{\rho_0^2}{4}\cdot\frac{1}{r^{2(d-2)}}$$

SO YES, THERE IS ACTUALLY A MASSIVE PARTICLE, BUT IT DIES QUICKLY. BUT WHAT DOES IT MEAN IN PARTICLE PHYSICS?

PARISI: it's a resonance (i.e. IT'S AN UNSTABLE PARTICLE).

SINCE

$$\begin{cases} G_\parallel(k) = \int d^d r\, \dfrac{e^{ikr}}{r^{2(d-2)}} \sim \dfrac{1}{k^{4-d}} = \dfrac{1}{k^\varepsilon} \\ G_\perp(k) = \dfrac{1}{k^2} \end{cases}$$

WE FIND

$$G_\parallel(k) = G_\perp(k)^{\varepsilon/2} \longrightarrow \underline{\chi_\parallel = \chi_\perp^{\varepsilon/2}}$$

FOR INSTANCE, IN $d=3$ IT'S

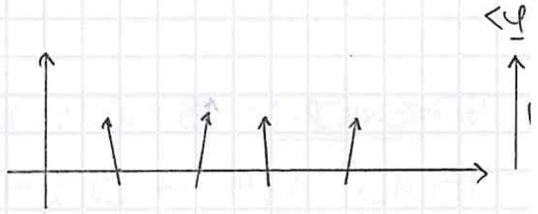
$$\chi_\parallel \sim \chi_\perp^{1/2}$$

★ LET'S KEEP ρ FIXED ($\delta\rho = 0$, MASSIVE FIELD). IN OTHER TERMS,

$$\| \underline{\sigma} \|^2 = 1$$

$$\underline{\varphi}^2 = \varphi_{\parallel}^2 + \varphi_{\perp}^2 = 1$$

THIS IS THE NONLINEAR σ MODEL. HENCE



$$\varphi_{\parallel}^2 = 1 - \varphi_{\perp}^2$$

$$\varphi_{\parallel} = 1 - \frac{1}{2}\varphi_{\perp}^2$$

LET'S ADD A FIELD IN THE $\parallel$ DIRECTION;

SPINS WILL ROTATE IN THE $\perp$ DIRECTION TO ALIGN. WE ACTUALLY GET

$$\langle \delta\varphi_{\parallel} \rangle > 0$$

THANKS TO $\chi_{\perp}$.

NOTE: $m_{\parallel} = \langle \delta\varphi_{\parallel} \rangle$, $\chi_{\parallel} = \frac{\partial m_{\parallel}}{\partial h_{\parallel}} \sim \langle \delta\varphi_{\parallel} \delta$
EVEN THOUGH ρ IS KEPT FIXED, WE DO SEE FLUCTUATIONS IN THE $\parallel$ DIRECTION.

- WHAT ABOUT THE FREE ENERGIES?

$$1) \quad g(\underline{m}) = -\frac{1}{\beta N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi})} \delta\left(\underline{m} - \frac{1}{N}\int d^d x\, \underline{\varphi}(x)\right)$$

$$P(\underline{m}) \sim e^{-\beta N g(\underline{m})} \qquad\qquad g(\underline{m}) \equiv g_1(\underline{m})$$

BUT I COULD BUILD INSTEAD

$$2) \quad g_2(\rho) = -\frac{1}{\beta N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi})} \delta\left(\rho - \frac{1}{N}\int d^d x\, |\underline{\varphi}(x)|\right)$$

$$P(\rho) \simeq e^{-g_2(\rho)}$$

CLEARLY

$$g_1(m) \neq g_2(\rho)$$

BECAUSE, EVEN THOUGH THEIR LANDAU VALUE IS THE SAME, THEY DIFFER IN THE FLUCTUATIONS. SIMILARLY,

$$f_1(\underline{h}) = -\frac{1}{\beta N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi}) + \underline{h}\cdot\int dx\, \underline{\varphi}(x)} \qquad (h_{\parallel}, h_{\perp})$$

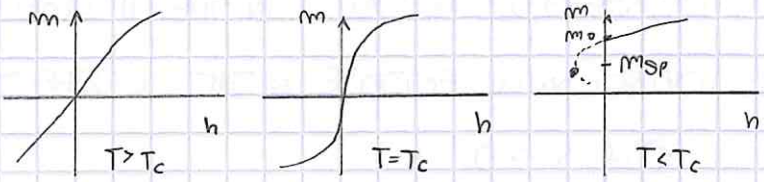
$$f_2(h_\rho) = -\frac{1}{\beta N} \ln \int \mathcal{D}\underline{\varphi}\, e^{-H(\underline{\varphi}) + h_\rho \cdot \int dx\, |\underline{\varphi}(x)|} \qquad (\rho, \theta)$$

BUT h_p IS NOT A PHYSICAL FIELD: YOU DON'T REALLY MEASURE SUCH A THING IN EXPERIMENTS. THIS PARTIALLY JUSTIFIES THE USE OF THE OTHER FIELD, h.

- **HOMEWORK**: GOING BALLISTIC

ISING ($\lambda\varphi^4$) → DISCRETE SSB

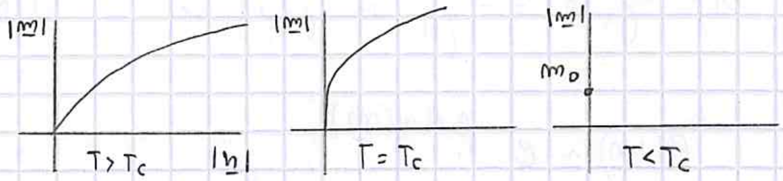
RECALL WE DEFINED A SPINODAL POINT BY WHICH THE METASTABLE BRANCH BEGINS:



$T < T_c$, SSB $\chi \sim 1$

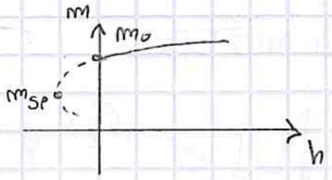
WE CAN LOWER m BELOW m_0 BY USING A SMALL NEGATIVE h.

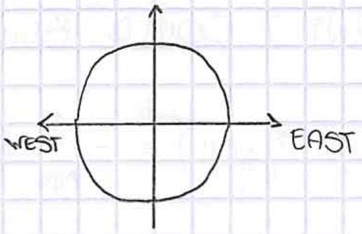
★ CONTINUOUS SSB ($d>2$)



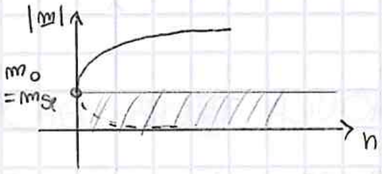
WHAT HAPPENS BELOW T_c?

LET'S MAKE A CONSERVATIVE GUESS: THE SAME AS DISCRETE, i.e. YOU CAN LOWER m BELOW (TO THE WEST) m_0 BY ADDING A SMALL WEST h. IF THIS IS POSSIBLE, $\chi_{\parallel} \sim 1$.





BUT IF THIS IS NOT POSSIBLE, THEN $m_0 = m_{sp}$ AND THERE'S NO METASTABLE PHASE.



$\chi_{\parallel} = \infty$

TEMPERATURE NEGATIVE

(SEMINARIO DOTTORANDI)

DEFINIZIONE DI T:

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

MA QUALE ENTROPIA?

$$S_B(E) = \ln \int dx \; \delta(H(x) - E) \qquad \text{(BOLTZMANN)}$$

$$S_G(E) = \ln \int_{H(x)<E} dx \qquad \text{(GIBBS)}$$

IN GENERE SI PENSA CHE

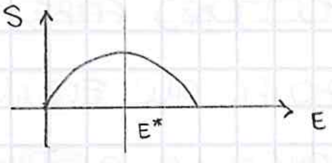
$$S_B(E) \simeq S_G(E)$$

INTANTO, $S_G(E)$ È UNA FUNZIONE CRESCENTE DI E, QUINDI

$$\frac{\partial S_G}{\partial E} > 0$$

CIÒ PORTA ALLA CONCLUSIONE $T > 0$.

MA È $S_B(E)$ A CONTARE IL NUMERO DI MICROSTATI ACCESSIBILI DAL SISTEMA; $S_G(E)$ NON HA UN VERO CORRISPETTIVO.



* SE PRENDO ISING,

$$H = - \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

ESISTE UNA E^* PER CUI $T = \infty$ E, PER $E > E^*$, LA TEMPERATURA ASSOLUTA DIVENTA NEGATIVA.

PROBLEMA:

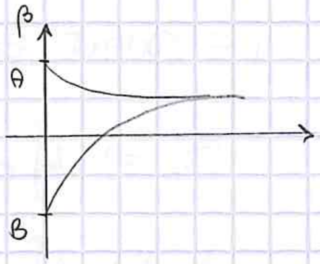
$e^{-\beta H} \rightarrow$ SE $\beta < 0$, Z ESPLODE.

QUINDI QUESTO IN EFFETTI SUCCEDE SOLO IN SISTEMI IL CUI SPAZIO DELLE FASI È LIMITATO (NON VERO PER LA MATERIA "NEWTONIANA")

UN ESEMPIO SONO I VORTICI PUNTIFORMI DI ONSAGER (1949).

* SI TRATTA DI UN MODELLINO DI VORTICI IN UN FLUIDO IN 2D. ALTRI ESEMPI SONO GLI SPIN NUCLEARI E I COLD ATOMS.

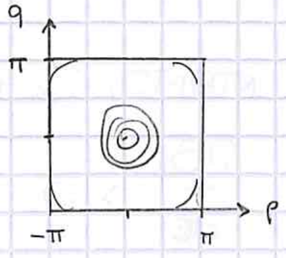
MA IL MONDO CHE CI CIRCONDA HA IN GENERE HAMILTONIANE LIMITATE (NON RELATIVISTICHE).
SIA 'A' UN OGGETTO PICCOLO DI MATERIA ORDINARIA, 'B' UN OGGETTO MOLTO GRANDE CHE AMMETTE $T<0$.
A CONTATTO TERMICO, LA TEMPERATURA DI EQUILIBRIO È SEMPRE POSITIVA. PERCHÉ? BENCHÉ 'A' SIA PIÙ PICCOLO, HA UNO Sdf MOLTO PIÙ GRANDE (HO PIÙ gdl CHE POSSONO CIUCCIARE ENERGIA).



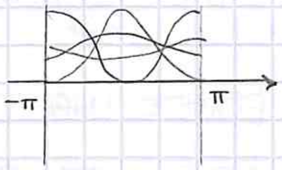
MODELLINO (VULPIANI, CERINO)

$$H = \sum_i (1-\cos p_i) + V(\{q_i\}) \xrightarrow{} \text{TIPO } \sum_{\langle ij \rangle} (1-\cos(q_i - q_j))$$

PER $p \ll 1$, $(1-\cos p) \sim \frac{p^2}{2}$. SCEGLIENDO q, p COME ANGOLI, PER PICCOLE ENERGIE IL SISTEMA È CIRCA UNA CATENA ARMONICA; MA AD UN CERTO PUNTO RAGGIUNGO IL BORDO.



SI POSSONO FARE SIMULAZIONI NUMERICHE SULLA DINAMICA E SI TROVA, ALL'EQUILIBRIO, QUANTO VISTO SOPRA.
A FIANCO LA DISTRIBUZIONE ALL'EQUILIBRIO PER VALORI DI $\beta>0$ E $\beta<0$. INOLTRE



$$\int e^{-\beta(1-\cos p)} \cos p \, dp \propto I_1(\beta)$$

(FUNZIONE DI BESSEL MODIFICATA DEL PRIMO TIPO), BEN NOTA E INVERTIBILE.

* CHE COSA DIVENTA IL MOTO BROWNIANO SE $\beta<0$?

$$H = \underbrace{K(\underline{P})}_{\text{COLLOIDE}} + \underbrace{\Sigma(\{p_i\})}_{\text{BAGNO}} + U(\underline{Q}) + V(\underline{Q}, \{q_i\})$$

IL NOSTRO ANSATZ È

$$\begin{cases} \dot{Q} = \frac{\partial H}{\partial P} \\ \dot{P} = -\frac{\partial U}{\partial Q} + \Gamma(P) + \sqrt{2D(P)}\,\xi_P \end{cases}$$

DRIFT DIFFUSIONE

SE QUESTO È VERO, BILANCIANDO LE CORRENTI DI PROBABILITÀ

$$J_Q = f(P,Q)\cdot\dot{Q}$$

$$J_P = f(P,Q)\cdot\dot{P}$$

E FACENDO I CONTI SI TROVA UNA RELAZIONE TRA DRIFT E DIFFUSIONI

$$\underline{\Gamma(P) = -\beta D(P)\frac{\partial}{\partial P}K(P)}$$

ASSUMENDO

$$D(P) = D$$

$$K(P) = \frac{P^2}{2}$$

SI RITROVA LA RELAZIONE DI EINSTEIN, MA SE NON È COSÌ $\Gamma(P)$ PUÒ IN GENERALE CAMBIARE SEGNO.

PARTIAMO DA UNA SIMULAZIONE CHE DÀ $P(t)$. DATA LANGEVIN

$$\dot{x} = F(x) + \sqrt{2D(x)}\,\xi$$

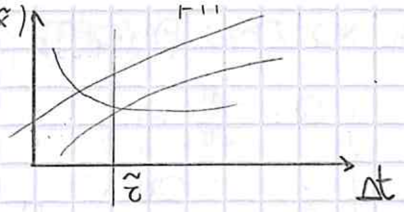
MI POSSO SEMPRE RICONDURRE ALLA DINAMICA TRAMITE

$$F(\tilde{x}) = \lim_{\Delta t\to 0}\left\langle \frac{\Delta x}{\Delta t}\,\middle|\, X(t_0)=\tilde{x}\right\rangle$$

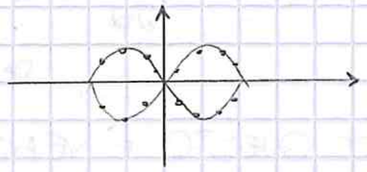
$$D(\tilde{x}) = \lim_{\Delta t\to 0}\left\langle \frac{(\Delta x - F\Delta t)^2}{2\Delta t}\,\middle|\, X(t_0)=\tilde{x}\right\rangle$$

QUESTO È SEMPRE UN MODO SOLIDO PER RICAVARSI I TERMINI DI UN'EQUAZIONE DI LANGEVIN PARTENDO DA UNA SERIE TEMPORALE. È VERO, PERÒ, SE IL PROCESSO È DAVVERO STOCASTICO; POICHÉ NELLA REALTÀ IL SISTEMA IN GENERE NON È DAVVERO STOCASTICO MA DETERMINISTICO, NON DEVO

CONSIDERARE Δt TROPPO PICCOLI ($\tilde{\tau}$ È DETTO TEMPO DI MARKOV-EINSTEIN).



LO SI È FATTO PER IL NOSTRO PROBLEMA E FUNZIONA: SI PUÒ FARE UN'EQUAZIONE DI LANGEVIN PER $\beta<0$.



MICELI HA STUDIATO SISTEMI A LUNGO RANGE (DOVE GLI ENSEMBLE CANONICO E MICROCANONICO NON SONO EQUIVALENTI) PER $\beta<0$.

· <u>OSS</u>: IL TEOREMA DI EQUIPARTIZIONE NON VALE SE $\beta<0$. SI ASSUME INFATTI CHE $S_G \simeq S_B$, NON VERO SE LO SPAZIO DELLE FASI È LIMITATO.