Stochastic Dynamics in Statistical Physics

Lecture notes by
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2019-2020
SISSA (Trieste)

STOCHASTIC DYNAMICS

09.10.2019

REFS:

N.G. VAN KAMPEN, "Stochastic processes in Physics and Chemistry", NORTH HOLLA

UWE TÄUBER, "Critical Dynamics: a field theoretical approach to equilibrium and non-eq. systems", CAMBRIDGE U.P.

ZINN-JUSTIN, "QFT & Critical phenomena"

JOHN CARDY, "Field theory & non eq. statistical mechanics", '98-'99 (LECTURE NOTES)

STOCHASTIC PROCESS

A RANDOM VARIABLE X WHICH DEPENDS ON A REAL PARAMETER $t \in \mathbb{R}$.

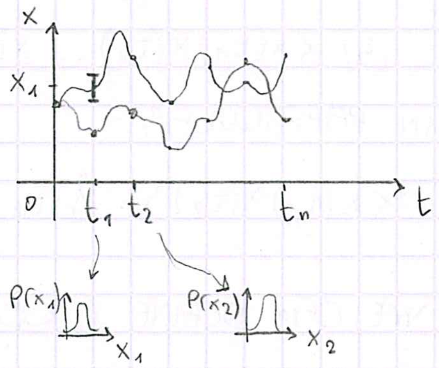
THIS MAY BE DISCRETIZED FOR CONVENIENCE,

$t_1 < t_2 < \ldots < t_n$

WHICH LEADS TO THE COLLECTION

$\{ x(t_1), x(t_2), \ldots, x(t_m) \}$

WHICH IS A RANDOM VECTOR CALLED A "REALIZATION OF THE STOCHASTIC PROCESS".



HOW CAN WE CHARACTERIZE COMPLETELY THE DRUNK-MAN PATH ON THE RIGHT?

WE NEED A JOINT PROBABILITY DISTRIBUTION, i.e. THE PROBABILITY ASSOCIATED TO THE RANDOM VECTOR AS A WHOLE:

$\{x_1 \ldots x_m\} \to P_m$

$$\text{Prob}\{ x(t_i) \in [x_i, x_i + dx_i], \; i = 1, \ldots, m \} \equiv P_m(x_1, t_1; \ldots; x_m, t_m)\, dx_1 \ldots dx_m$$

WHAT ARE THE PROPERTIES OF THE JOINT DENSITY?

(0) $P_m \geq 0$ (NON-NEGATIVE)

(1) $P_1(x_1, t_1) \to \int dx_1\, P_1(x_1, t_1) = 1$ (NORMALIZABLE)

(2) HIERARCHY:

$$\int dx_m\, P_m(x_1,t_1;\, x_2,t_2;\, \dots;\, x_m,t_m) = P_{m-1}(x_1,t_1;\, \dots;\, x_{m-1},t_{m-1})$$

IT'S A SIMPLE EXERCISE TO SHOW THAT

$$\int dx_1 \dots dx_m\, P_m(\dots) = 1$$

NOTE: 1.

ONCE WE HAVE THIS, WE CAN COMPUTE THE m-TIME <u>CORRELATION FUNCTION</u>

$$\langle X(t_1)\, X(t_2) \dots X(t_m)\rangle = \int dx_1 \dots dx_m\, x_1 \dots x_m\, P_m(x_1,t_1;\, \dots;\, x_m,t_m)$$

$X_1^{(1)}$ $X_2^{(1)}$ ... $X_m^{(1)}$ 1st REALIZATION

$X_1^{(2)}$ $X_2^{(2)}$ ... $X_m^{(2)}$ 2nd REALIZATION

THE SIMPLEST IS

$$\langle X(t_1)\, X(t_2)\rangle$$

A <u>STATIONARY PROCESS</u> IS SUCH THAT

$$P_m(x_1,t_1+\tau;\, x_2,t_2+\tau;\, \dots x_m,t_m+\tau) = P_m(x_1,t_1;\, \dots;\, x_m,t_m) \quad \forall \tau$$

THEN IT'S SIMPLE TO SHOW THAT

(a) $P_1(x_1,\cancel{t_1})$

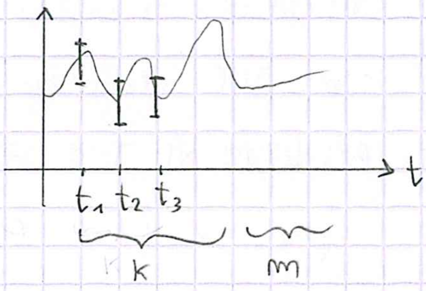
(b) $\langle X(t_1)X(t_2)\dots X(t_m)\rangle = \langle X(0)X(t_2-t_1)\dots X(t_m-t_1)\rangle$

IN PARTICULAR

$$\langle X(t_1)X(t_2)\rangle = f(t_1-t_2)$$

NOTE: JUST LOOK AT THE DEFINITION ABOVE.

WE CAN DEFINE A CONDITIONED PROBABILITY AS



$$P_{m|k}(x_{k+1},t_{k+1};\, \dots;\, x_{k+m},t_{k+m} \,\|\, x_1,t_1;\, \dots;\, x_k,t_k)$$
$$\cdot\, dx_{k+1} \dots dx_{k+m}$$
$$= \text{Prob}\{X(t_i)\in[x_i,\, x_i+dx_i],\ i=k+1,\dots,k+m \text{ KNOWING THAT } X(t_i)\in[x_i,\, x_i+dx_i],\ i=1,\dots,k\}$$

BY BAYES THM,

$$P\left(\frac{m}{} \,\middle|\, \frac{k}{}\right) = \frac{P_{m+k}\left(\frac{m}{},\frac{k}{}\right)}{P_k\left(\frac{k}{}\right)}$$

- <u>UNCORRELATED PROCESS</u>

$$P_{m|k}(\underline{m}, \underline{k}) = P_m(\underline{m})$$

BY USING AGAIN BAYES THM, ONE FINDS THAT

$$P_{m+k}(\underline{k}, \underline{m}) = P_m(\underline{m}) P_k(\underline{k})$$

A PROCESS IS <u>FULLY UNCORRELATED</u> WHEN

$$P_m(x_1, t_1; \dots x_m, t_m) = \prod_{i=1}^{m} P_1(x_i, t_i)$$

i.e. WHEN THE P_m FACTORIZES INTO THE P_1.

AN EXAMPLE IS A BERNOULLI TRIAL (DICE, COINS), FOR WHICH

$$P_1(x_i, \cancel{t_i}) = P_1(x_i)$$

- <u>MARKOV PROCESSES</u>

THE DISTRIBUTION OF THE VARIABLE AT TIME t_{i+1} DEPENDS ONLY ON ITS VALUE AT TIME t_i. THIS CORRESPONDS TO SHORT-TIME MEMORY.

$$P_{1|m-1}(x_m, t_m | x_1, t_1; \dots; x_{m-1}, t_{m-1}) = P_{1|1}(x_m, t_m | x_{m-1}, t_{m-1})$$

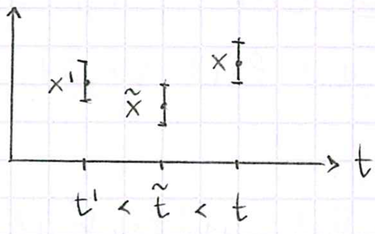
EXERCISE: PROVE THAT NOTE: ②.

$$P_m(x_1, t_1; \dots; x_m, t_m) = P_{1|1}(x_m, t_m | x_{m-1}, t_{m-1}) P_{1|1}(x_{m-1}, t_{m-1} | x_{m-2}, t_{m-2}) \cdot \dots \cdot P_{1|1}(x_2, t_2 | x_1, t_1) P_1(x_1, t_1)$$

THIS WAY THE P_m (WHICH IS ALL WE NEED TO KNOW) IS EXPRESSED IN TERMS OF P_1 AND $P_{1|1}$, THE TRANSITION PROBABILITY: A M.P. IS COMPLETELY SPECIFIED BY P_1 AND $P_{1|1}$.

BUT DO ANY PAIR P_1, $P_{1|1}$ DEFINE A MARKOV PROCESS?

IMAGINE WE ADD A MIDDLE POINT $\tilde{t}$ BETWEEN t' AND t. THEN, INTUITIVELY,



$$P_{1|1}(x, t | x', t') \overset{\forall \tilde{t}}{=} \int d\tilde{x} \, P_{1|1}(x, t | \tilde{x}, \tilde{t}) P_{1|1}(\tilde{x}, \tilde{t} | x', t')$$

WHICH COULD BE PROVEN FORMALLY BY NOTICING THAT

$$P_2(x',t';x,t) \overset{B.T.}{=} P_{1|1}(x,t|x',t')\,P_1(x',t')$$
$$\parallel$$
$$\int d\tilde{x}\, P_3(x',t';\tilde{x},\tilde{t};x,t) \overset{M.P.}{=} \int d\tilde{x}\, P_{1|1}(x,t|\tilde{x},\tilde{t})\,P_{1|1}(\tilde{x},\tilde{t}|x',t')\,P_1(x',t')$$

THIS IS CALLED THE CHAPMAN-KOLMOGOROV CONDITION.

ANOTHER NECESSARY CONDITION IS

NOTE: THIS IS PROBABILITY CONSERVATION.

$$P_1(\tilde{x},\tilde{t}) = \int dx'\, P_2(x',t';\tilde{x},\tilde{t}) \overset{B.T.}{=} \int dx'\, P_{1|1}(\tilde{x},\tilde{t}|x',t')\,P_1(x',t')$$

A MARKOV PROCESS IS THEN DEFINED BY THE 2 CONDITIONS

$$P_1,\ P_{1|1} \geq 0$$

$$\begin{cases} P_{1|1}(x_3,t_3|x_1,t_1) \overset{\forall t_2}{=} \int dx_2\, P_{1|1}(x_3,t_3|x_2,t_2)\,P_{1|1}(x_2,t_2|x_1,t_1) & \text{(C-K)} \\ P_1(\tilde{x},\tilde{t}) = \int dx'\, P_{1|1}(\tilde{x},\tilde{t}|x',t')\,P_1(x',t') \end{cases}$$

NOTE: IT SEEMS TO ME THAT THIS LAST PROPERTY SHOULD HOLD FOR A NON-MARKOV PROCESS ALSO.

EXAMPLES

(A) $t \geq 0,\ t_2 > t_1$

(i) $P_1(x,t=0) = \delta(x)$

(ii) $$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{1}{\sqrt{2\pi(t_2-t_1)}}\exp\left\{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\right\}$$

THIS DEFINES A NON STATIONARY MARKOV PROCESS CALLED THE WIENER PROCESS. IT CAN BE USED TO DESCRIBE THE POSITION OF A BROWNIAN PARTICLE.

EX: CALCULATE P_1 AT A GIVEN TIME AND CHECK IF C-K IS SATISFIED.

(B) $t \in \mathbb{R},\ \tau \equiv (t_2 - t_1) > 0$

(i) $$P_1(x,t=0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(ii) $$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}}\exp\left\{-\frac{(x_2-x_1e^{-\tau})^2}{2(1-e^{-2\tau})}\right\}$$

THIS DEFINES A STATIONARY M.P. CALLED <u>ORNSTEIN-UHLENBECK</u>.
IT DESCRIBES THE VELOCITY OF A BROWNIAN PARTICLE.
<u>EX</u>: CALCULATE $P_1(x,t)$, CHECK C-K AND ITS STATIONARITY.

Ⓒ $t \in \mathbb{R}$, $\tau = (t_2 - t_1) > 0$

$$P_{1|1}(x_2, t_2 | x_1, t_1) = \frac{1}{\pi} \frac{\tau}{(x_2 - x_1)^2 + \tau^2}$$

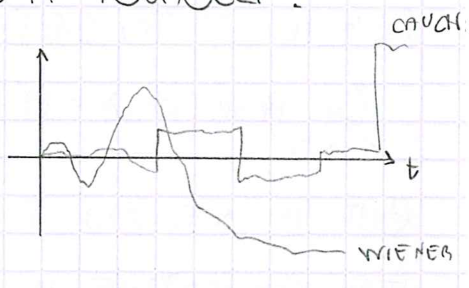
<u>EX</u>: VERIFY C-K. CAN YOU FIND AN INITIAL CONDITION s.t. THE PROCESS IS STATIONARY?

THIS DEFINES THE <u>CAUCHY PROCESS</u>. THIS BELONGS TO THE FAMILY OF THE "STABLE DISTRIBUTIONS" UNDER CONVOLUTION (THE DISTRIBUTION OF THE SUM LOOKS THE SAME, AND THIS IS CALLED LEVI-STABILITY).

QUESTION: HOW DOES THE TRAJECTORIES LOOK LIKE?
USING Mathematica, PLACE A δ-FUNCTION IN ZERO AND GENERATE A VARIABLE FROM

$$P_{1|1}(x_1, \Delta t | 0, 0)$$

THEN MAKE Δt SMALLER AND SMALLER. DO IT YOURSELF!

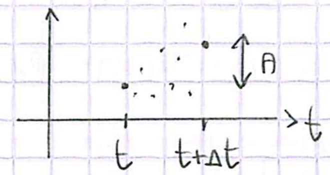


DIFFERENTIABILITY

11.10.19

I MAY WONDER:

$$\text{Prob}\left(|x(t+\Delta t) - x(t)| > A\right) \xrightarrow[\Delta t \to 0]{} 0 \quad ?$$



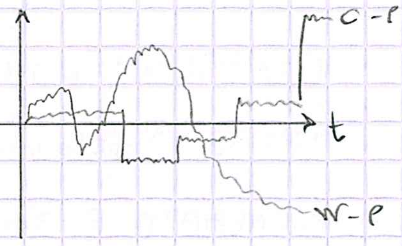
BUT EVEN IF SO, DOES IT MEAN THAT THE TRAJECTORY GETS "CONTINUOUS" FOR $\Delta t \to 0$? NOT REALLY: THIS PROPERTY IS SATISFIED BY BOTH CAUCHY AND WIENER.

WHERE IS THE CATCH? IN BETWEEN t AND $t+\Delta t$ THERE ARE MANY POINTS, SO WHAT WE ASK IS THAT THE MAXIMUM AMONG THESE GOES TO ZERO AT LEAST AS Δt. THIS IMPLIES

$$\frac{\text{Prob}\left(|x(t+\Delta t) - x(t)| > A\right)}{\Delta t} \xrightarrow[\Delta t \to 0]{} 0$$

NOTE: THE PROOF IS LEFT AS AN EXERCISE IN PROBLEM SET #1.

THIS IS THE <u>LINDEBERG CRITERION</u> (AND IT IS NOT SATISFIED BY A CAUCHY PROCESS).



LET'S CALL

$$d_\Delta x = \frac{1}{\Delta}\left(x(t+\Delta) - x(t)\right)$$

ONE COULD SEARCH FOR THE CUMULANT $\text{Prob}(d_\Delta x < A)$ AND DERIVE IT TO GET THE DISTRIBUTION. FOR $\Delta t \to 0$,

$$\lim_{\Delta \to 0} \text{Prob}(d_\Delta x < A) = \text{Prob}(\text{"}\dot{x}\text{"} < A)$$

BUT IF WE DO IT FOR WIENER YOU GET $\frac{1}{2}$, MEANING THAT ITS DERIVATIVE IS ALWAYS $\pm\infty$. THIS IS AN HEURISTIC WAY TO SEE THAT A W-P IS CONTINUOUS, BUT NOWHERE DIFFERENTIABLE. ON THE OTHER HAND, THIS QUANTITY IS ALWAYS FINITE FOR A C-P (EVEN IF IT IS NOT CONTINUOUS!). THE REASON IS THESE DISCONTINUITIES ARE "RARE".

* WHY ARE STOCHASTIC PROCESSES USEFUL IN PHYSICS?

SOME PHENOMENA LOOKS UNPREDICTABLE. SOMETIMES THIS IS INTRINSIC (QUANTUM MECHANICS), SOMETIMES THIS IS DUE TO LACK OF INFORMATION.

IF RELAXATION HAPPENS ON SHORTER TIMESCALES THAN OUR OBSERVATION RESOLUTION, THAN IT MAKES SENSE TO ADOPT A MARKOV PROCESS. IT IS NOT INHERENT (PHYSICS IS INHERENTLY NON-MARKOVIAN).

THIS IS THE CASE FOR BROWNIAN MOTION (1827, ROBERT BROWN)

BROWNIAN MOTION

BROWN WAS A BOTANIST WHO OBSERVED POLLEN INTO WATER ($L \sim 1\ \mu m$) WITH A MICROSCOPE. HE EXCHANGED THE MOVEMENT FOR "LIFE" BECAUSE HE COULDN'T SEE THE SOLVENT, i.e. THE WATER MOLECULES. BUT HOW COULD HE CHECK IF THIS WAS DUE TO POLLEN? HE TRIED AGAIN WITH ASHES FROM THE FIREPLACE (THERE'S ONE IN EVERY ENGLISH HOUSE!) : THEN IT COULDN'T BE LIFE.

ACTUALLY THE FIRST OBSERVATION DATES BACK TO 1789 BY INGEN-HOUSZ, WHO WATCHED DUST PARTICLES AND BLAMED THIS EFFECT ON WATER EVAPORATION, WHICH HE WAS STUDYING.

REMEMBER THE ATOMIC HYPOTHESIS WAS STILL ONLY AN HYPOTHESIS. IN 1905, EINSTEIN GAVE A THEORETICAL INTERPRETATION ON

Ann der Physik 17, P. 549

BUT HE ADMITTED HE LACKED EXPERIMENTAL DATA.

JEAN-BAPTIST PERRIN PERFORMED QUANTITATIVE EXPERIMENTS IN 1908 AND GOT THE NOBEL PRIZE IN 1926 FOR PROVING THE ATOMISTIC THEORY.

EISTEN USED SORT OF MASTER EQUATIONS, AND ONLY LATER SMOLUCHOWSKI GAVE AN EXPLANATION IN TERMS OF LANGEVIN EQUATIONS.

IN ORDER FOR THIS TO WORK, YOU NEED A CLEAR SEPARATION OF SPACE AND TIME SCALES IN FAST AND SLOW DEGREES OF FREEDOM.

HOW DO I OBSERVE STOCHASTICITY?

i) PARTITIONING A TIME TRAJECTORY (BUT IT RELIES ON STATIONARIETY).

ii) ENSEMBLE (I PREPARE, SAY, 3000 POLLENS).

MASTER EQUATION

WE CAN ALWAYS EXPRESS

$$P_1(x, t+\tau) = \int dx' \, P_2(x, t+\tau; x', t) \qquad \tau > 0$$

$$\overset{\text{B.T.}}{=} \int dx' \, P_{1|1}(x, t+\tau | x', t) P_1(x', t)$$

$$\frac{\partial}{\partial t} P_1(x,t) = \lim_{\tau \to 0} \frac{1}{\tau} \left[P_1(x, t+\tau) - P_1(x,t) \right]$$

$$= \int dx' \lim_{\tau \to 0} \frac{1}{\tau} \left[P_{1|1}(x, t+\tau | x', t) - P_{1|1}(x, t | x', t) \right] P_1(x', t) \qquad \text{(I)}$$

BUT, IF $\tau \to 0$,

$$P_{1|1}(x, t+\tau | x', t) = A \delta(x-x') + \tau \underbrace{W(x' \to x, t)}_{\text{TRANSITION RATES}} + O(\tau^2)$$

WHERE WE REQUIRE

$$1 \equiv \int dx'' \, P_{1|1}(x'', t+\tau | x', t) = A + \tau \int dx'' \, W(x' \to x'', t) + O(\tau^2)$$

SO AS TO DETERMINE

$$A = 1 - \tau \int dx'' \, W(x' \to x'', t) + O(\tau^2).$$

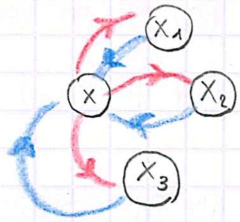
BY PLUGGING IT INTO (I), ONE FINDS

$$\frac{\partial}{\partial t} P_1(x,t) = \int dx' \left\{ -\delta(x-x') \int dx'' W(x' \to x'', t) + W(x' \to x, t) \right\} P_1(x',t)$$

$$= \int dx' \left[W(x' \to x, t) P_1(x',t) - W(x \to x', t) P_1(x,t) \right] \qquad (x'' \to x') \text{ CHANGED}$$

WE FOUND THE MASTER EQUATION

$$\frac{\partial}{\partial t} P_1(x,t) = \int dx' \Big[\underbrace{W(x' \to x, t) P_1(x',t)}_{\text{GAIN}} - \underbrace{W(x \to x', t) P_1(x,t)}_{\text{LOSS}} \Big] \qquad (\text{II})$$



THIS IS THE EQUIVALENT OF A CONTINUITY EQUATION FOR FLUIDS, OR THE KIRCHHOFF'S LAW, WHERE

$$J(x' \to x, t) \equiv W(x' \to x, t) P_1(x',t) - W(x \to x', t) P_1(x,t)$$

IS THE NET PROBABILITY CURRENT.

BY INTEGRATING OVER x ANY OF THE TWO SIDES OF (II), ONE GETS ZERO BY PROBABILITY CONSERVATION.

- EXERCISE

NOTE: ①

DETERMINE THE TRANSITION RATE (HENCE THE MASTER EQUATION) FOR THE WIENER AND ORNSTEIN-UHLENBECK PROCESSES.

NOTICE IN GENERAL W DEPENDS ON P_1, HENCE (II) IS NOT LINEAR IN P_1. IT BECOMES SUCH FOR A MARKOVIAN PROCESS: FOR A M-P, THE W'S ARE GIVEN.

GIVEN W, WE CAN CONSTRUCT P_1 VIA (II). BUT WHAT ABOUT $P_{1|1}$?

- EXERCISE

NOTE: ②

PROVE THAT THE MASTER EQUATION FOR $P_{1|1}$ IS THE SAME. JUST TAKE THE C-K EQUATION AS YOUR STARTING POINT. ACTUALLY, SINCE

$$P_{1|1}(x, t_0 | x', t_0) = \delta(x - x')$$

THE $P_{1|1}$ IS THE GREEN FUNCTION OF THE MASTER EQUATION.

IF THE STATES OF THE SYSTEM ARE DISCRETE, (II) CAN BE CAST IN MATRIX FORM AS

$$\frac{\partial}{\partial t} \underline{\psi} = M \underline{\psi}$$

BY DISCRETIZING TIME AS WELL, WE GET A MARKOV CHAIN.

- <u>GENERAL PROPERTIES OF M.E.</u>

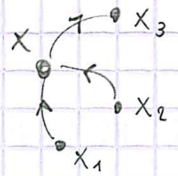
IF $w(x \to x', \cancel{t})$, THEN

(a) $\exists$ AT LEAST 1 STATIONARY SOLUTION $P_s(x)$.

(b) IF $P_s(x)$ IS UNIQUE, THEN $\lim_{t \to +\infty} P_1(x,t) = P_s(x)$.

IN THE STATIONARY STATE, THE R.H.S. OF THE M.E. IS NULL:

$$0 = \int dx' \, J_s(x' \to x), \quad \forall x$$



THIS IS NOTHING BUT KIRCHHOFF LAW.

- <u>FOCUS</u>: M.E. FOR THE PROPAGATOR (VAN KAMPEN P. 97)

"EQUATION (II) MUST BE INTERPRETED AS FOLLOWS. TAKE A TIME t_1 AND A VALUE x_1, AND CONSIDER THE SOLUTION OF (II) THAT IS DETERMINED FOR $t \geq t_1$ BY THE INITIAL CONDITION $P_1(x, t_1) = \delta(x - x_1)$. THIS SOLUTION IS THE TRANSITION PROBABILITY $P_{1|1}(x, t \,|\, x_1, t_1)$ OF THE MARKOV PROCESS - FOR ANY CHOICE OF t_1 AND x_1. THE MASTER EQUATION IS not MEANT AS AN EQUATION FOR THE SINGLE-TIME DISTRIBUTION $P_1(x,t)$!".

EQUILIBRIUM SOLUTIONS

16.10.19

LAST TIME WE WROTE

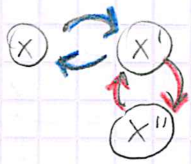
$$\partial_t P_1(x,t) = \int dx' \left\{ P_1(x',t)\, w(x' \to x, t) - P_1(x,t)\, w(x \to x', t) \right\} \qquad (\mathrm{I})$$

IF THE TRANSITION RATES DO NOT DEPEND ON t, THEN THE SYSTEM MAY ADMIT A STATIONARY STATE (NOT NECESSARILY: THINK OF BROWNIAN MOTION). IF IT EXISTS AND IT IS UNIQUE, THEN

$$\exists\, P_s(x) \mid \text{rhs} = 0 \mid \lim_{t \to +\infty} P_1(x,t) = P_s(x)$$

SOME STATIONARY SOLUTIONS ARE ALSO EQUILIBRIUM SOLUTIONS

$$P^*(x): \quad P^*(x)\, w(x \to x') = P^*(x')\, w(x' \to x) \qquad \forall x, x'$$

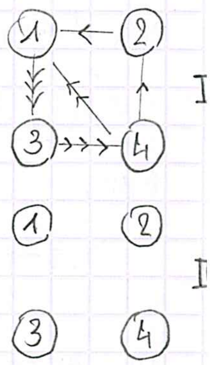


(DETAILED BALANCE). THIS MEANS THE CURRENT $J(x,t) = 0$.

IF $P^*(x)$ SATISFIES D.B., THEN IT IS STATIONARY BY (I).

★ IMAGINE A CONFIGURATION STATE

$x \in \{1,2,3,4\}$



AND SOMEONE GIVES ME A DYNAMICS I $\{ w^{(1)}(x \to x') \}$ (WITH ITS STATIONARY STATE) AND DYNAMICS II $\{ w^{(2)}(x \to x') \}$ WHICH SATISFIES THE DB (SO IT'S A $P^*(x)$).

HOW ARE I AND II DIFFERENT?

A) $P^*(x)$ IS A STATE INVARIANT UNDER TIME REVERSAL.

ANALOGY: A RIVER VS A LAKE.

*NOTE: KULLBACK - LEIBLER.

B) GIVEN $P^*(x)$, CONSTRUCT THE K-L* DIVERGENCE

$$S(t) = -\left\langle \ln \frac{P_1(x,t)}{P^*(x)} \right\rangle = -\int dx\, P_1(x,t) \ln \frac{P_1(x,t)}{P^*(x)}$$

WHICH IS A TIME DEPENDENT ENTROPY. THEN

$$\dot{S}(t) = -\int dx\, \partial_t P_1(x,t) \left\{ \ln \frac{P_1(x,t)}{P^*(x)} + 1 \right\} \qquad (\mathrm{II})$$

NOW, $\partial_t P_1$ IS GIVEN BY (I) AND

$$P^*(x)\, w(x \to x') = P^*(x')\, w(x' \to x)$$

NOTE: FROM WHAT I KNEW, DETAILED BALANCE ISN'T REALLY REQUIRED TO PROVE THIS H THEOREM (INDEED, NOTICE HERE W USE IT INSIDE OF $\int dx'$!).

SO WE CAN REPLACE $W(x' \to x)$ IN (I) WITH

$$\partial_t P_1(x,t) = \int dx' \left\{ P_1(x',t) \frac{P^*(x) W(x \to x')}{P^*(x')} - P_1(x,t) W(x \to x') \right\}$$

$$= \int dx' \left\{ \frac{P_1(x',t)}{P^*(x')} - \frac{P_1(x,t)}{P^*(x)} \right\} P^*(x) W(x \to x')$$

REPLACING THIS INTO (II) GIVES

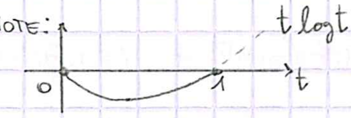
$$\dot{S}(t) = -\int dx \int dx' \left\{ \frac{P_1(x',t)}{P^*(x')} - \frac{P_1(x,t)}{P^*(x')} \right\} \overbrace{P^*(x') W(x' \to x)}^{\text{INVARIANT UNDER } (x \leftrightarrow x')} \left[\ln \frac{P_1(x,t)}{P^*(x)} + 1 \right]$$

$$= \frac{1}{2}(\cdot) + \frac{1}{2}(\cdot)(x \leftrightarrow x')$$

$$= -\frac{1}{2} \int dx\, dx' \underbrace{\left\{ \frac{P_1(x',t)}{P^*(x')} - \frac{P_1(x,t)}{P^*(x)} \right\} \cdot \left[\ln \frac{P_1(x,t)}{P^*(x)} - \ln \frac{P_1(x',t)}{P^*(x')} \right]}_{= (A-B)(\log B - \log A) \le 0} \underbrace{P^*(x') W(x' \to x)}_{\ge 0}$$

$$\ge 0$$

NOTE:

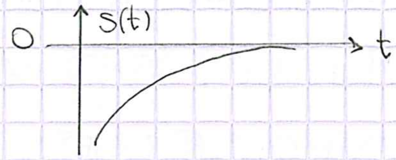


WE FOUND THAT

$\underline{\dot{S}(t) \ge 0}$ $\Rightarrow$ THE APPROACH TO $P^*(x)$ IS IRREVERSIBLE.

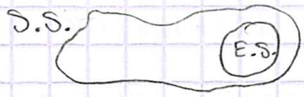
AND

$S(t \to +\infty) = 0$



PROPERTIES (A) + (B) CHARACTERIZE AN EQUILIBRIUM STATE

$P^*(x) = P_{eq}(x)$



★ BUT GIVEN THE W'S, HOW DO I KNOW IF THERE EXISTS A $P^*(x)$ WHICH SATISFIES THE D.B.?

THIS PROBLEM IS ADDRESSED BY <u>KOLMOGOROV CRITERION</u> (Math. Ann. 112, 155 (1936)).

(a) ASSUME D.B. IS SATISFIED, i.e.

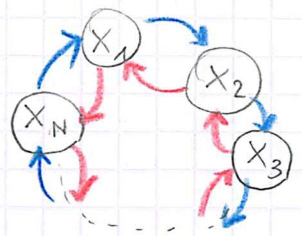
$\exists\, P^*$ s.t. $P^*(x) W(x \to x') = P^*(x') W(x' \to x)$

WE ORDER THE CONFIGURATIONS $\{x_1, x_2, \dots, x_N\}$ SO THAT THE TRANSITION RATES NEVER VANISH.

NOTE: THE x_m'S ARE NOT NECESSARILY DISTINGUISHED.

FOR AN ADJACENT PAIR,

$$p^*(x_i)W(x_i \to x_{i+1}) = p^*(x_{i+1})W(x_{i+1} \to x_i)$$



TAKE ON BOTH SIDES THE PRODUCTS

$$\prod_{i=1}^{N}$$

IDENTIFYING $x_{N+1} \equiv x_1$, THE p^*'S CANCEL AND WE GET

$$\prod_{i=1}^{N} W(x_i \to x_{i+1}) = \prod_{i=1}^{N} W(x_{i+1} \to x_i)$$

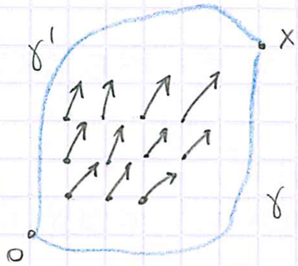
$$\Pi_+ = \Pi_-$$

THIS MIGHT AS WELL BE CHECKED WITHOUT LOOKING AT p^* (WE ONLY NEED TO KNOW IT EXISTS).

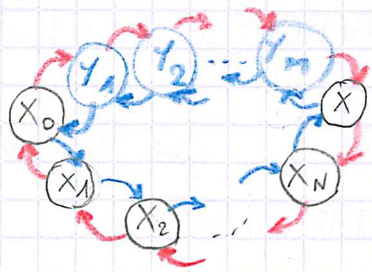
(b) ASSUME THAT $\Pi_+ = \Pi_-$ FOR ALL POSSIBLE SEQUENCES $\{x_1, \dots, x_N\}$
(i.e. ASSUME THE RATES TO BE s.t. $\Pi_+ = \Pi_-$).

ANALOGY: OUT OF A VECTOR FIELD, AM I ABLE TO BUILD A FUNCTION OF POSITIONS ONLY, i.e. THE POTENTIAL?

I SHOULD CHECK IF $\int_\gamma = \int_{\gamma'}$ FOR EVERY CHOICE OF $\gamma - \gamma'$, OR IN OTHER WORDS IF $\oint = 0$.



SIMILARLY, CHOOSE HERE A "REFERENCE" CONFIGURATION x_0. FOR ANY SPECIFIC x, LET'S BUILD



$$x \to f_{x_0}(x \mid \{x_1, \dots, x_N\}) = \prod_{i=0}^{N} \frac{W(x_i \to x_{i+1})}{W(x_{i+1} \to x_i)}$$

$$x_{N+1} \equiv x$$

STATEMENT: THIS QUANTITY IS INDEPENDENT OF $\{x_1, \dots, x_N\}$.

TO PROVE IT, CHOOSE ANOTHER PATH:

$$f_{x_0}(x \mid \{y_1, \dots, y_M\}) = \prod_{j=0}^{M} \frac{W(y_i \to y_{i+1})}{W(y_{i+1} \to y_i)}$$

$$y_0 = x_0, \; y_{M+1} = x$$

HOW DO WE CHECK IF THEY ARE EQUAL?

WE EVALUATE THE RATIO

$$\frac{f_{x_0}(x \mid \{x_1, \dots, x_N\})}{f_{x_0}(x \mid \{y_1, \dots, y_M\})} = \frac{\prod_{i=0}^{N} W(x_i \to x_{i+1}) \prod_{j=0}^{M} W(y_{j+1} \to y_j)}{\prod_{i=0}^{N} W(x_{i+1} \to x_i) \prod_{j=0}^{M} W(y_j \to y_{j+1})}$$

$$= \frac{\Pi_+ (x_1 \div x_N, \, y_M \div y_1)}{\Pi_- (y_1 \div y_M, \, x_N \div x_1)} = 1.$$

THEN WE CAN SIMPLY TAKE

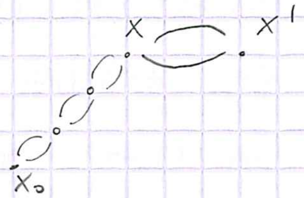
$$P^*(x) \equiv A_{x_0} f_{x_0}(x)$$

WHERE A_{x_0} IS FIXED BY REQUIRING NORMALIZATION.

IS $P^*(x)$ s.t. D.B. IS SATISFIED? WE NEED TO CHECK IF

$$f_{x_0}(x) W(x \to x') \stackrel{?}{=} f_{x_0}(x') W(x' \to x)$$

BUT THIS IS SIMPLE, BECAUSE BY DEFINITION



$$f_{x_0}(x') = f_{x_0}(x) \cdot \frac{W(x \to x')}{W(x' \to x)}$$

WHICH IS PRECISELY D.B., WHICH IS THEREFORE EQUIVALENT TO

$$\Pi_+ = \Pi_-$$

WHICH IS KOLMOGOROV CRITERION, AND IT IS A TEST ON THE RATES ONLY, D.B. IS A PROPERTY OF THE TRANSITION RATES!

THIS ALSO PROVIDES A WAY TO CONSTRUCT $P^*(x)$ IN TERMS OF THE RATES.

★ TO CONTINUE THE ANALOGY, IF WE CALL

$$\frac{W(x \to x')}{W(x' \to x)} = e^{\underline{A}} \quad \rightsquigarrow \quad \frac{P^*(x)}{P^*(x')} = e^{\int_{\gamma(x' \to x)} \underline{A} \cdot d\underline{l}}$$

$$\frac{\Pi_+}{\Pi_-} = 1 \equiv e^{\int_{\gamma(x' \to x)} \underline{A} \cdot d\underline{l} - \int_{\gamma'(x' \to x)} \underline{A} \cdot d\underline{l}} = e^{\oint \underline{A} \cdot d\underline{l}}$$

THIS IMPLIES

$$\underline{A} = \underline{\nabla} \phi$$

AND FINALLY

$$\frac{\rho^*(x)}{\rho^*(x')} = e^{\int_{\gamma(x' \to x)} \nabla\phi \cdot d\underline{l}} = e^{\phi(x) - \phi(x')} \Rightarrow \rho^*(x) \propto e^{\phi(x)}$$

WHERE $\phi(x)$ IS "MORALLY" THE HAMILTONIAN OF THE EQUILIBRIUM STATE.

☆ SOMETIMES YOU WANT A MONTE CARLO SIMULATION TO GENERATE A ρ^* THAT IS STATIONARY, BUT NON-EQUILIBRIUM (NONZERO PROBABILITY CURRENT). CAN WE DO IT?

Zia, Schmittman, J. Phys A 39 (2006), L407

J. Stat. Mech. (2007), P07012

YES: WE CAN CREATE ALGORITHMS WHICH DO NOT SATISFY D.B..

· EXERCISE

NOTE: SOLVED IN THE FINAL EXERCISE

MARKOV PROCESS $\{c_1, \dots, c_N\}$. ITS MASTER EQUATION CAN BE WRITTEN AS

$$\frac{\partial}{\partial t} P_1(c,t) = \sum_{c'} L_{c,c'} P_1(c',t) \qquad L \to N \times N \text{ MATRIX}$$

i) WRITE THE ELEMENTS OF $L_{c,c'}$ IN TERMS OF $W(c \to c')$.

(ii) $L_{c,c'}$ IS IN GENERAL NOT SYMMETRIC: LEFT AND RIGHT EIGENVECTORS ARE DIFFERENT.

(iii) SHOW THAT

$$\sum_c L_{c,c'} = 0 \quad \forall c'$$

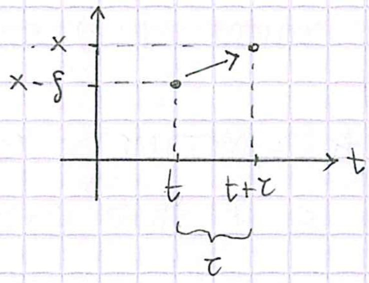
HENCE THERE IS AT LEAST ONE LEFT EIGENVECTOR WITH EIGENVALUE $\emptyset$, THE CONSTANT ONE. THE ASSOCIATED RIGHT EIGENVECTOR IS THE STATIONARY STATE.

KRAMERS - MOYAL EXPANSION

18.10.19

SOMETIMES A STOCHASTIC PROCESS IS SUCH THAT NOT MUCH HAPPENS FOR SMALL TIME INTERVALS τ. IF SO, WE CAN WRITE

$$P_1(x,t+\tau) = \int d\xi \; P_{1|1}(x,t+\tau \,|\, x-\xi,t)\, P_1(x-\xi,t)$$



FOR EXAMPLE, IN A DIFFUSION PROCESS

BIAS: $|\xi| \sim \tau$

NO BIAS: $|\xi| \sim \tau^{1/2}$

ASSUME $P_{1|1}$ TO BE TIME - TRANSLATIONAL INVARIANT, i.e.

$$P_{1|1}(x,t+\tau \,|\, x-\xi,t) \equiv P_{Tr}(x-\xi,\xi,\tau)$$

THEN THE PRODUCT

$$P_{Tr}(x-\xi,\xi,\tau)\, P_1(x-\xi,t)$$

NOTE: $P_{Tr}(\underset{\text{STARTING POINT}}{\downarrow}, \underset{\text{JUMP}}{\swarrow}, \underset{\text{TIME INTERVAL}}{\swarrow})$

IS A SMOOTH FUNCTION OF x WHICH CAN BE EXPANDED AROUND x:

$$P_1(x,t+\tau) = \int d\xi \left\{ P_{Tr}(x,\xi,\tau)P_1(x,t) + \sum_{k=1}^{+\infty} \frac{(-\xi)^k}{k!} \frac{\partial^k}{\partial x^k}\left[P_{Tr}(x,\xi,\tau)P_1(x,t)\right]\right\}$$

NOTE THIS IS NOT AN EXPANSION AROUND $\xi = 0$, BECAUSE THIS OBJECT IS SINGULAR FOR $\xi = 0$; NEVERTHELESS, ξ IS SMALL. HENCE

$$P_1(x,t+\tau) \overset{*}{=} P_1(x,t) + \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!}\frac{\partial^k}{\partial x^k}\Big\{ \underbrace{\Big[\int d\xi\; \xi^k P_{Tr}(x,\xi,\tau)\Big]}_{= \langle \xi^k \rangle_{x,\tau}} P_1(x,t)\Big\}$$

NOW JUST TAKE

*NOTE: $\left[\int d\xi\, P_{Tr}(x,\xi,\tau)\right] P_1(x,t)$. $\left[\int d\xi\, P_{Tr}(x,\xi,\tau)\right] = 1$ (BY BAYES THM, IF YOU LIKE).

$$\partial_t P_1(x,t) = \lim_{\tau\to 0} \frac{P_1(x,t+\tau) - P_1(x,t)}{\tau}$$

$$= \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!}\frac{\partial^k}{\partial x^k}\left\{\alpha_k(x) P_1(x,t)\right\} \qquad \text{(I)}$$

WHERE

$$\alpha_k(x) = \lim_{\tau\to 0} \frac{1}{\tau}\int d\xi\; \xi^k P_{Tr}(x,\xi,\tau)$$

NOTE $= \lim_{\tau\to 0} \frac{1}{\tau}\langle \xi^k\rangle_{x,\tau}$.

THIS IS THE (FORWARD) KRAMERS - MOYAL EXPANSION (THERE EXISTS A BACKWARD VERSION, FOR A BACKWARD "JUMP").

THIS IS NOTHING ELSE THAN AN ALTERNATE FORM FOR THE MASTER EQUATION.

· EXERCISE

NOTE: I'VE DONE IT IN THE EXERCISES, BUT ACTUALLY K-M IS A 1 ORDER D.E. IN t AND $P_1 = P_{1|1}$ WITH A GIVEN B.C..

PROVE THAT, FOR A MARKOV PROCESS, $P_{1|1}$ SATISFIES THE SAME K-M EXPANSION EQUATION AS P_1.

NOTICE THAT (I) STARTS WITH $K=1$. IT CAN BE REWRITTEN AS

$$\partial_t P_1(x,t) = -\frac{\partial}{\partial x} J_1(x,t)$$

$$J_1(x,t) \equiv \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k!} \cdot \frac{\partial^{k-1}}{\partial x^{k-1}} \left[\alpha_k(x) P_1(x,t) \right]$$

WHICH IS A CONTINUITY EQUATION FOR P_1, LIKE

$\partial_t P_1 = -\underline{\nabla} \cdot \underline{J}$

AND GUARANTEES PROBABILITY CONSERVATION.

NOTICE ALSO THERE APPEARS A $\lim_{\tau\to 0} \frac{1}{\tau}$ (MOMENTS) IN THE DEFINITION OF $\alpha_k(x)$, HENCE ALL THE MOMENTS SMALLER THAN $O(\tau)$ DON'T COUNT.

IF $\alpha_{k>2}(x) = 0$, THEN WE GET THE <u>FOKKER-PLANCK EQUATION</u> (OR DRIFT-DIFFUSION EQUATION)

$$\frac{\partial}{\partial t} P_1(x,t) = -\frac{\partial}{\partial x}\left[\alpha_1(x) P_1(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_2(x) P_1(x,t)\right]$$

$$\alpha_1(x) = \lim_{\tau\to 0} \frac{\langle \xi \rangle_{x,\tau}}{\tau} \qquad \alpha_2(x) = \lim_{\tau\to 0} \frac{\langle \xi^2 \rangle_{x,\tau}}{\tau} = \lim_{\tau\to 0} \frac{\langle (\xi - \langle \xi \rangle)^2 \rangle_{x,\tau}}{\tau}$$

THIS RESEMBLES A SCHRÖDINGER EQUATION FOR IMAGINARY TIMES, WITH SOME DUE DIFFERENCES: FOR INSTANCE, THE S-E WOULD PRESERVE P_1^2 (NOT P_1) AND ITS HAMILTONIAN MUST BE HERMITIAN (WE'LL COME BACK TO IT).

ACCORDING TO PAWULA'S THEOREM, EITHER THE K-M EXPANSION TAKES THE FORM OF A F-P EQUATION, OR ELSE IT CONTAINS INFINITELY MANY TERMS.

· PAWULA'S THEOREM (Phys. Rev. 162, 186 (1967))
IF $P_{Tr} \geq 0$, THE K-M EXPANSION MAY EITHER STOP AFTER THE FIRST OR SECOND TERM (i.e., $\alpha_{n>2} \equiv 0 \rightarrow$ F-P EQUATION) OR IT MUST CONTAIN ∞-MANY TERMS.

PROOF:

$$f, g \rightarrow (f|g) \equiv \int d\xi\, f(\xi)\, g(\xi)\, P_{Tr}(x,\xi,\tau)$$

FOR ANY $\lambda \in \mathbb{R}$,

$$(\lambda f + g | \lambda f + g) \geq 0$$

WHENCE THE SCHWARTZ INEQUALITY

$$(f|g)^2 \leq (f|f)(g|g)$$

NOW TAKING

$$f(\xi) = \xi^{\mu} \qquad \mu \text{ INTEGER}, \ \mu \geq 0$$

$$g(\xi) = \xi^{\nu} \qquad \nu > \mu \text{ INTEGER}$$

WE FIND THAT

$$\left(\int d\xi\, \xi^{\mu+\nu} P_{Tr}(x,\xi,\tau)\right)^2 \leq \left(\int d\xi\, \xi^{2\mu} P_{Tr}(x,\xi,\tau)\right)\left(\int d\xi\, \xi^{2\nu} P_{Tr}(x,\xi,\tau)\right)$$

FOR $\mu = 0$ AND IF $\tau \rightarrow 0$,

$$(\tau \alpha_{\nu}(x))^2 \leq \tau \alpha_{2\nu}(x) \xrightarrow{\tau \rightarrow 0} 0 \leq \alpha_{2\nu}(x) \quad \text{(TRIVIAL)}$$

FOR $1 \leq \mu < \nu$ AND $\tau \rightarrow 0$,

$$(\tau \alpha_{\mu+\nu}(x))^2 \leq (\tau \alpha_{2\mu}(x))(\tau \alpha_{2\nu}(x)) \Rightarrow \underline{\alpha_{\mu+\nu}^2 \leq \alpha_{2\mu}\alpha_{2\nu}}$$

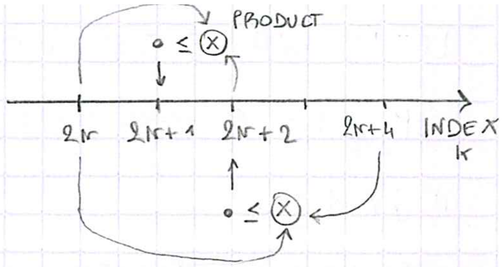
LET'S SPECIALIZE THIS TO THE CASE OF CONSECUTIVE μ, ν, i.e.

$$\mu = k, \quad \nu = k+1 \qquad \Rightarrow \qquad \alpha_{2k+1}^2 \leq \alpha_{2k}\alpha_{2k+2}$$

$$k \geq 1$$

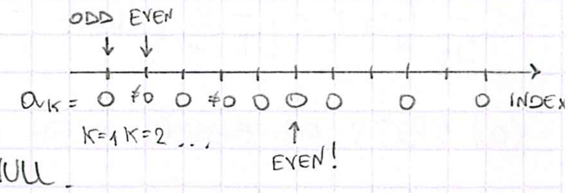
TAKING INSTEAD $\mu = k,\ \nu = k+2$,

$$\alpha^2_{2k+2} \leq \alpha_{2k}\,\alpha_{2k+4}$$



IMAGINE A PROCESS WHICH IS SYMMETRIC UNDER REFLECTION, i.e.

$$P_{Tr}(x,\xi,\tau) = P_{Tr}(x,-\xi,\tau)$$



SO THAT ALL THE ODD COEFFICIENTS ARE NULL.

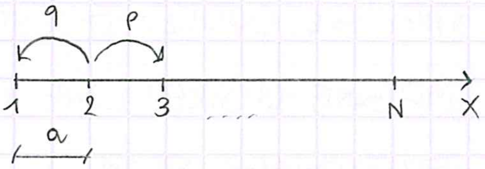
ASSUME, BY CONTRADICTION, THAT THE EVEN COEFFICIENTS START TO BE ZERO AS WELL AT A CERTAIN POINT: IT'S EASY TO SEE THAT THIS CONTRADICTS THE INEQUALITIES WE DERIVED ABOVE. THE ONLY CASE WHERE IT DOES NOT IS WHEN $k=0$ (FOR WHICH THE INEQUALITIES DO NOT HOLD), WHICH LEADS TO F-P EQUATION:

$$a_{k>2} = 0$$

FINALLY, NOTICE THE SAME APPLIES IF WE RELAX THE ASSUMPTION THAT THE ODD COEFFICIENTS BE NULL. $\therefore$

• EXAMPLES OF M.P.

(A) 1-d RANDOM WALKER ON A LATTICE



FINITE LATTICE WITH SPACING a. LET'S ASSOCIATE TO EACH j IN CONFIGURATION SPACE A VALUE

$$j \rightarrow x_j = ja$$

LET'S CALL p/q THE PROBABILITY OF A FORWARD/BACKWARD JUMP IN A TIME INTERVAL τ. THEN THE RATES ARE

$$W(j \rightarrow j+1) = \frac{p}{\tau}$$
$$W(j \rightarrow j-1) = \frac{q}{\tau}$$

BOUNDARY CONDITIONS CAN BE:

a) PERIODIC: $N+1 \equiv 1,\ 0 \equiv N$

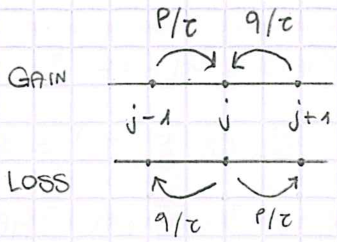
b) REFLECTING: $W(1 \rightarrow 2) = W(N \rightarrow N-1) = \frac{1}{\tau}$

LET'S WRITE THE MASTER EQUATION FOR

$P_j(t)$ = PROBABILITY FOR THE PARTICLE TO BE IN j AT TIME t

IF $j \neq 1, N$,

$$\frac{\partial}{\partial t} P_j(t) = \left[\frac{p}{\tau} P_{j-1}(t) + \frac{q}{\tau} P_{j+1}(t)\right] - \left(\frac{p}{\tau} + \frac{q}{\tau}\right) P_j(t)$$



(a) PBC: AS ABOVE, WITH $N+1 \equiv 1$ AND $0 \equiv N$.

(b) RBC

$$\frac{\partial}{\partial t} P_1(t) = \frac{q}{\tau} P_2(t) - \frac{1}{\tau} P_1(t)$$

$$\frac{\partial}{\partial t} P_2(t) = \frac{1}{\tau} P_1(t) + \frac{q}{\tau} P_3(t) - \left(\frac{p}{\tau} + \frac{q}{\tau}\right) P_2(t)$$

AND SIMILARLY ON THE OTHER SIDE.

NOW THIS CAN BE CAST, IN PRINCIPLE, IN THE FORM

$$\partial_t \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix} = \begin{pmatrix} & & \\ & \cdots & \\ & & \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix}$$

AND WE COULD SOLVE IT TO GET THE DYNAMICS.

IMAGINE INSTEAD WE ARE ONLY INTERESTED IN FINDING THE STATIONARY STATE.

a) PBC. IF $p \neq q$, WE EXPECT A CURRENT (BOTH A PARTICLE CURRENT AND A PROBABILITY CURRENT IN CONFIGURATION STATE), HENCE NON EQUILIBRIUM. IF $p = q$, THERE CAN BE EQUILIBRIUM. IN BOTH CASES, THE DENSITY IS CONSTANT:

$q = p \rightarrow P_{eq}$ = CONST., PROB. CURRENT = 0

$q \neq p \rightarrow P_{neq}$ = CONST., PROB. CURRENT $\neq 0$

b) RBC $\rightarrow$ EQUILIBRIUM, P_{eq} NOT CONST.

LET'S CHECK THE DETAILED BALANCE USING KOLMOGOROV CRITERIUM. A VIOLATION IS FOUND IF WE CONSIDER THE

WHOLE LOOP WITH PBC, WHENCE

$$\Pi_+ = \left(\frac{p}{\tau}\right)^N$$

$$\Pi_- = \left(\frac{q}{\tau}\right)^N$$

SO

$$\Pi_+ = \Pi_- \qquad \Rightarrow \qquad q = p.$$

NOTICE ANY OTHER LOOP, LIKE

$$\{1, 2, 3, 2\}$$

GIVES TRIVIALLY $\Pi_+ = \Pi_-$.

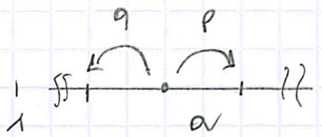
WITH PBC, YOU ARE FORCED TO TAKE AS MANY STEPS TO THE RIGHT AS TO THE LEFT IN CONFIGURATION SPACE, IF YOU WANT TO GO BACK TO THE INITIAL POINT:

$$\Pi_+ = \left(\frac{p}{\tau}\right)^{N/2}\left(\frac{q}{\tau}\right)^{N/2} \overset{?}{\underset{\checkmark}{=}} \Pi_- = \left(\frac{q}{\tau}\right)^{N/2}\left(\frac{p}{\tau}\right)^{N/2}$$

RANDOM WALK

28.10.19

SMALL RECAP:

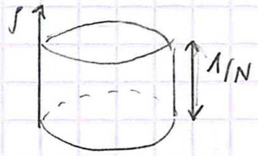


$$W(i \to i+1) = \frac{p}{\tau}$$

$$W(i \to i-1) = \frac{q}{\tau}$$

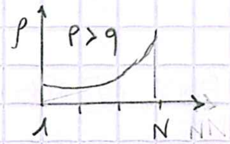
PBC $\nearrow$ NEQ-SS if $p \neq q$

PBC $\searrow$ EQ-SS if $p = q$



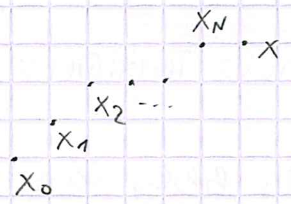
PBC

RBC $\to$ EQ-SS IRRESPECTIVE OF p, q



RBC

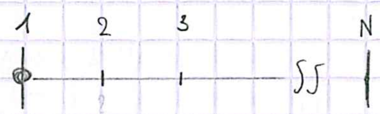
WHAT IS P_{eq}? BY KOLMOGOROV CRITERIUM,



$$f_{x_0}(x \mid \{x_1, \dots, x_N\}) = \prod_{\substack{i=0 \\ x_{N+1} \equiv x}}^{N} \frac{W(x_i \to x_{i+1})}{W(x_{i+1} \to x_i)}$$

WE SHOWED THAT, IF D-B IS SATISFIED, THEN f_{x_0} IS INDEPENDENT OF $\{x_1, \dots, x_N\}$ AND

$$P_{eq}(x) \propto f_{x_0}(x).$$

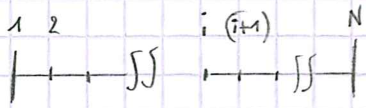


LET'S DO IT IN RBC CASE. CHOOSE $x_0 \equiv 1$, SO THAT

$$f(2) = \frac{W(1 \to 2)}{W(2 \to 1)} = \left(\frac{1/\tau}{q/\tau}\right) = \frac{1}{q}$$

$$P_{eq}(1) = A f(1) = A$$

$$P_{eq}(2) = A f(2) = \frac{A}{q}$$



FOR ANY i FAR FROM THE BOUNDARIES,

$$P_{eq}(i) = P_{eq}(2) \cdot \left(\frac{p/\tau}{q/\tau}\right)^{i-2} = \frac{A}{q}\left(\frac{p}{q}\right)^{i-2}$$

$i = 2, \dots, N-1$

FINALLY

$$P_{eq}(N) = P_{eq}(N-1) \cdot \left(\frac{p/\tau}{1/\tau}\right) = \frac{A}{q}\left(\frac{p}{q}\right)^{N-3} \cdot p$$

· EXERCISE:

FIX A s.t. $\sum_{i=1}^{N} P_{eq}(i) = 1$

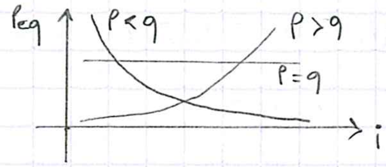
NOTE:

$$A^{-1} = 1 + \frac{1}{q-p} + \left(\frac{p}{q}\right)^{N-2}\left(1 - \frac{1}{q-p}\right)$$

THIS GIVES P_{eq} WITHOUT SOLVING THE DYNAMICS (JUST BY D-B).

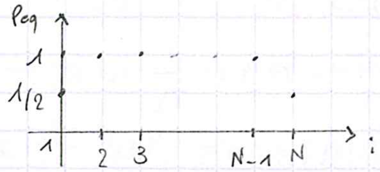
WE FOUND

$$P_{eq}(i \neq 1, N) \propto \left(\frac{p}{q}\right)^{i-2} \propto e^{i \cdot \ln\left(\frac{p}{q}\right)}$$



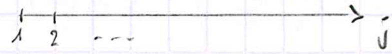
IF $p = q$,

$$\begin{cases} P_{eq}(1) = P_{eq}(N) = \dfrac{1}{2(N-1)} \\ P_{eq}(i \neq 1, N) = \dfrac{1}{N-1} \end{cases}$$



☆ LET'S GO BACK TO THE MASTER EQUATION

$$\frac{\partial}{\partial t} P_j(t) = \frac{p}{\tau} P_{j-1}(t) + \frac{q}{\tau} P_{j+1}(t) - \frac{p+q}{\tau} P_j(t)$$



$x_j = j \cdot a$

WHAT IF WE WANT TO FORGET ABOUT THE LATTICE, AND GIVE INSTEAD A COARSE GRAINED DESCRIPTION? IN THE CONTINUUM LIMIT,

$$P_j(t) \to P(x_j, t)$$

$$P_{j\pm1}(t) \to P(x_j \pm a, t)$$

AND

$$\frac{\partial}{\partial t} P(x,t) = \frac{p}{\tau} P(x-a,t) + \frac{q}{\tau} P(x+a,t) - \frac{p+q}{\tau} P(x,t).$$

ASSUMING $a \to 0$, WE CAN EXPAND

$$\frac{\partial}{\partial t} P(x,t) = \frac{p}{\tau}\left[P(x,t) - a\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2}P(x,t) + O(a^3)\right]$$
$$+ \frac{q}{\tau}\left[P(x,t) + a\frac{\partial}{\partial x}P(x,t) + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2}P(x,t) + O(a^3)\right] - \frac{p+q}{\tau}P(x,t).$$

THE $P(x,t)$ TERM ON THE RIGHT CANCELS OUT (WE WOULD HAVE PROBLEMS WITH PROBABILITY CONSERVATION OTHERWISE!) AND

$$\frac{\partial}{\partial t} P(x,t) = -\underbrace{\frac{(p-q)a}{\tau}}_{v}\frac{\partial P}{\partial x} + \underbrace{\frac{1}{2}\frac{a^2}{\tau}(p+q)}_{D}\frac{\partial^2}{\partial x^2}P(x,t) + O\left(\frac{a^3}{\tau}\right)$$

THIS TAKES THE FORM OF A FOKKER-PLANCK EQUATION (ALMOST), WITH

v: DRIFT VELOCITY

D: DIFFUSION COEFFICIENT

BUT IS THE LAST TERM REALLY NULL?

FIRST WE NOTICE THAT, IN ORDER TO GET FINITE v AND D, THEN

$$\tau \sim a^2$$

AND, NECESSARILY,

$$(p-q) \sim \frac{\tau}{a} \sim a$$

NOTE: THIS MEANS THAT, IF YOU WANT TO MAKE a SMALL, YOU NEED TO REDUCE THE BIAS $(p-q)$ IN ORDER NOT TO GET TRIVIAL RESULTS.

OTHERWISE THE PROCESS TRIVIALIZES (YOU LOSE THE BIAS OR THE DIFFUSION). IF YOU DO THIS, THEN

$$O\left(\frac{a^3}{\tau}\right) \sim O(a)$$

AND WE ACTUALLY HAVE A F-P EQUATION FOR $a \to 0$:

$$\frac{\partial}{\partial t} P_1(x,t) = -v \frac{\partial}{\partial x} P_1(x,t) + D \frac{\partial^2}{\partial x^2} P_1(x,t). \qquad (I)$$

BEING THE PROCESS MARKOVIAN, THEN THE SAME EQUATION HOLDS FOR $P_{1|1}$; THE SOLUTION WITH INITIAL DATUM $\delta(x)$ IS THE $P_{1|1}$,

$$P_1(x,t=0) \equiv \delta(x) \qquad \leftrightarrow \qquad P_{1|1}(x,t=0|0,0) = \delta(x).$$

TO SOLVE (I), BETTER TO SIT ON THE PARTICLE:

$$P_1(x,t) = g(x-vt,t)$$

$$\frac{\partial}{\partial t} P_1 = \frac{\partial}{\partial t} g - v \frac{\partial g}{\partial x} \qquad \Rightarrow \qquad \frac{\partial}{\partial t} g(x,t) = D \frac{\partial^2}{\partial x^2} g(x,t)$$

NOW THIS IS EASILY SOLVED BY FOURIER TRANSFORM:

$$\frac{\partial}{\partial t} \tilde{g}(q,t) = -Dq^2 \tilde{g}(q,t) \qquad + \text{I.C. } \tilde{g}(q,t=0) = 1$$

$$\tilde{g}(q,t) = e^{-Dq^2 t} \qquad \Rightarrow \qquad g(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

THIS IS ALSO THE $P_{1|1}$, AND WE RECOGNIZE IN $g(x,t)$ THE $P_{1|1}$ OF A <u>WIENER PROCESS</u>, WHICH IS THUS THE CONTINUUM LIMIT OF A RANDOM WALK.

FINALLY, ADDING BACK THE DRIFT,

$$\underset{(1|1)}{P_1}(x,t\underset{|0,0)}{\,}) = \frac{1}{\sqrt{4\pi D t}} \exp\left\{-\frac{(x-vt)^2}{4Dt}\right\}.$$

IT'S EASY TO SHOW THAT

$$\langle x \rangle = v t$$

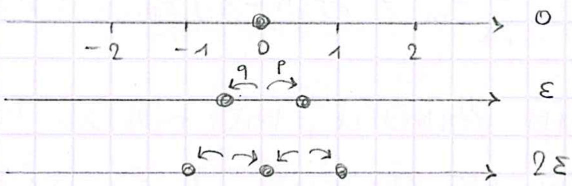
$$\langle (\Delta x)^2 \rangle = \langle (x - \langle x \rangle)^2 \rangle = 2 D t \qquad \text{(FICK'S LAW OF DIFFUSION).}$$

IF ONE CALCULATES THE α_K'S OF KRAMERS-MOYAL'S EXPANSION STARTING FROM $P_{1|1}$, THEN NOT SURPRISINGLY $\alpha_{K>2} = 0$.

- <u>RANDOM WALK, ANOTHER WAY</u>

DISCRETE TIME, ON A LATTICE, $p+q=1$.

WHAT IS THE PROBABILITY OF FINDING THE PARTICLE AT SITE j AFTER K STEPS?



ℓ = # STEPS TO THE LEFT

r = # STEPS TO THE RIGHT

UP TO "TIME" K, i.e. $\ell + r = K$

$r - \ell = j \quad \Rightarrow \quad r = \frac{1}{2}(K+j), \ \ell = \frac{1}{2}(K-j)$

SO IT'S ESSENTIALLY A BERNOULLI TRIAL PROBLEM:

$$P(j,K) = \binom{K}{\ell} p^r q^\ell \qquad \text{IF } r, \ell \text{ ARE INTEGERS, 0 OTHERWISE}$$

- <u>EXERCISE</u>:

REINSTATE THE DIMENSIONS OF TIME AND SPACE (τ, a), TAKE THE PROPER LIMITS AND RECOVER THE GAUSSIAN IN THE CONTINUUM LIMIT (BY STIRLING'S FORMULA).

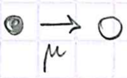
- <u>POPULATION DYNAMICS</u>: BRANCHING & DECAY PROCESS

(EXAMPLE (B) OF A MARKOV PROCESS)

THIS IS A 0-DIMENSIONAL STATISTICAL PROBLEM.

m INDIVIDUALS ($m \in \mathbb{N}$) WITH A MARKOVIAN DYNAMICS:

(a) BRANCH  $W(m \to m+1) = \sigma m$

(b) DECAY  $W(m \to m-1) = \mu m$

$m=0$ IS THE <u>ABSORBING STATE</u> FOR THE DYNAMICS: ONCE YOU REACH IT, NO FLUCTUATIONS (NO DYNAMICS). THIS CREATES A VIOLATION OF TIME REVERSAL, AND THUS A NON-EQUILIBRIUM BEHAVIOR.

THIS IS A "GAUSSIAN MODEL" (IN A SENSE WHICH WILL BE CLARIFIED): LET'S TRY TO SOLVE IT. THE M-E READS

$$\frac{\partial}{\partial t} P_m = \sigma(m-1)\,P_{m-1} + \mu(m+1)\,P_{m+1} - (\mu+\sigma)\,m\,P_m \qquad m \neq 0. \qquad (*)$$

IN PRINCIPLE, THIS HOLDS FOR $m > 1$, BUT IT WORKS ANYWAY.
IF WE ASSUME $P_{-1} \equiv 0$, THEN THE SAME EQUATION CAN BE USED FOR $m=0$ AS WELL.
MORALLY, WE FOUND A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WHICH CAN BE SIMPLIFIED THROUGH A FOURIER TRANSFORM.
SIMILARLY, HERE WE DEFINE A <u>GENERATING FUNCTION</u>

$$g(x,t) = \sum_{m=0}^{+\infty} x^m \, P_m(t)$$

NOTE: $= \langle x^m \rangle_{P_m}$.

THIS IS USEFUL BECAUSE:

(i) THE COEFFICIENTS OF ITS TAYLOR EXPANSION AROUND $x=0$ GIVE US BACK $P_m(t)$.

(ii) IT PROVIDES DIRECT INFO ON THE MOMENTS. FOR INSTANCE

$$\langle m \rangle = \left.\frac{\partial g}{\partial x}\right|_{x=1}$$

$$\langle m(m-1) \rangle = \left.\frac{\partial^2 g}{\partial x^2}\right|_{x=1}.$$

(iii) BECAUSE OF NORMALIZATION,

$$g(x=1,t) \equiv 1 \quad \forall t.$$

(iv) THE PROBABILITY OF BEING IN THE ABSORBING STATE IS

$$P_0(t) = g(x=0,t).$$

APPLYING ON (*)

$\sum_{m=0}^{\infty} x^m \left(LHS = RHS \right)$

NOTE: IN THE FIRST TERM, THE $m=0$ CONTRIBUTION IS NULL BECAUSE $P_{-1} = 0$ AND THE $m=1$ IS KILLED BY $(m-$ WHENCE x^2.

WE FIND

$$\frac{\partial}{\partial t} g(x,t) = \sum_{m=0}^{\infty} \sigma(m-1) P_{m-1} x^m + \sum_{m=0}^{\infty} \mu(m+1) P_{m+1} x^m - \sum_{m=0}^{\infty} (\mu+\sigma) m P_m x^m$$

$$= \underbrace{\sum_{m=0}^{\infty} \sigma m P_m x^{m+1}}_{m = m+1 \quad (P_{-1} \equiv 0)} + \underbrace{\sum_{m=0}^{+\infty} \mu m P_m x^{m-1}}_{m = m-1} - \sum_{m=0}^{\infty} (\mu+\sigma) m P_m x^m$$

HENCE

$$\frac{\partial}{\partial t} g(x,t) = \sigma x^2 \frac{\partial}{\partial x} g + \mu \frac{\partial}{\partial x} g - (\mu+\sigma) x \frac{\partial}{\partial x} g$$

$$= (\mu - \sigma x)(1-x) \frac{\partial}{\partial x} g$$

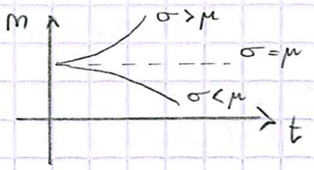
WHICH IS NOT SURPRISING, BECAUSE $g(x=1,t) = 1 \ \forall t$, SO $\frac{\partial g}{\partial t}$ SHOULD VANISH IN $x=1$ TO PRESERVE NORMALIZATION.

- POPULATION DYNAMICS

30.10.19

NOTICE, AT FIRST SIGHT,

$$\dot{m} = \sigma m - \mu m \qquad (\text{HERE } m \to \langle m \rangle).$$

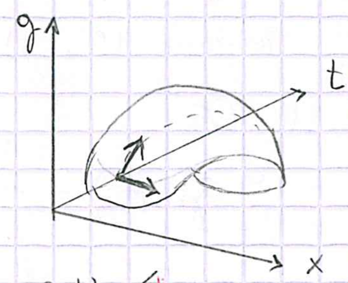


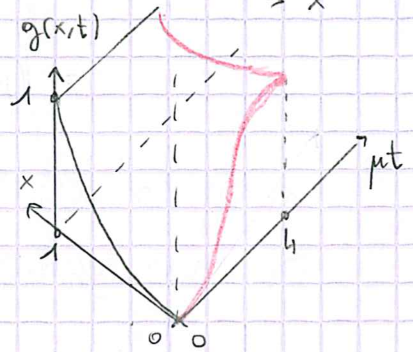
BY WRITING A DIFFERENTIAL EQUATION FOR g, WE FOUND

$$\frac{\partial g}{\partial t} = \mu (1-x)\left(1 - \frac{\sigma}{\mu} x\right) \frac{\partial g}{\partial x}$$

WHICH WE ARE GOING TO SOLVE BY THE METHOD OF CHARACTERISTICS.

THE SOLUTION WILL BE A SURFACE $g(x,t)$, AND WE KNOW A RELATION BETWEEN THE GRADIENTS IN THE x AND t DIRECTIONS. WE CAN RECONSTRUCT THE SURFACE BY BUILDING UP ITS CONTOUR LINES.





WHAT DO WE KNOW ABOUT g? IF $m(0) = m_0$,

$$g(x, t=0) = \sum_{m=0}^{\infty} P_m(0) x^m = x^{m_0}$$

$$g(x=1, t) = 1.$$

A CONTOUR LINE HAS THE FORM $(x(\ell), t(\ell))$,

$g(x(\ell), t(\ell))$ IS INDEPENDENT OF ℓ

$$\frac{dg}{d\ell}(x(\ell), t(\ell)) = 0 = x' \frac{\partial g}{\partial x} + t' \frac{\partial g}{\partial t}.$$

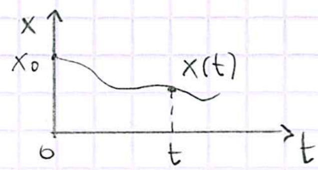
A CONVENIENT CHOICE IS TO TAKE $\ell = t$, SO THAT $t' = 1$, $x' = \dot{x}$ AND

$$\dot{x} = -\frac{\partial g}{\partial t} \cdot \left(\frac{\partial g}{\partial x}\right)^{-1} = -\mu(1-x)\left(1 - \frac{\sigma}{\mu} x\right).$$

THIS IS ACTUALLY DINI'S THEOREM. SEPARATING VARIABLES,

$$\frac{dx}{(1-x)\left(1-\frac{\sigma}{\mu}x\right)} = -\mu\, dt \quad \Rightarrow \quad \int_{x_0}^{x(t)} \frac{d\xi}{(1-\xi)\left(1-\frac{\sigma}{\mu}\xi\right)} = -\mu t.$$

THIS DEFINES IMPLICITLY THE CONTOUR LINE OF ALTITUDE $g(x_0, t=0) = x_0^{m_0}$.



GIVEN x, t, ONE CAN USE IT TO FIND $x_0(x,t)$ AND

$$g(x,t) = g(x_0(x,t), 0) \qquad \left(= (x_0(x,t))^{m_0} \text{ FOR US, IN RED}\right).$$

· <u>EXERCISE</u>: CALCULATE $g(x,t)$. NOTE: (A).

FINALLY, THE EXTINCTION PROBABILITY IS GIVEN BY

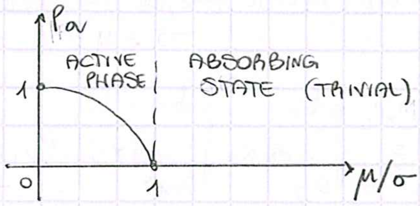
$$P_0(t) = g(x=0,t) = \begin{cases} \left[\dfrac{e^{(\mu-\sigma)t} - 1}{e^{(\mu-\sigma)t} - \sigma/\mu}\right]^{m_0} & \text{IF } \sigma \neq \mu \rightarrow \tau_c = \dfrac{1}{|\mu-\sigma|} \\ \left(\dfrac{\mu t}{1+\mu t}\right)^{m_0} & \text{IF } \sigma = \mu . \end{cases}$$

WE NOTICE THAT THE CHARACTERISTIC TIME SCALE τ_c DIVERGES AS $\mu \to \sigma$, PRETTY MUCH AS IT HAPPENS IN CRITICAL PHENOMENA.

TO SEE IT BETTER, LET'S LOOK AT P_{AV} (BEING ACTIVE IN THE STAT. STATE),

(THERE'S REALLY NO STATIONARY STATE!)

$$P_{av} = 1 - P_0(t \to +\infty) = \begin{cases} 0, & \mu > \sigma \\ 1 - \left(\dfrac{\mu}{\sigma}\right)^{m_0}, & \mu < \sigma \end{cases}$$



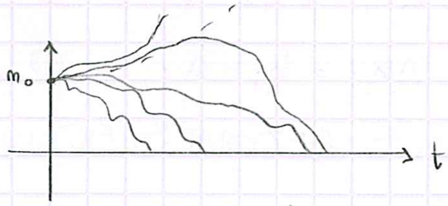
EXPANDING AROUND 1,

NOTE: $\left(\frac{\mu}{\sigma}\right)^{m_0} = \left(1-\left(1-\frac{\mu}{\sigma}\right)\right)^{m_0} \simeq 1 - m_0\left(1-\right.$

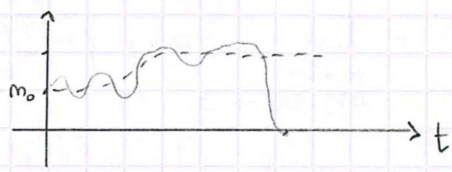
$$P_{av} \sim m_0\left(1 - \frac{\mu}{\sigma}\right) \quad \Rightarrow \quad \text{"}\beta = 1\text{"} \quad \left(\text{MEAN FIELD: } \beta = \frac{1}{2}\right.$$

BUT HOW COMES WE GOT A PHASE TRANSITION IN ZERO DIMENSIONS (THE THERMODINAMIC LIMIT IS ACHIEVED, MORALLY, BY SENDING $t \to \infty$).

· <u>EXERCISE</u>: FIND $\langle m(t) \rangle$. (NOTE: (B).)



IMAGINE WE SIMULATE $m(t)$. IN THE ACTIVE PHASE, THERE IS A FINITE PROBABILITY OF SURVIVING.
IF WE ADD AN "INTERACTION TERM" WHICH LIMITS THE GROWTH OF m, THEN $m(t)$ CAN ASINTOTICALLY STABILIZE.



HOWEVER, SOONER OR LATER A FLUCTUATION WILL TAKE $m(t)$ TO ZERO: THERE ARE NO PHASE TRANSITIONS.
WE CAN STILL HAVE TRANSITIONS IF WE ADD MORE SPACE DIMENSIONS: THEN "MIGRATION OF RABBITS" CAN HEAL THE LOSS OF LOCAL ORDER DUE TO "$m(x,t) = 0$ AT A SPECIFIC x ONLY".

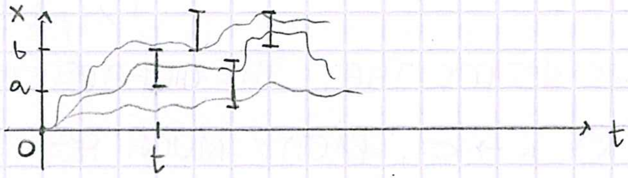
· <u>WIENER MEASURE</u>

LAST TIME WE FOUND P_1 AND $P_{1|1}$ BOTH SATISFY

$$\frac{\partial}{\partial t} P_1 = -v \frac{\partial P_1}{\partial x} + D \frac{\partial^2 P_1}{\partial x^2} \quad \overset{v=0}{\rightsquigarrow} \quad P_{1|1}(x_2,t_2|x_1,t_1) = \frac{e^{-\frac{(x_2-x_1)^2}{4D(t_2-t_1)}}}{\left[4\pi D(t_2-t_1)\right]^{1/2}}$$

WHICH IS NOTHING BUT A WIENER PROCESS. IF $X(0)=0$, THEN

$$\mathrm{Prob}\left(x(t)\in[a,b]\right) = \int_a^b dx\, P_{1|1}(x,t|0,0).$$



PUTTING ANOTHER GATE,

$$\mathrm{Prob}\left(x(t_1)\in[a_1,b_1], x(t_2)\in[a_2,b_2]\right) = \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} dx_1\, P_2(x_2,t_2;x_1,t_1)$$

$$\overset{\text{M.P.}}{=} \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} dx_1\, P_{1|1}(x_2,t_2|x_1,t_1)\, P_{1|1}(x_1,t_1|0,0).$$

IN FACT, FOR A MARKOV PROCESS(*) P_2 "=" $P_{1|1} \times P_{1|1}$. AND SO ON,

$$\mathrm{Prob}\left(x(t_1)\in[a_1,b_1],\dots,x(t_N)\in[a_N,b_N]\right) = \int_{a_N}^{b_N} dx_N \dots \int_{a_1}^{b_1} dx_1\, P_{1|1}(x_N,t_N|x_{N-1},t_{N-1})\dots P_{1|1}(x_1,t_1|0,0)$$

BY SENDING $N\to\infty$ AND

$$\Delta t_i = t_{i+1} - t_i \to 0$$

$$\Delta x_i = b_i - a_i \to 0$$

(*) NOTE: P_2 "=" $P_{1|1} \cdot P_{1|1} \cdot P_1(0,0)$, BUT IN THIS CASE $X(0)$.

NOTE: WE'LL USE $\int_{a_1}^{b_1} dx_1 (\cdot) \simeq (\cdot)\Delta x_1$.

THIS BECOMES ACTUALLY THE PROBABILITY OF A TRAJECTORY:

$$\mathrm{Prob}(\text{TRAJ.}) = \lim_{\substack{N\to\infty \\ \Delta t_i\to 0 \\ \Delta x_i \to 0}} \Delta x_N \dots \Delta x_1 \frac{1}{\sqrt{4\pi D(t_N - t_{N-1})}} \cdot \ldots \cdot \frac{1}{\sqrt{4\pi D \Delta t_1}} \cdot e^{-\sum_{k=0}^{N-1} \frac{(x_{i+1}-x_i)^2}{4D(t_{i+1}-t_i)}}$$

$$= \lim_{\substack{N\to\infty \\ \Delta t_i, \Delta x_i \to 0}} \exp\left\{ -\sum_{k=0}^{N-1} \frac{1}{4D} \left(\frac{x_{i+1}-x_i}{t_{i+1}-t_i}\right)^2 \underbrace{(t_{i+1}-t_i)}_{\Delta t_i} \right\} \prod_{k=1}^{N-1} \frac{\Delta x_i}{\sqrt{4\pi D \Delta t_i}}$$

$$\underset{\substack{\uparrow \\ \text{FORMAL}}}{=} \exp\left\{ -\int_0^t d\tau \frac{(\dot{x}(\tau))^2}{4D} \right\} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}}$$

$$\equiv d_W X(\tau).$$

NOTE: IT'S THE PROBABILITY DENSITY OF HAVING A TRAJECTORY WITH SHAPE $X(\tau)$ COMING FROM A WIENER PROCESS.

THIS DEFINES (FORMALLY) THE WIENER MEASURE.

CLEARLY WE HAVE A PROBLEM: TIPICALLY, A PROCESS IS NOT

DIFFERENTIABLE, WHILE HERE WE ARE USING $\dot{X}(\tau)$ AS A STATISTICAL WEIGHT. THIS IS COMPENSATED BY A BADLY DEFINED DIFFERENTIAL.. THIS IS ANALOGOUS TO USING THE INVERSE OF THE KINETIC TERM AS A PROPAGATOR IN QUANTUM MECHANICS, WHICH IS NOT WELL DEFINED (IT'S A FEYNMAN PATH INTEGRAL).

NOW WE CAN EVALUATE

$$\mathrm{PROB}\left(x(t) \in [a,b]\right) = \int_{x(0)=0}^{x(t)\in[a,b]} d_w x(\tau)$$

$$= \int_{x(0)=0}^{x(t)\in[a,b]} \prod_\tau \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}} \exp\left\{-\frac{1}{4D}\int_0^t d\tau\, \dot{x}^2\right\} = \int_a^b dx \; P_{1|1}(x,t\,|\,0,0).$$

ANOTHER SIMPLE PATH INTEGRAL IS

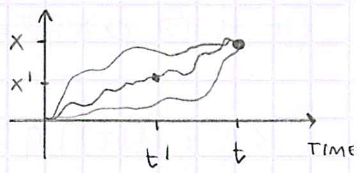
$$\int_{x(0)=0}^{x(t)\ \text{ARBITRARY}} d_w x(\tau) = \int_{-\infty}^{+\infty} dx \; P_{1|1}(x,t\,|\,0,0) = 1.$$

THIS IS THE UNCONDITIONED WIENER MEASURE: IN GENERAL,

$$\int_{x(0)=0}^{x(t)\ \text{ARBITRARY}} d_w x(\tau)\cdot \underset{\uparrow\atop \text{FUNCTIONAL}}{(\cdot)} = \int_{-\infty}^{+\infty} dx_t \int_{x(0)=0}^{x(t)=x_t} d_w x(\tau)\cdot(\cdot)$$

WHILE C-K GIVES

$$\int_{x(0)=0}^{x(t)=x_t} d_w x(\tau)\cdot(\cdot) \overset{\text{M.P.}}{\underset{0<t'<t}{=}} \int_{-\infty}^{+\infty} dx' \int_{x(0)=0}^{x(t')=x'} d_w x(\tau) \int_{x(t')=x'}^{x(t)=x_t} d_w x(\tau)\,(\cdot).$$



· <u>THEOREM</u> (WIENER)

THE SET OF DISCONTINUOUS AS WELL AS THE SET OF DIFFERENTIABLE FUNCTIONS HAS ZERO MEASURE.

NOTE: USING THIS MEASURE, OBVIOUSLY.

THERE REMAIN THE FUNCTIONS WHICH ARE CONTINUOUS AND NOWHERE DIFFERENTIABLE.

NOTE: INTERESTINGLY,

$$P_{1|1}(x_t,t\,|\,x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_w x(\tau)$$

WIENER MEASURE AND QUANTUM MECHANICS

CONSIDER THE SINGLE-PARTICLE HAMILTONIAN

$H = H_0 + V$.

FOR $t \in \mathbb{R}$, WE CAN CONSTRUCT THE PROPAGATOR IN QM VIA

$$\mathcal{U}(t) = e^{-i\frac{H}{\hbar}t}.$$

IN FACT, CALLING q THE POSITION, THE PROPAGATOR IS

$$\langle q' | \mathcal{U}(t) | q \rangle = \int_{Q(0)=q}^{Q(t)=q'} \mathcal{D}Q \, e^{i\frac{S}{\hbar}}$$

WHICH IS FEYNMAN'S PATH INTEGRAL FORMULATION, AND WHERE

$$S = \int_0^t dt' \left[\frac{1}{2} m \dot{Q}^2 - V(Q) \right]$$

IS THE CLASSICAL ACTION ASSOCIATED TO THE MOTION OF THE PARTICLE IN THAT POTENTIAL.

IT IS SOMEWHAT CONVENIENT TO CONSIDER THE EUCLIDEAN VERSION OF THIS,

$$\mathcal{U}(\tau) = e^{-\frac{H}{\hbar}\tau} \qquad t = -i\tau$$

(IF H IS BOUNDED FROM BELOW, THEN τ IS POSITIVE), SO THAT

$$\langle q' | \mathcal{U}(\tau) | q \rangle = \int_{Q(0)=q}^{Q(\tau)=q'} \mathcal{D}Q \, e^{-\frac{S_E}{\hbar}}$$

$$S_E = \int_0^\tau d\tau' \left[\frac{1}{2} m \dot{Q}^2(\tau') + V(Q) \right]$$

NOTE: $d\tau = i\,dt$, $\frac{\partial}{\partial \tau} = -i \frac{\partial}{\partial t}$.

WHICH RESEMBLES THE WIENER PROCESS (BROWNIAN MOTION) IF $V=0$:

WIENER MEASURE $\leftrightarrow$ QM IN IMAGINARY TIME (FREE PARTICLE)

(NOTICE IN QM YOU GET TRANSITION AMPLITUDES, AND NOT TRANSITION PROBABILITIES). INDEED, THE ANALYTIC CONTINUATION OF SCHRÖDINGER'S EQUATION FOR IMAGINARY TIMES GIVES THE DIFFUSION EQUATION.

* BUT WHAT HAPPENS IF I SWITCH ON $V(x)$? IN QM,

$$i\partial_t \Psi = H\Psi \qquad\qquad H \propto -\partial_x^2 + V .$$

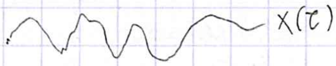
IN EUCLIDEAN TIME,

$$\partial_\tau \Psi = \left(-\partial_x^2 + V\right)\Psi \qquad \text{(I)}$$

NOTE: IT CAN'T BE CAST IN THE FORM OF A DIVERGENCE OF A PROBABILITY CURRENT.

WHICH DOESN'T LOOK LIKE A FOKKER-PLANCK EQUATION: WHAT'S WORSE, IT VIOLATES PROBABILITY CONSERVATION!

SO $V(x)$ IS BETTER INTERPRETED AS THE PROBABILITY OF "KILLING" A PARTICLE.



TO CHECK THIS, LET'S CONSIDER A BROWNIAN PARTICLE, DESCRIBED BY $x(\tau)$, WITH AN "EVAPORATION" RATE $V(x(\tau),\tau)$.

FOR SMALL Δt, THE PROBABILITY OF SURVIVAL UP TO $(t+\Delta t)$ IS

$$P_S(t+\Delta t) = P_S(t) - \Delta t\, V(x(t),t)\, P_S(t)$$

$$\frac{dP_S(t)}{dt} = -V\big(x(t),t\big)P_S(t) \overset{\circledast}{\Rightarrow} P_S(t) = e^{-\int_0^t d\tau\, V(x(\tau),\tau)} .$$

LAST TIME WE EXPRESSED (WITH NO EVAPORATION)

$$P^0_{1|1}(x_t,t|x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau) .$$

IN THE PRESENCE OF $V(x)$, THIS WILL SIMPLY BECOME

$$P^V_{1|1}(x_t,t|x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau)\, \underbrace{e^{-\int_0^t d\tau\, V(x(\tau),\tau)}}_{P_S(t)} . \qquad \text{(II)}$$

DOES THIS $P^V_{1|1}$ SATISFY (I)? LET'S CHECK IT:

$$\frac{\partial}{\partial t} P^V_{1|1}(x_t,t|x_0,0) = \left[D\frac{\partial^2}{\partial x_t^2} - V(x_t,t)\right] P^V_{1|1}(x_t,t|x_0,0) . \qquad \text{(III)}$$

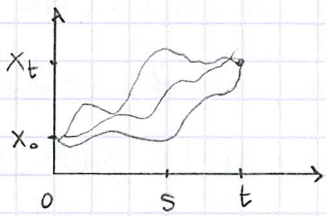
THIS IS KNOWN AS <u>BLOCH EQUATION</u>, AND IT CANNOT COME FROM A KRAMERS-MOYAL EXPANSION BECAUSE OF PROBABILITY VIOLATION. WE TAKE AS BOUNDARY CONDITIONS

$$P^V_{1|1}(x_t,0|x_0,0) = \delta(x_t - x_0) .$$

$\circledast$ NOTE: $P_S(t)$ IS NOT A FUNCTION OF POSITION AS WELL BUT RATHER A FUNCTIONAL OF THE PATH $x(\tau)$.

HOW DO WE CHECK IT? WE KNOW IT IS SATISFIED IF $V=0$, SO WE WILL TRY TO WRITE SOMETHING LIKE A DYSON'S EQUATION.

· <u>OBSERVATION</u>: IF $V=0$,



$$P^{0}_{1|1}(x_t,t|x_0,0) \underset{0<s<t}{=} \int_{-\infty}^{\infty} dx_s\, P^{0}_{1|1}(x_t,t|x_s,s)\, P^{0}_{1|1}(x_s,s|x_0,0).$$

BUT THIS ALSO HOLDS FOR $V \neq 0$:

$$P^{V}_{1|1}(x_t,t|x_0,0) = \int_{x(0)=x_0}^{x(t)=x_t} d_W x(\tau)\, e^{-\int_0^t d\tau\, V(x(\tau),\tau)}$$

$$= \int_{-\infty}^{\infty} dx_s \int_{x(0)=x_0}^{x(s)=x_s} d_W x(\tau) \int_{x(s)=x_s}^{x(t)=x_t} d_W x(\tau)\, e^{-\int_0^s d\tau\, V(x(\tau),\tau)}\, e^{-\int_s^t d\tau\, V(x(\tau),\tau)}$$

$$= \int_{-\infty}^{+\infty} dx_s\, P^{V}_{1|1}(x_t,t|x_s,s)\, P^{V}_{1|1}(x_s,s|x_0,0).$$

· <u>OBSERVATION</u>: BY CONSTRUCTION (LOOK AT THE EXPONENTIAL),

$$P^{V}_{1|1}(x_t,0|x_0,0) = \delta(x_t - x_0)$$

SO $P^{V}_{1|1}$ SATISFIES THE SAME BOUNDARY CONDITION AS $P^{0}_{1|1}$.

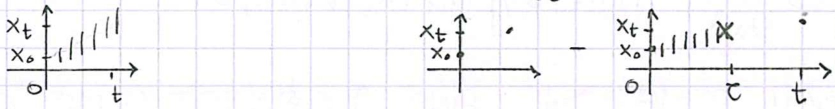
· <u>OBSERVATION</u>:

$$\frac{d}{dt} e^{-\int_0^t ds\, V(x(s),s)} = -V(x(t),t)\, e^{-\int_0^t ds\, V(x(s),s)}.$$

INTEGRATING BOTH SIDES FROM 0 TO t,

$$e^{-\int_0^t ds\, V(x(s),s)} - 1 = -\int_0^t d\tau\, V(x(\tau),\tau)\, e^{-\int_0^\tau ds\, V(x(s),s)}$$

$$e^{-\int_0^t ds\, V(x(s),s)} = 1 - \int_0^t d\tau\, V(x(\tau),\tau)\, e^{-\int_0^\tau ds\, V(x(s),s)}$$



WHERE IN THE LAST TERM THE "EVAPORATION" ACTS ONLY UP UNTIL TIME τ (LIKE IN DYSON'S EQUATION).

PUTTING EVERYTHING BACK INTO (II),

$$\rho^{V}_{11}(x_t,t\,|\,x_0,0) = \rho^{0}_{11}(x_t,t\,|\,x_0,0) - \int_0^t d\tau \int_{x(0)=x_0}^{x(t)=x_t} d_W x \; V(x(\tau),\tau)\, e^{-\int_0^\tau ds\, V(x(s),s)}$$

$$= \rho^{0}_{11}(x_t,t\,|\,x_0,0) - \int_0^t d\tau \int_{-\infty}^{+\infty} dx_\tau \int_{x(0)=x_0}^{x(\tau)=x_\tau} d_W x \int_{x(\tau)=x_\tau}^{x(t)=x_t} d_W x \; V(x_\tau,\tau)\, e^{-\int_0^\tau ds\, V(x(s),s)}$$

WHERE $V(x(\tau),\tau)$, WHICH WAS A FUNCTIONAL, HAS BECOME A NUMBER:

$$\rho^{V}_{11}(x_t,t\,|\,x_0,0) = \rho^{0}_{11}(x_t,t\,|\,x_0,0) - \int_0^t d\tau \int_{-\infty}^{+\infty} dx_\tau\, \rho^{0}_{11}(x_t,t\,|\,x_\tau,\tau)\, V(x_\tau,\tau)\, \rho^{V}_{11}(x_\tau,\tau\,|\,x_0,0)$$

NOW WE CAN CONCLUDE THE PROOF OF THE <u>FEYNMAN - KAC THEOREM</u>.
WE RECOGNIZE UP HERE THE FORM OF DYSON'S EQUATION; TAKING ITS TIME DERIVATIVE, SINCE BY CONSTRUCTION

$$\partial_t \rho^{0}_{11}(x_t,t\,|\,x_0,0) = D\,\partial^2_{x_t} \rho^{0}_{11}(x_t,t\,|\,x_0,0)$$

THEN

$$\partial_t \rho^{V}_{11}(x_t,t\,|\,x_0,0) = D\,\partial^2_{x_t} \rho^{0}_{11}(x_t,t\,|\,x_0,0) - \int_{-\infty}^{\infty} dx_\tau\, \underbrace{\rho^{0}_{11}(x_t,t\,|\,x_\tau,t)}_{\delta(x_t - x_\tau)}\, V(x_\tau,t)\, \rho^{V}_{11}(x_\tau,t\,|\,x_0,0)$$

$$- \int_0^t d\tau \int_{-\infty}^{+\infty} dx_\tau\, D\,\partial^2_{x_t} \rho^{0}_{11}(x_t,t\,|\,x_\tau,\tau)\, V(x_\tau,\tau)\, \rho^{V}_{11}(x_\tau,\tau\,|\,x_0,0)$$

$$= D\,\partial^2_{x_t} \rho^{V}_{11}(x_t,t\,|\,x_0,0) - V(x_t,t)\, \rho^{V}_{11}(x_t,t\,|\,x_0,0).$$

WE CONCLUDE THAT THE PATH INTEGRAL (II) SATISFIES BLOCH'S EQUATION (III).

- <u>DETOUR</u> : DIRECTED POLYMERS

DUE TO ELECTROSTATIC REASONS (THEY'RE ALSO CALLED POLYELECTROLITE), THEY TEND TO ALIGN IN STRAIGHT CONFIGURATIONS, MODULO FLUCTUATIONS.
THE COST OF EACH FLUCTUATION IS

$$\Delta E = \frac{1}{2}\gamma\,(x_{i+1} - x_i)^2$$

THE TYPICAL FLUCTUATION OF THE TOTAL CHAIN IS

$$\Delta t \propto \int_0^t ds \; \frac{1}{2}\left(\frac{dx}{ds}\right)^2$$

AND THE TOTAL ENERGY IS

$$E[\{x(s)\}] = \int_0^t ds \left[\frac{1}{2}\gamma\left(\frac{dx}{ds}\right)^2 + V(x(s),s)\right].$$

IN EQUILIBRIUM AT TEMPERATURE T,

$$P[\{x(\tau)\}] \propto e^{-\frac{E[\{x(s)\}]}{T}}$$

SO THAT THE PARTITION FUNCTION LOOKS LIKE

$$\int_{x(0)=x_0}^{x(t)=x_t} \mathcal{D}x(s) \exp\left\{-\frac{1}{T}\int_0^t ds\left[\frac{1}{2}\gamma\dot{x}^2(s) + V(x(s),s)\right]\right\} = P_{1|1}^{V}(x_t,t\,|\,x_0,0)$$

SO WE ALREADY KNOW IT SATISFIES A BLOCH EQUATION:

$$\partial_t Z(x_t,t) = \frac{T}{2}\partial_{x_t}^2 Z(x_t,t) - \frac{V(x_t,t)}{T} Z(x_t,t).$$

IT COMES HANDY TO EXPRESS (COLE-HOPF TRANSFORMATION)

$$Z(x,t) = e^{\frac{\lambda}{T}h(x,t)}$$

WHERE h IS MORALLY A FREE ENERGY. IT TURNS OUT

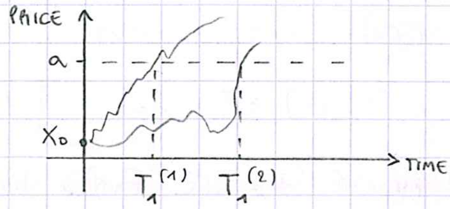
$$\partial_t h = \frac{T}{2}\partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 - \frac{V(x,t)}{\lambda}.$$

ASSUMING $V(x,t)$ TO BE AN ANNEALED RANDOM VARIABLE, THIS IS NOTHING THAN THE KARDAN-PARISI-ZHANG EQUATION. THIS IS A COMMONLY USED MAPPING.

APPLICATION: FIRST PASSAGE (CROSSING) TIME

07. 11. 19

- EXAMPLES



a) PRICE OF A SHARE

WHAT IS THE FIRST TIME AT WHICH THE PRICE EXCEEDS a?

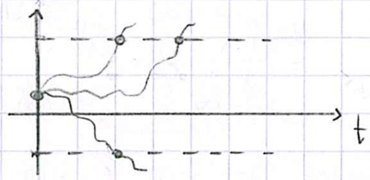
$T_1 = T_1(a)$ IS A RANDOM VARIABLE

WHAT IS THE DISTRIBUTION OF T_1?

(b) FLUCTUATIONS OF THE MAGNETIZATION OF THE ISING MODEL

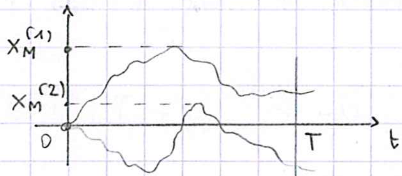
(c) EXTINCTION OF A POPULATION

- RELATED PROBLEMS



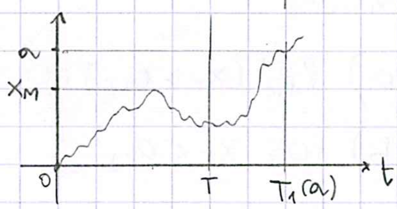
(i) ESCAPE FROM AN INTERVAL

(ii) DISTRIBUTION OF EXTREMA



WHAT IS THE DISTRIBUTION OF X_M?

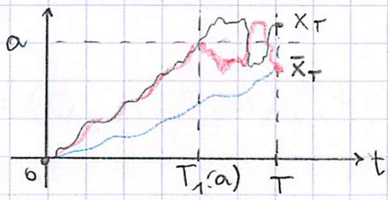
AND WHY ARE THEM ANALOGOUS PROBLEMS?



X_M IS A MAXIMUM IF ANY VALUE $a > X_M$ IS FIRST REACHED AT A TIME $T_1(a) > T$:

$$\mathrm{Prob}(X_M < a) = \mathrm{Prob}(T_1(a) > T).$$

- ELEMENTARY APPROACH (REFLECTION PRINCIPLE)



CLAIM: ONLY IF $T_1(a) < T$ DOES $P(X_T > a) = P(X_T < a)$.

INDEED, I COULD CONSTRUCT ANOTHER PROCESS (•) WHERE I INVERT THE SIGN OF THE STEPS AFTER $T_1(a)$. NOTICE THIS IS NOT TRUE IF $T_1(a) > T$: INDEED, I COULD CONSTRUCT A TRAJECTORY WHICH DIRECTLY GOES TO $\bar{X}_T$ AND HAS NO COUNTERPART IN THE REFLECTED PLANE ABOVE a (•). THEN

$$\mathrm{Prob}(X_T < a, T_1(a) < T) = \mathrm{Prob}(X_T > a, T_1(a) < T)$$

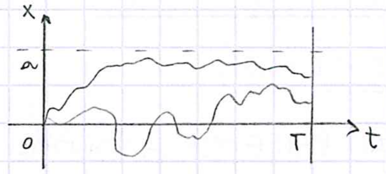
NOTE: IT'S THE TOTAL PROBABILITY TH

$$\begin{aligned}\mathrm{Prob}(T_1(a) < T) &= \mathrm{Prob}(T_1(a) < T, X_T > a) + \mathrm{Prob}(T_1(a) < T, X_T < a)\\ &= 2\,\mathrm{Prob}(T_1(a) < T, X_T > a)\\ &= 2\,\mathrm{Prob}(X_T > a)\end{aligned}$$

$(X_T > a \Rightarrow T_1(a) < T)$.

PATH INTEGRAL APPROACH

CONSIDERING THAT



$$T_1(a) > T \iff \{x(\tau) : x(\tau) < a, \ \forall t \in [0,T]\}$$

HOW DO WE FIND $\mathrm{Prob}(T_1(a) > T)$? WE INTRODUCE

$$\mathcal{X}[\{x(\tau)\}] = e^{-\int_0^T d\tau \, V_a(x(\tau))}$$

$$V_a(x) = \begin{cases} 0 & \text{FOR } x < a \\ +\infty & \text{FOR } x > a \end{cases}$$

THAT IS AN ABSORBING POTENTIAL FOR $x > a$. THEN

$$P = \mathrm{Prob}(T_1(a) > T) = \int_{x(0)=0}^{x_T \text{ ARB.}} d_w x \, \mathcal{X}[\{x(\tau)\}]$$

NOTE: RECALL $\int_{x(0)=0}^{x_T \text{ ARB.}} d_w x = 1$. IF THERE'S A SINGLE $t^* \in [0,T]$ S.T. $x(t^*) > a$ THEN $\mathcal{X} = 0$ FOR THAT WHOLE PATH $x(\tau)$.

$$\overset{*}{=} \int_{-\infty}^{\infty \ (a)} dx_T \int_{x(0)=0}^{x(T)=x_T} d_w x(\tau) \, e^{-\int_0^T d\tau \, V_a(x(\tau))} \equiv \int_{-\infty}^{a} P_{1|1}^{V_a}(x_T, T | 0, 0).$$

HOW DO WE FIND $P_{1|1}$? WE KNOW IT SOLVES BLOCH'S EQUATION

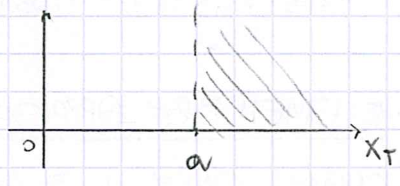
$$\partial_T P_{1|1}^{V_a}(x_T, T|0,0) = \left[D \partial_{x_T}^2 - V_a(x_T)\right] P_{1|1}^{V_a}(x_T, T|0,0)$$

WHERE

a) $P_{1|1}^{V_a}(x_T > a, T|0,0) = 0 \quad \overset{\text{CONTINUITY}}{\Rightarrow} \quad P_{1|1}^{V_a}(a, T|0,0) = 0.$

b) FOR $x_T < a$,

(*) NOTE: WE'RE APPLYING THE RULE FOR THE UNCONDITIONED WIENER MEASURE.



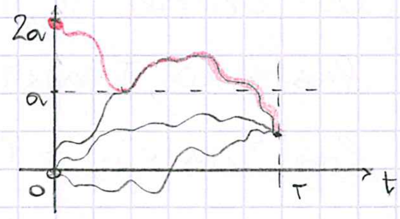
$$\partial_T P_{1|1}^{V_a}(x_T, T|0,0) = D \partial_{x_T}^2 P_{1|1}^{V_a}(x_T, T|0,0)$$

WHOSE SOLUTION IS

$$P_{1|1}^{0}(x_T, T|x_0, 0) = \frac{1}{\sqrt{4\pi D T}} e^{-\frac{(x_T - x_0)^2}{4DT}}$$

WHICH CLEARLY DOESN'T SATISFY THE BOUNDARY CONDITIONS, SO I USE THE <u>METHOD OF IMAGES</u>:

$$P_{1|1}^{V_a}(x_T, T|0,0) = P_{1|1}^{0}(x_T, T|0,0) - P_{1|1}^{0}(x_T, T|2a, 0).$$



MORALLY, IN ORDER TO SUPPRESS TRAJECTORIES ABOVE a, I PUT IN $2a$ A NEGATIVE SOURCE OF TRAJECTORIES (i.e. WITH NEGATIVE WEIGHTS).

NOTICE THIS WORKS BECAUSE $p^0_{111}(x_T, T \mid 2a, 0)$ IS A FUNCTION OF $(x_T - 2a)^2$ (REFLECTION SYMMETRY). BUT THEN

$$p = \int_{-\infty}^{a} dx_T \left[p^0_{111}(x_T, T \mid 0,0) - p^0_{111}(x_T, T \mid 2a, 0) \right]$$

$$= \int_{-\infty}^{a} dx_T \, p^0_{111}(x_T, T \mid 0,0) - \int_{-\infty}^{-a} dx_T \, p^0_{111}(x_T, T \mid 0,0)$$

$$= 1 - \int_{a}^{+\infty} dx_T \, p^0_{111}(x_T, T \mid 0,0) - \int_{a}^{+\infty} dx_T \, p^0_{111}(x_T, T \mid 0,0)$$

$$= 1 - 2\int_{a}^{+\infty} dx_T \, p^0_{111}(x_T, T \mid 0,0) = 1 - 2\,\mathrm{Prob}(x_T > a)$$

AND

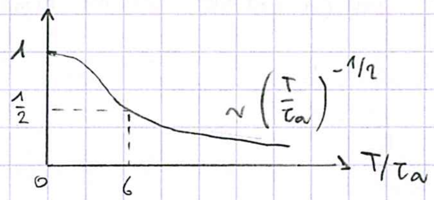
$$\mathrm{Prob}(T_1(a) < T) = 1 - p = 2\,\mathrm{Prob}(x_T > a)$$

AND WE RECOVERED THE RESULT OF THE ELEMENTARY APPROACH IN THE PRESENCE OF A PROPAGATOR WITH REFLECTION SYMMETRY.

* LET'S COMPLETE THE CALCULATION:

$$p = \mathrm{Prob}(T_1(a) > T) = \frac{2}{\sqrt{\pi}} \int_0^{a/\sqrt{4DT}} dx \, e^{-x^2}$$

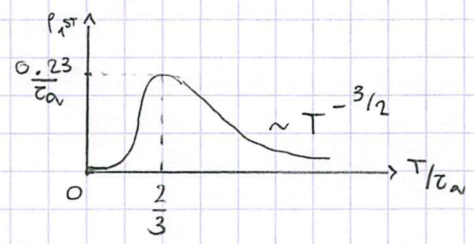
$$= \mathrm{Erf}\left[\left(\frac{T}{\tau_a} \right)^{-1/2} \right]$$



$$\tau_a \equiv \frac{a^2}{4D}$$

AND

$$p_{1st}(T) = -\frac{d}{dT} \mathrm{Prob}(T_1(a) > T) = \frac{1}{\sqrt{\pi}} \left(\frac{T}{\tau_a} \right)^{-\frac{3}{2}} \frac{1}{\tau_a} e^{-\tau_a/T}$$



WE NOTICE AN ESSENTIAL SINGULARITY IN $T/\tau_a = 0$: IT IS A REMNANT OF THE FACT THAT, IN A DISCRETE WALK, THERE IS A MINIMUM AMOUNT OF STEPS YOU MUST TAKE TO REACH a.

THE TAIL IS VERY SLOW, AND IN FACT

$$\langle T_1 \rangle = \int_0^{+\infty} dT \cdot T \cdot p_{1st}(T) = +\infty \qquad \forall a$$

NO MATTER HOW SMALL a IS.

· OBSERVATIONS

(i) ONCE YOU LEAVE A POINT, IT TAKES ON AVERAGE ∞ TIME TO VISIT IT AGAIN.

(ii) RECURRENCE OF RW IN $d=1,2$: EACH POINT IS VISITED ∞ - MANY TIMES.

IN THE DISCRETE, THE TOTAL NUMBER OF RETURNS (STARTING FROM ZERO) IS

$$N = \sum_{r=1}^{+\infty} \underset{\uparrow}{P_r} = \sum_{\substack{r=1 \\ (\text{EVEN } r)}}^{+\infty} \left(\frac{1}{2}\right)^r \binom{r}{r/2} = +\infty$$

PROBABILITY TO GO BACK TO ZERO AT THE r-TH STEP (i.e. PROB($\frac{r}{2}$ STEPS TO THE RIGHT)).

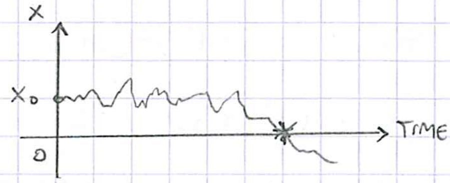
· EXERCISE: WHAT HAPPENS IN THE PRESENCE OF A BIAS $\left(p \neq \frac{1}{2}\right)$?

NOTE: ONE GETS, IF q IS THE BIAS,

$$N = \sum_{r=1} \binom{r}{r/2} q^{r/2} (1-q)^{r/2}$$

WHICH SEEMS TO ME EQUALLY INFINITE (RATHER COUNTERINTUITIVELY).

PERSISTENCE PROBABILITY



$$\mathcal{P}(t) \equiv \text{PERSISTENCE PROBABILITY}$$
$$= \text{PROB}\{x_0 \cdot x(\tau) > 0,\ \forall \tau \in [0,t]\}.$$

CLEARLY

$$\frac{d\mathcal{P}}{dt} = -\rho_{1ST}(t)$$

AT BARRIER $a = -x_0$

$$\mathcal{P}(t) = \text{PROB}\left(T_{1ST}(-x_0) > t\right).$$

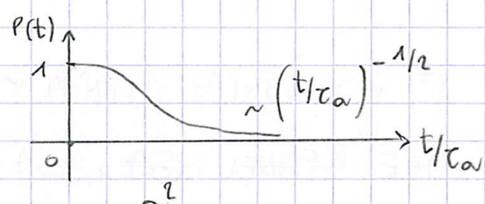
THERE ARE CASES IN WHICH

$$\mathcal{P}(t) \underset{t \gg \tau_a}{\sim} t^{-\theta}$$

WHERE θ IS A UNIVERSAL QUANTITY IN THE SENSE OF CRITICAL PHENOMENA, AND IT IS CALLED PERSISTENCE EXPONENT.

LAST TIME WE OBSERVED, FOR A WIENER PROCESS, THAT

$$\theta_{W.P.} = \frac{1}{2}$$



$$\tau_a = \frac{a^2}{4D}$$

WHICH REMINDS US OF GAUSSIAN THEORY,
WHICH IS INDEED WHAT WE OBSERVED IN THE PATH INTEGRAL FORMULATION OF THE WIENER PROCESS.

HOW CAN WE GO BEYOND GAUSSIANITY?

i) NON-GAUSSIAN TERMS. BUT WE HAVE ALREADY SEEN THAT THEY BECOME POTENTIALS AND VIOLATE PROBABILITY CONSERVATION

ii) STAY GAUSSIAN, BUT ADD NON-MARKOVIAN TERMS.

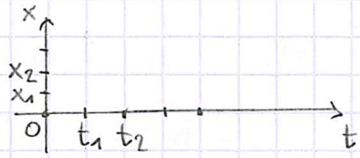
- PERSISTENCE EXPONENT FOR A GAUSSIAN, NON-MARKOVIAN PROCESS

REF: K. OERDING, S.J. CORNELL, A.J. BRAY PRE 56, R25 (1997)

WHAT IS A GAUSSIAN PROCESS? IT IS A PROCESS COMPLETELY

SPECIFIED BY $\langle x(t)x(s)\rangle$ (ALL THE REST CAN BE BUILT VIA WICK'S THEOREM). LET'S DEFINE x SO THAT

$\langle x(t)\rangle = 0 \quad \forall t.$



DISCRETIZING SPACE AND TIME, WE WOULD FIND

$$p(\{x_1,\dots,x_N\}) \propto \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N} x_i G_{ij} x_j\right\}$$

$G^{-1} \equiv$ CORRELATION MATRIX. G CAN BE TAKEN SYMMETRIC.

ON THE CONTINUUM,

$$p(\{x(\tau)\}) \propto \exp\left\{-\frac{1}{2}\int_0^t d\tau_1 \int_0^t d\tau_2 \, x(\tau_1) G(\tau_1,\tau_2) x(\tau_2)\right\}.$$

- OBSERVATION:

WIENER AND O-U ARE GAUSSIAN PROCESSES. FOR WIENER,

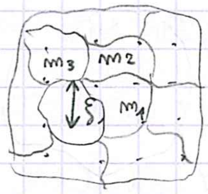
$G(\tau_1,\tau_2) \propto \delta''(\tau_1-\tau_2)$

NOTE: INTEGRATING BY PARTS, YOU GET $\dot{x}(\tau_1)^2$.

IT FOLLOWS SIMPLY THAT $p_{1|1}(x_2,t_2|x_1,t_1)$ IS A GAUSSIAN. THE KERNEL $G(\tau_1,\tau_2)$ CONTAINS INFORMATION ABOUT THE STATIONARITY OF THE PROCESS (BUT THIS HAS TO DO WITH THE BOUNDARY CONDITIONS AS WELL: THIS MAKES WIENER NON STATIONARY, AND O-U STATIONARY).

- EXAMPLE: GLOBAL MAGNETIZATION OF ISING MODEL

$$M = \sum_{i=1}^{V} S_i$$



IMAGINE YOUR SAMPLE TO BE DIVIDED IN PATCHES OF THE SAME SIZE OF THE CORRELATION LENGTH ξ:

$$M = \sum_{k=1}^{V/\xi^d} m_k \xrightarrow{V \gg \xi^d} \text{GAUSSIAN}.$$

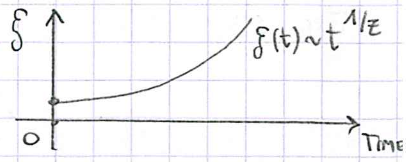
*NOTE: IN FACT, $\xi = \infty$ FROM THE START.

THIS IS IMPOSSIBLE TO ACHIEVE AT THE CRITICAL POINT*; HOWEVER, YOU CAN DO A QUENCH FROM AN INITIAL STATE WITH ξ OF THE

SIZE OF THE LATTICE SPACING, TO $T = T_c$.

THIS ENSURES M TO BE GAUSSIAN AT

ALL TIMES: YOU TAKE THE THERMODYNAMIC LIMIT FIRST.*



BEING AT A CRITICAL POINT, WE ALSO EXPECT SOME KIND OF SCALE INVARIANCE. ALL IN ALL,

Y $(\equiv M)$ IS GAUSSIAN & HAS A SCALING BEHAVIOR

IN THE SENSE THAT

*NOTE: IN PRACTICE, ξ WILL BE FINITE FOR ANY FINITE TIME t.

$$\langle Y(t_1) Y(t_2) \rangle \underset{t_2 > t_1}{=} t_1^{\alpha} \, \phi\left(\frac{t_1}{t_2}\right)$$

i.e. THERE IS NO INTRINSIC TIME SCALE.

NOTE: SOLVED IN THE EXERCIS

· <u>EXERCISE</u>: CHECK IF THIS IS TRUE FOR THE WIENER PROCESS.

NOTICE THE PROCESS IS IN GENERAL NON STATIONARY:

$$\langle Y(t_1) Y(t_1) \rangle = t_1^{\alpha} \underbrace{\phi(1)}_{\sim \text{ FINITE}}$$

THIS SHOULD BE TIME INDEPENDENT, BECAUSE IT IS A 1-TIME QUANTITY. WE MIGHT CONSIDER DIVIDING IT BY ITS VARIANCE:

DEFINE

$$X(t) \equiv \frac{Y(t)}{\sqrt{\langle Y^2(t) \rangle}}$$

$$\langle X(t_1) X(t_2) \rangle = \frac{t_1^{\alpha} \, \phi(t_1/t_2)}{t_1^{\alpha/2} \, t_2^{\alpha/2} \, \phi(1)} = \left(\frac{t_1}{t_2}\right)^{\alpha/2} \frac{\phi(t_1/t_2)}{\phi(1)} .$$

INTRODUCING THE LOGARITHMIC TIME, THIS BECOMES A FUNCTION OF $(T_2 - T_1)$:

$$t \equiv e^{T} \qquad \Rightarrow \qquad \langle X(t_1) X(t_2) \rangle = A(T_2 - T_1).$$

★ IMAGINE NOW THE PERSISTENCE PROBABILITY OF THE ORIGINAL PROCESS LOOKS LIKE

$$P_Y(t) \sim t^{-\theta}$$

$$\underset{\parallel\!\parallel}{} $$

$$P_X(t) \xrightarrow{\text{log TIME}} P_X(T) \sim e^{-\theta T}.$$

THE PROCESS IS GAUSSIAN, SO WE CAN INTRODUCE AN ACTION

$$S[\{x(\tau)\}] = \frac{1}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \; x(\tau_1)\, G(\tau_1,\tau_2)\, x(\tau_2)$$

AND

$$P(T) = \text{PERSISTENCE PROBABILITY} = \frac{\int_{x>0} \mathcal{D}x\, e^{-S}}{\int \mathcal{D}x\, e^{-S}}. \qquad \text{(I)}$$

NOTICE

$$\langle x(T_1)\, x(T_2)\rangle = A(T_2 - T_1) = G^{-1}$$

(PAYING ATTENTION TO THE BOUNDARY CONDITIONS).

LAST TIME WE SAW HOW TO COMPUTE (I) FOR A MARKOVIAN PROCESS. IMAGINE $x(T)$ IS NON-MARKOVIAN, BUT "CLOSE" TO A MARKOVIAN PROCESS $x_0(T)$ IN THE SENSE THAT

$$G = G_0 + \varepsilon g$$

WHERE ε MAY HAVE TO DO WITH THE USUAL $\varepsilon = d_c - d$ IN FIELD THEORY.

* BUT HOW DOES G_0 LOOK LIKE?

IF YOU TAKE A <u>STATIONARY</u>, <u>GAUSSIAN</u> AND <u>MARKOVIAN</u> PROCESS, IT CAN BE SHOWN THAT YOU GET THE O-U PROCESS. WE WILL SEE LATER IN THE COURSE THAT IT MUST SATISFY

$$\frac{d}{dT} x_0 = -\mu x_0 + \xi(T)$$

WHICH IS A LANGEVIN EQUATION IN LOG TIME; THEN

$$\langle x_0(T_1)\, x_0(T_2)\rangle = e^{-\mu|T_1 - T_2|} \equiv A_0(T_1 - T_2). \qquad \text{(II)}$$

THEN G_0 IS THE INVERSE OF THIS A_0.

NOW, WE CAN EXPAND THE ACTION AS

$$S = S_0 + \varepsilon S_1$$

WHERE

$$S_0 = \frac{1}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \; x(\tau_1) G_0(\tau_1,\tau_2)\, x(\tau_2)$$

$$S_1 = \frac{1}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \; x(\tau_1) g(\tau_1,\tau_2)\, x(\tau_2)$$

SO THAT

$$P(T) = \frac{\int_{x>0} \mathcal{D}x \; e^{-S_0 - \varepsilon S_1}}{\int \mathcal{D}x \; e^{-S_0 - \varepsilon S_1}}.$$

LET'S START FROM THE NUMERATOR:

$$\int_{x>0} \mathcal{D}x \; e^{-S_0 - \varepsilon S_1} = \int_{x>0} \mathcal{D}x \; e^{-S_0}\left[1 - \varepsilon S_1\right] + O(\varepsilon^2)$$

$$= \int_{x>0} \mathcal{D}x \; e^{-S_0} - \frac{\varepsilon}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \; g(\tau_1,\tau_2) \cdot \int_{x>0} \mathcal{D}x \; x(\tau_1)\, x(\tau_2)\, e^{-S_0} + O(\varepsilon^2)$$

$$= \int_{x>0} \mathcal{D}x \; e^{-S_0} \left\{ 1 - \underbrace{\frac{\varepsilon}{2}\int_0^T d\tau_1 \int_0^T d\tau_2 \; g(\tau_1,\tau_2) \underbrace{\frac{\int_{x>0}\mathcal{D}x \; e^{-S_0} x(\tau_1)\, x(\tau_2)}{\int_{x>0}\mathcal{D}x \; e^{-S_0}}}_{\equiv A_0^+(\tau_1,\tau_2)}}_{\equiv \gamma^+(T)} \right\} + O(\varepsilon^2)$$

HENCE

$$P(T) \simeq \frac{e^{-\frac{\varepsilon}{2}\gamma^+(T)} \int_{x>0} \mathcal{D}x \; e^{-S_0} + O(\varepsilon^2)}{e^{-\frac{\varepsilon}{2}\gamma(T)} \int \mathcal{D}x \; e^{-S_0} + O(\varepsilon^2)}.$$

BUT

$$P_0(T) = \frac{\int_{x>0} \mathcal{D}x \; e^{-S_0}}{\int \mathcal{D}x \; e^{-S_0}}$$

IS THE PERSISTENCE PROBABILITY OF THE MARKOVIAN PROCESS,

$$P_0(T) \sim e^{-\mu T}.$$

THEN

$$P(T) \simeq e^{-\mu T - \frac{\varepsilon}{2}\left[\gamma^+(T) - \gamma(T)\right]} + O(\varepsilon^2)$$

WHERE WE CAN READ

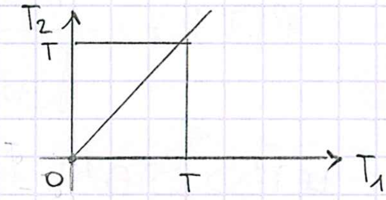
NOTE: $P(T) \sim e^{-\theta T}$ FOR LARGE T.

$$\theta = \mu + \frac{\varepsilon}{2} \lim_{T\to\infty} \frac{\gamma^+(T) - \gamma(T)}{T} + O(\varepsilon^2)$$

$$= \mu + \frac{\varepsilon}{2} \lim_{T\to\infty} \frac{1}{T} \int_0^T d\tau_1 \int_0^T d\tau_2 \; g(\tau_1,\tau_2) \left[A_0^+(\tau_1,\tau_2) - A_0(\tau_1,\tau_2) \right].$$

BUT HOW DO WE KNOW THAT THIS INTEGRAL IS LINEAR IN T?

SINCE $A_0(\tau_1,\tau_2) = A_0(\tau_1-\tau_2)$, WE KNOW IT IS SYMMETRICAL AROUND THE BISECTOR. THEN



$$\int_0^T d\tau_1 \int_0^T d\tau_2 \; f(\tau_1-\tau_2) \simeq T \cdot \int_{-\infty}^{+\infty} du \; f(u)$$

$u = \tau_1 - \tau_2$
$v = \tau_2 + \tau_1 - T$

NOTE: WELL, ACTUALLY $\tau_1 = \frac{1}{2}(u+v+T)$, $\tau_2 = \frac{1}{2}(v-u+T)$, $|J| = 1/2$ AND $I = \int_{-T}^{T} dv \int_{-T}^{T} du \cdot \frac{1}{2} f(u) = T \int_{-T}^{T} f(u)\, du$. THEN $T\to\infty$.

WE EXPECT THE OVERALL FUNCTION TO BECOME STATIONARY FOR LONG TIMES ANYWAYS (IT WILL HAVE FORGOTTEN THE INITIAL CONDITION). THEN

NOTE: I THINK THIS JUSTIFIES $g^{ST}(\tau)$.

$$\theta \simeq \mu + \frac{\varepsilon}{2} \int_{-\infty}^{+\infty} d\tau \; g^{ST}(\tau) \left[A_0^{+,ST}(\tau) - A_0^{ST}(\tau) \right]$$

$$= \mu + \frac{\varepsilon}{2} \int_{-\infty}^{+\infty} d\omega \; \tilde{g}^{ST}(\omega) \left[\tilde{A}_0^{+,ST}(\omega) - \tilde{A}_0^{ST}(\omega) \right]. \qquad \text{(III)}$$

AFTER SOME CALCULATIONS (WHICH YOU CAN DO BY EXERCISE),

$$A_0^+(T) = \int_0^{+\infty} dx_1 \int_0^{+\infty} dx_2 \; x_1 x_2 \; P^+(x_2, T; x_1, 0)$$

$$= \frac{2}{\pi} \left[3(1-e^{-2\mu T})^{1/2} + (e^{\mu T} + 2e^{-\mu T}) \arcsin e^{-\mu T} \right].$$

TYPICALLY, $\tilde{g}^{ST}(\omega)$ IS NOT EASILY ACCESSIBLE. WHAT IS ITS RELATION WITH THE 2-POINT FUNCTION? IN THE STATIONARY LIMIT,

$$G = G_0 + \varepsilon g \quad \rightsquigarrow \quad \tilde{G} = \tilde{G}_0 + \varepsilon \tilde{g}$$

$$\downarrow$$

$$A = A_0 + \varepsilon a \quad \overset{(*)}{\Longrightarrow} \quad \tilde{g}(\omega) = -\frac{\tilde{a}(\omega)}{\tilde{A}_0^2(\omega)}.$$

(*) NOTE: SEE NEXT PAGE.

FROM (II),

$$\tilde{A}_0(\omega) = \frac{2\mu}{\mu^2+\omega^2}.$$

NOTE: (II) IS
$A_0(\tau) = e^{-\mu|\tau|}$.
YOU CAN USE
$\theta(\tau)e^{-\mu\tau} \xrightarrow[FT]{} \frac{1}{\mu+i\omega}$
(ALL FT'S WITHOUT 2π).

THEN

$$\tilde{g}^{ST}(\omega) = -\left(\frac{\mu^2+\omega^2}{2\mu}\right)^2 \tilde{\alpha}^{ST}(\omega)$$

WHICH CAN BE PLUGGED INTO (III) TO OBTAIN

$$\theta = \mu - \varepsilon \frac{2\mu^2}{\pi}\int_0^\infty d\tau \frac{\alpha(\tau)}{\left(1-e^{-2\mu\tau}\right)^{3/2}} + O(\varepsilon^2).$$

⊛NOTE: INVERTING LOOKS LIKE TAKING THE RECIPROCAL IN FOURIER SPACE, SO THAT

$$G = A^{-1}$$

$$\tilde{G} = \frac{1}{\tilde{A}} = \frac{1}{\tilde{A}_0+\varepsilon\tilde{\alpha}} = \frac{1}{\tilde{A}_0\left(1+\varepsilon\frac{\tilde{\alpha}}{\tilde{A}_0}\right)} \simeq \frac{1}{\tilde{A}_0}\left(1-\varepsilon\frac{\tilde{\alpha}}{\tilde{A}_0}\right) = \frac{1}{\tilde{A}_0} - \varepsilon\frac{\tilde{\alpha}}{\tilde{A}_0^2} \equiv \tilde{G}_0 + \varepsilon\tilde{g}.$$

- LANGEVIN EQUATIONS 13.11.19

MACROSCOPIC WORLD: ~mm/m BALLS

MESOSCOPIC WORLD: ~μm COLLOIDS

MICROSCOPIC WORLD: ~0.5 mm ATOMS, MOLECULES

} INTRODUCTION OF RANDOMNESS

THERE IS A CLEAR TIME AND LENGTH SCALE SEPARATION. WE WRITE

$$\dot{x} \text{ or } \ddot{x} = f(x, \dot{x}) + (\text{RANDOM, NOISE}) . \qquad (\text{I})$$

CAN WE GO FURTHER UP, TO THE MACRO WORLD? WELL, EVERY TIME YOU COARSE GRAIN, THE NOISE GETS SMALLER.

EQUATION (I) IS A LANGEVIN-LIKE STOCHASTIC DIFFERENTIAL EQUATION.

- PARADIGM: BROWNIAN MOTION

COLLOID

$m, v(t)$

$$m\dot{v} = \overbrace{-\gamma v}^{\text{AVERAGE EFFECT OF MICROSCOPIC COLLISIONS (UNBALANCE BETWEEN SIDES)}} + \underbrace{f(t)}_{\text{RANDOM FORCE (WHAT IS NOT CAPTURED BY } \gamma v)}$$

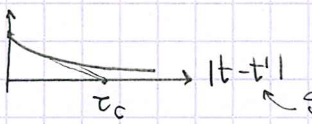
THIS IS A LINEAR LANGEVIN EQUATION.

AVERAGING OVER THE NOISE, SINCE THE EQUATION IS LINEAR,

$$m\langle \dot{v} \rangle = -\gamma \langle v \rangle + \langle f \rangle \quad \Rightarrow \quad \langle f \rangle = 0 .$$

BUT WHAT ABOUT THE SECOND MOMENT? WE EXPECT

$$\langle f(t) f(t') \rangle \equiv$$



$$\equiv C(t-t')$$

STATIONARITY

SINCE $f(t)$ COMES FROM THE SUM OF MANY VARIABLES, WE ASSUME IT TO BE A GAUSSIAN STOCHASTIC PROCESS:

$$P[\{f(t)\}] \propto \exp\left\{-\frac{1}{2}\int dt \int dt' \, f(t)\, C^{-1}(t-t')\, f(t')\right\}.$$

ASSUMING THE OBSERVATION TIME $\gg \tau_c$, WE GET

$$C(t) \simeq \lambda \delta(t) \quad \Rightarrow \quad f \text{ IS A WHITE NOISE.}$$

SETTING $m=1$,

$$\dot{v} = -\gamma v + f.$$

ITS HOMOGENEOUS SOLUTION IS GIVEN BY

$$v_H(t) = v_0 e^{-\gamma t}.$$

FOR THE NON-HOMOGENEOUS, DEFINE THE GREEN FUNCTION G s.t.

$$\dot{G} = -\gamma G + \delta(t).$$

*NOTE: IN THE SENSE THAT $\delta(t\sim 0) \gg \gamma G(t\sim 0)$.

IF G IS BOUNDED, CLOSE TO $t=0$ WE HAVE*

$$\dot{G} = \delta(t) \qquad \rightarrow \qquad G(t) = \theta(t)\, g(t), \quad g(0)=1$$

AND ACTUALLY

$$G(t) = \theta(t)\, e^{-\gamma t},$$

SO THAT

$$v_{NH}(t) = \int_0^{+\infty} dt'\, G(t-t')\, f(t') \qquad \left(\text{ACTUALLY } \int_0^t, \text{ BECAUSE OF } \theta(t)\right)$$

WHICH IS WHY G IS ALSO CALLED <u>LINEAR RESPONSE FUNCTION</u>.

THE FULL SOLUTION READS

$$v(t) = v_0 e^{-\gamma t} + \int_0^t dt'\, e^{-\gamma(t-t')} f(t').$$

NOTICE THAT, IF $f(t)$ IS GAUSSIAN, THEN v AS WELL HAS A GAUSSIAN DISTRIBUTION (IT IS A SUM OF GAUSSIAN VARIABLES), AND

$P_{1|1}(v_2, t_2 | v_1, t_1)$ IS GAUSSIAN.

WE JUST NEED TO COMPUTE ITS FIRST TWO MOMENTS:

$$\langle v(t) \rangle = v_0 e^{-\gamma t}$$

$$\langle v(t_2) v(t_1) \rangle = \underbrace{v_0^2 e^{-\gamma(t_1+t_2)}}_{=\langle v(t_1)\rangle\langle v(t_2)\rangle} + \int_0^{t_1} dt_1'\, e^{-\gamma(t_1-t_1')} \int_0^{t_2} dt_2'\, e^{-\gamma(t_2-t_2')} \underbrace{\langle f(t_1') f(t_2')\rangle}_{=\lambda\delta(t_1'-t_2')}$$

$$= v_0^2 e^{-\gamma(t_1+t_2)} + \lambda \int_0^{\min\{t_1,t_2\}} dt'\, e^{-\gamma(t_1+t_2-2t')}$$

$$= v_0^2 e^{-\gamma(t_1+t_2)} + \frac{\lambda}{2\gamma} e^{-\gamma(t_1+t_2)} \left(e^{2\gamma \min\{t_1,t_2\}} - 1 \right).$$

CALLING

$$t_< = \min\{t_1, t_2\}$$

$$t_1 + t_2 - 2t_< = (t_> + t_<) - 2t_< = t_> - t_< = |t_1 - t_2|$$

HENCE

$$\langle v(t_2) v(t_1) \rangle = \left(v_0^2 - \frac{\lambda}{2\gamma} \right) e^{-\gamma(t_1+t_2)} + \frac{\lambda}{2\gamma} e^{-\gamma|t_1 - t_2|} .$$

THE FIRST TERM IS NON-STATIONARY, BUT IT DIES OUT. IT'S A SIMPLE EXERCISE (WHICH YOU SHOULD DO) TO SHOW THAT $P_{1|1}(v_2, t_2 | v_1, t_1)$ BECOMES THAT OF THE ORNSTEIN-UHLENBECK PROCESS. NOTE: (A).

* IN THE STATIONARY STATE, i.e. $t_{1,2} \gg \gamma^{-1}$,

$$\langle v(t_2) v(t_1) \rangle = \frac{\lambda}{2\gamma} e^{-\gamma|t_2 - t_1|}$$

$$P(v) \propto \exp\left\{ -\frac{1}{2} \cdot \frac{v^2}{(\lambda/2\gamma)} \right\} .$$

BUT WE KNOW THAT, AT EQUILIBRIUM,

$$P(v) \propto \exp\left\{ -\frac{\text{KINETIC ENERGY}}{K_B T} \right\} = \exp\left\{ -\frac{1}{2} \cdot \frac{v^2}{K_B T} \right\} .$$

BY COMPARISON,

$$\underline{\lambda = 2\gamma K_B T} \qquad (\lambda = 2\gamma m K_B T \text{ IF } m \neq 1)$$

WHICH IS A FORM OF <u>EINSTEIN RELATION</u>, OR F-D-THEOREM (γ IS IN FACT THE FRICTION COEFFICIENT).

* SO FAR SO GOOD FOR OUR MODEL. BUT DOES IT HAVE ANY PREDICTIVE POWER? WE ALREADY KNEW EQUILIBRIUM. A RELEVANT QUANTITY WE CAN PREDICT IS

$$\Delta X(t) = X(t) - X(0) = \int_0^t dt' \, v(t') .$$

THEN

$$\langle \Delta x(t) \rangle = \int_0^t dt' \langle v(t') \rangle = \frac{v_0}{\gamma}\left(1 - e^{-\gamma t}\right)$$

NOTE: THIS IS ONLY ZERO (AND CORRESPONDS TO THE WIENER PROCESS) IN THE OVERDAMPED LIMIT, AND NOT IN GENERAL!

$$\langle (\Delta x(t))^2 \rangle = \int_0^t dt_1' \int_0^t dt_2' \langle v(t_1') v(t_2') \rangle$$

$$= \left(v_0^2 - \frac{\lambda}{2\gamma}\right)\left(\frac{1 - e^{-\gamma t}}{\gamma}\right)^2 + \frac{\lambda}{\gamma^2}\left(t - \frac{1 - e^{-\gamma t}}{\gamma}\right)$$

WHICH PRESENTS TWO REGIMES:

$\gamma t \ll 1$: $\langle \Delta x^2 \rangle \simeq v_0^2 t^2 + O(t^3)$ — BALLISTIC PROPAGATION

$\gamma t \gg 1$: $\langle \Delta x^2 \rangle \simeq \frac{\lambda}{\gamma^2} t \equiv 2Dt$ — DIFFUSION

THE SECOND IS KNOWN AS <u>FICK'S LAW</u>, WITH

$$D = \frac{\lambda}{2\gamma^2} \underset{\substack{\uparrow \\ \text{REINSTATE } m}}{=} \frac{k_B T}{m\gamma}.$$

- <u>EXERCISE</u>: CALCULATE

  $P_{1|1}(x_2, t_2 | x_1, t_1)$

  UNDER THE ASSUMPTION THAT $\left(\frac{\text{VELOCITY}}{\gamma} \ll \text{DISPLACEMENT}\right)$; THIS IS THE OVERDAMPED REGIME.

  NOTE: (B). I THINK IT'S REALLY IMPORTANT, BECAUSE IT CLARIFIES GAUSSIAN NOISE ↔ WIENER.

  (a) VERIFY THIS IS THE WIENER PROCESS.

  (b) START INSTEAD FROM THE LANGEVIN EQUATION AND ASSUME

  $|\dot{v}| \ll \gamma |v|$

  DERIVE THE EQUATION FOR Δx.

- <u>OBSERVATION</u>

  T. Franosh et al, Nature 478, 85 (2011)

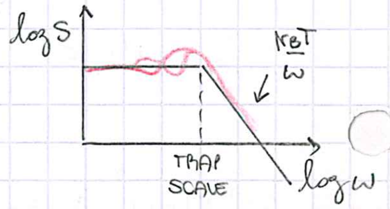
  IN REAL EXPERIMENTS (OPTICAL TWEEZERS AND FAST CAMERAS),

  $\langle f(t) f(t') \rangle = 2k_B T \gamma \delta(t - t')$ — IDEAL (UP TO KHz IT'S FINE)

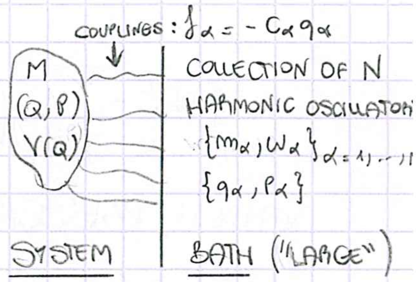
  $\langle f(t) f(t') \rangle \underset{t \neq t'}{=} -k_B T \gamma \frac{1}{\sqrt{4\pi}\, \tau_f}\left(\frac{t - t'}{\tau_f}\right)^{-3/2}$ — ACTUAL: $\tau_f \simeq 3\mu s$.

YOU START SEEING THIS EFFECT WITH MH2 CAMERAS.

$\langle x(t)\, x(0) \rangle \rightarrow S(\omega)$.



· DERIVATION OF THE GENERALIZED LANGEVIN EQUATION



CONSIDER THE SYSTEM ILLUSTRATED HERE.

THE TOTAL FORCE IS

$$\sum_{\alpha=1}^{N} f_\alpha = -\sum_{\alpha=1}^{N} C_\alpha q_\alpha$$

$$\Rightarrow U_{INT}(Q) = -Q \sum_{\alpha=1}^{N} C_\alpha q_\alpha .$$

WE CAN CONSTRUCT (FOR A CLASSICAL SYSTEM)

$$\begin{cases} H_{SYS} = \dfrac{P^2}{2M} + V(Q) \\ H_{BATH} = \displaystyle\sum_{\alpha=1}^{N} \left\{ \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 q_\alpha^2 \right\} \\ H_{S-B} = -Q \displaystyle\sum_{\alpha=1}^{N} C_\alpha q_\alpha \end{cases}$$

$$H = H_{SYS} + H_{BATH} + H_{S-B} .$$

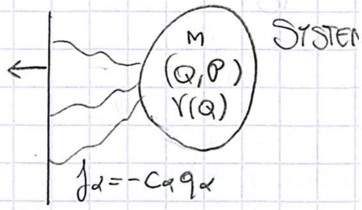
FOR QUANTUM SYSTEMS, SOMETHING SIMILAR CAN BE DONE: SEE

Caldera-Leggett for quantum dissipation, PRL 46, 211 (1981)

Feynman-Vernon, influence functionals.

GENERALIZED LANGEVIN EQUATION

18.11.1



LARGE "THERMAL" BATH, N HARMONIC OSCILLATORS

$\{p_\alpha, q_\alpha\}_{\alpha=1,\dots,N}$ $\qquad m_\alpha, w_\alpha$

HENCE

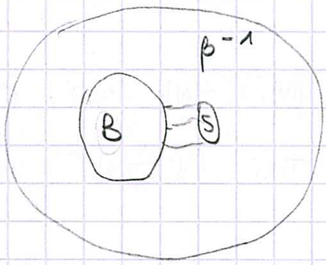
$$H_{SYS} = \frac{P^2}{2M} + V(Q)$$

$$H_{BATH} = \sum_{\alpha=1}^{N} \left\{ \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha w_\alpha^2 q_\alpha^2 \right\}$$

$$H_{S-B} = Q \sum_{\alpha=1}^{N} C_\alpha q_\alpha$$

AND

$$H = H_{SYS} + H_{BATH} + H_{S-B} .$$



* ASSUME (S+B) IS IN EQUILIBRIUM AT TEMPERATURE β^{-1} (AS IF IT WERE IMMERSED INTO A LARGER BATH). THEN

$$P_{eq}(P,Q) = \frac{1}{Z} \int \left(\prod_\alpha dp_\alpha dq_\alpha \right) e^{-\beta H} \propto e^{-\beta\left[\frac{P^2}{2M} + V(Q) - \sum_{\alpha=1}^{N} \frac{C_\alpha^2}{m_\alpha w_\alpha^2} \cdot \frac{Q^2}{2}\right]} .$$

IN FACT

$$\int dq_\alpha \, e^{-\beta\left[\frac{1}{2} m_\alpha w_\alpha^2 q_\alpha^2 + C_\alpha Q q_\alpha\right]} \propto e^{-\beta\left[-\frac{C_\alpha^2}{2 m_\alpha w_\alpha^2} Q^2\right]}$$

NOTE: HOW IS IT THAT I NEVE[R] THOUGHT OF THIS BEFORE?
$C_\alpha Q q_\alpha = 2 \cdot \left(\sqrt{\frac{m_\alpha}{2}} w_\alpha q_\alpha\right) \cdot X$
HENCE
$X = \frac{C_\alpha Q}{\sqrt{2 m_\alpha} w_\alpha}$.

(I COMPLETED THE SQUARE). DEFINING

$$\gamma(t) \equiv \sum_{\alpha=1}^{N} \frac{C_\alpha^2}{m_\alpha w_\alpha^2} \cos(w_\alpha t)$$

WE SEE IT'S AS IF THE PARTICLE (Q,P) WERE SUBJECT TO

$$V_{eq}(Q) = V(Q) - \gamma(0) \frac{Q^2}{2}$$

WHERE THE LAST IS A "COUNTER-TERM", AND IT'S NEGATIVE: IF $V(Q)$ IS NOT STRONG ENOUGH, THEN $V_{eq}(Q)$ MIGHT GET UNBOUNDED.

* ASSUME INSTEAD NON-EQUILIBRIUM. THEN THE PARTICLE EXPERIENCES

$$\begin{cases} \dot{Q} = \dfrac{P}{M} \\ \\ \dot{P} = -V'(Q) - \sum\limits_{\alpha=1}^{N} c_\alpha q_\alpha \end{cases} \qquad \text{(I)}$$

WHILE FOR THE BATH

$$\begin{cases} \dot{q}_\alpha = \dfrac{p_\alpha}{m_\alpha} \\ \\ \dot{p}_\alpha = -m_\alpha \omega_\alpha^2 q_\alpha - c_\alpha Q(t) \end{cases} \qquad \text{(II)}$$

IMAGINE WE KNOW $Q(t)$: THEN WE CAN SOLVE (II) AND PLUG THE RESULT INTO (I). WE WOULD GET

$$m_\alpha \ddot{q}_\alpha = -m_\alpha \omega_\alpha^2 q_\alpha - c_\alpha Q(t)$$

$$q_\alpha^{Hom}(t) = A\cos(\omega_\alpha t) + B\sin(\omega_\alpha t).$$

THE NON-HOMOGENEOUS PART CAN BE FOUND VIA GREEN EQUATION, OR VIA VARIATION OF CONSTANTS: THE LATTER GIVES

$$\ddot{q}_\alpha = -\omega_\alpha^2 q_\alpha - \frac{c_\alpha}{m_\alpha} Q(t)$$

$$q_\alpha^{NH}(t) = A(t)\cos(\omega_\alpha t) + B(t)\sin(\omega_\alpha t)$$

$$\dot{q}_\alpha^{NH}(t) = -\omega_\alpha A\sin(\omega_\alpha t) + \omega_\alpha B\cos(\omega_\alpha t) + \underbrace{\dot{A}\cos(\omega_\alpha t) + \dot{B}\sin(\omega_\alpha t)}_{\equiv 0}$$

$$\ddot{q}_\alpha^{NH}(t) = \underbrace{-\omega_\alpha^2 A\cos(\omega_\alpha t) - \omega_\alpha^2 B\sin(\omega_\alpha t)}_{= -\omega_\alpha^2 q_\alpha^{NH}(t)} \underbrace{- \omega_\alpha \dot{A}\sin(\omega_\alpha t) + \omega_\alpha \dot{B}\cos(\omega_\alpha t)}_{\equiv -\frac{c_\alpha}{m_\alpha} Q(t)}$$

THAT IS

$$\begin{cases} \dot{A}\cos(\omega_\alpha t) + \dot{B}\sin(\omega_\alpha t) = 0 \\ \\ -\dot{A}\sin(\omega_\alpha t) + \dot{B}\cos(\omega_\alpha t) = -\dfrac{c_\alpha}{m_\alpha \omega_\alpha} Q(t) \end{cases}$$

WHENCE

$$\begin{cases} \dot{B} = -\dfrac{c_\alpha}{m_\alpha w_\alpha} Q(t)\cos(w_\alpha t) \\ \dot{A} = \dfrac{c_\alpha}{m_\alpha w_\alpha} Q(t)\sin(w_\alpha t) \end{cases}$$

AND FINALLY

$$\begin{cases} B(t) = -\dfrac{c_\alpha}{m_\alpha w_\alpha} \displaystyle\int_0^t dt'\, Q(t')\cos(w_\alpha t') \\ A(t) = \dfrac{c_\alpha}{m_\alpha w_\alpha} \displaystyle\int_0^t dt'\, Q(t')\sin(w_\alpha t') . \end{cases}$$

THE COMPLETE SOLUTION OF (II) READS

$$q_\alpha(t) = A\cos(w_\alpha t) + B\sin(w_\alpha t) - \frac{c_\alpha}{m_\alpha w_\alpha}\int_0^t dt'\, Q(t') \underbrace{\sin[w_\alpha(t-t')]}_{\text{GREEN FUNCTION}} . \quad \text{(III)}$$

A, B ARE FIXED BY THE INITIAL CONDITIONS:

$$A \equiv q_\alpha(0) \qquad\qquad B \equiv \frac{p_\alpha(0)}{m_\alpha} .$$

NOW LOOK AT (III): WE ARE SEARCHING FOR A FRICTION TERM (i.e. $\propto \dot{Q}$), SO WE INTEGRATE BY PARTS

$$\int_0^t dt'\, Q(t')\sin(w_\alpha(t-t'))$$

$$= \frac{1}{w_\alpha}\int_0^t dt' \left\{ \frac{d}{dt'}\left[Q(t')\cos(w_\alpha(t-t'))\right] - \dot{Q}(t')\cos(w_\alpha(t-t')) \right\}$$

$$= \frac{1}{w_\alpha}\left\{ Q(t) - Q(0)\cos(w_\alpha t) - \int_0^t dt'\, \dot{Q}(t')\cos(w_\alpha(t-t')) \right\}$$

AND WE GET

$$q_\alpha(t) = q_\alpha(0)\cos(w_\alpha t) + \frac{p_\alpha(0)}{m_\alpha w_\alpha}\sin(w_\alpha t)$$

$$- \frac{c_\alpha}{m_\alpha w_\alpha^2}\left\{ Q(t) - Q(0)\cos(w_\alpha t) - \int_0^t dt'\, \dot{Q}(t')\cos(w_\alpha(t-t')) \right\} .$$

THIS CAN BE SUBSTITUTED INTO EQUATION (I) FOR THE PARTICLE:

$$\dot{P} = -V'(Q) - \sum_{\alpha=1}^{N} c_\alpha q_\alpha(t)$$

$$= \underbrace{-V'(Q) + \gamma(0)Q(t)}_{= -V'_{eq}(Q)} - \int_0^t dt' \, \dot{Q}(t')\gamma(t-t') + \eta(t).$$

NOTICE A DISSIPATION WITH MEMORY EMERGES, BECAUSE $\gamma(t-t')$ IS NONLOCAL IN TIME. WE ALSO CALLED $\eta(t)$ THE REMAINING PART, i.e.

$$\eta(t) \equiv -\sum_{\alpha=1}^{N} c_\alpha \left\{ q_\alpha(0)\cos(\omega_\alpha t) + \frac{p_\alpha(0)}{m_\alpha \omega_\alpha}\sin(\omega_\alpha t) + \frac{c_\alpha}{m_\alpha \omega_\alpha^2} Q(0)\cos(\omega_\alpha t) \right\}$$

$$= -\sum_{\alpha=1}^{N} c_\alpha \left\{ q_\alpha(0)\cos(\omega_\alpha t) + \frac{p_\alpha(0)}{m_\alpha \omega_\alpha}\sin(\omega_\alpha t) \right\} - \gamma(t)Q(0).$$

*NOW LET'S MAKE SOME ASSUMPTIONS:

a) THE BATH OF OSCILLATORS IS IN CANONICAL EQUILIBRIUM AT TEMPERATURE β^{-1} WITH THE PARTICLE FIXED IN ITS INITIAL POSITION $Q(0)$.

b) N IS LARGE ENOUGH THAT THE PARTICLE DOES NOT AFFECT SIGNIFICANTLY THE BATH (i.e. "THE BATH IS ALWAYS IN EQUILIBRIUM").

c) N IS LARGE ENOUGH TO PROVIDE A "GOOD SAMPLING" OF THE INITIAL DISTRIBUTION OF THE BATH.

THIS ALLOWS US TO CONCLUDE THAT, UNDER THESE ASSUMPTIONS, $\eta(t)$ LOOKS LIKE NOISE!
ITS "STATISTICS" IS DETERMINED BY THE INITIAL DISTRIBUTION.

AS WE SAID,

$$\rho\left[\{q_\alpha(0), p_\alpha(0)\}\right] \propto e^{-\beta\left[H_{BATH} + H_{S-B}|_{t=0}\right]} \propto e^{-\beta \sum_{\alpha=1}^{N} \left\{ \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2(0) - c_\alpha Q(0) q_\alpha(0) + \frac{p_\alpha^2(0)}{2m_\alpha} \right\}}$$

WHENCE

$$\langle q_\alpha(0)\rangle_0 = -\frac{c_\alpha}{m_\alpha \omega_\alpha^2} Q(0)$$

NOTE: INDEED,

$$Z = \int \prod_\alpha dq_\alpha \, e^{-\beta \sum_\alpha \left[\frac{1}{2} m_\alpha \omega_\alpha^2 q_\alpha^2(0) + c_\alpha Q(0) q_\alpha(0)\right]}$$

$$= \mathcal{N} e^{\beta \sum_\alpha \frac{c_\alpha^2 Q^2(0)}{2 m_\alpha \omega_\alpha^2}}$$

$$\langle q_1(0) \dots q_m(0)\rangle_0^{(c)} = \frac{\partial^m \log Z}{\partial(-\beta c_1 Q(0)) \dots \partial(-\beta c_m Q(0))}$$

AND WE MAY REWRITE

$$\eta(t) = -\sum_{\alpha=1}^N c_\alpha \left[\underbrace{\left(q_\alpha(0) - \langle q_\alpha(0)\rangle_0\right)}_{=\delta q_\alpha(0)} \cos(\omega_\alpha t) + \frac{p_\alpha(0)}{m_\alpha \omega_\alpha} \sin(\omega_\alpha t) \right]$$

WHERE IT APPEARS THAT

$$\langle \eta(t)\rangle_0 = 0.$$

MOREOVER, SINCE $q_\alpha(0)$, $p_\alpha(0)$ HAVE A GAUSSIAN DISTRIBUTION, THEN $\eta(t)$ IS A <u>GAUSSIAN PROCESS</u>.

LOOKING AT $P[\{q_\alpha(0), p_\alpha(0)\}]$ WE DEDUCE

$$\langle \delta q_\alpha(0) p_\beta(0)\rangle = 0$$

$$\langle p_\alpha(0) p_\beta(0)\rangle = \delta_{\alpha\beta} \frac{m_\alpha}{\beta}$$

$$\langle \delta q_\alpha(0) \delta q_\beta(0)\rangle = \delta_{\alpha\beta} \frac{1}{\beta m_\alpha \omega_\alpha^2}.$$

NOTE: BASED ON WHAT WE DID UP HERE,

$$P[\{q_\alpha\}] \propto e^{-\beta \sum_\alpha \frac{1}{2} m_\alpha \omega_\alpha^2 \left(q_\alpha(0) - \langle q_\alpha(0)\rangle_0\right)^2}.$$

THEN

$$\langle \eta(t)\eta(t')\rangle = \sum_{\alpha=1}^N \Big\{ c_\alpha^2 \frac{1}{\beta m_\alpha \omega_\alpha^2} \cos(\omega_\alpha t)\cos(\omega_\alpha t')$$

$$+ c_\alpha^2 \frac{m_\alpha}{\beta m_\alpha^2 \omega_\alpha^2} \sin(\omega_\alpha t)\sin(\omega_\alpha t') \Big\}$$

$$= \beta^{-1} \gamma(t-t')$$

WHICH IS THE <u>FLUCTUATION-DISSIPATION THEOREM</u> (γ WAS THE KERNEL OF DISSIPATION). IT'S ALSO A GENERALIZATION OF EINSTEIN'S RELATION (AND IT COMES FROM THE ASSUMPTION OF INITIAL EQUILIBRIUM).

* IN SUMMARY,

$$\underline{M\ddot{Q} = -V'_{eq}(Q) - \int_0^t dt'\, \gamma(t-t')\dot{Q}(t') + \eta(t)}$$

WHERE

$$\langle \eta(t) \rangle = 0 \qquad \langle \eta(t)\eta(t') \rangle = \beta^{-1}\gamma(t-t').$$

THIS IS A GENERALIZED LANGEVIN EQUATION WITH COLORED NOISE AND MEMORY IN THE FRICTION.

BUT WHAT IS γ?

$$\gamma(t) = \sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2} \cos(\omega_\alpha t)$$

$$= \frac{2}{\pi} \int_0^{+\infty} d\omega \frac{\cos(\omega t)}{\omega} \underbrace{\frac{\pi}{2} \sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2} \delta(\omega - \omega_\alpha)}_{\equiv J(\omega),\ \text{SPECTRAL DENSITY}}$$

WHERE GENERALLY

$$J(\omega) \xrightarrow[\omega \sim 0]{} \omega^S$$

AND FOR

$S = 1$: OHMIC $\rightsquigarrow \delta(t)$, MARKOVIAN

$S > 1$: SUPER OHMIC

$0 < S < 1$: SUB OHMIC.

STOCHASTIC INTEGRALS

20.11.19

CONSIDER

$$\dot{x} = a(x,t) + b(x,t)\eta(t) \qquad \langle \eta(t)\eta(t')\rangle = \delta(t-t')$$

THIS EQUATION CONTAINS A MULTIPLICATIVE NOISE (HERE WHITE).

MORE RIGOROUSLY,

$$dx = a(x,t)\,dt + b(x,t)\underbrace{\eta(t)\,dt}_{dW(t)}$$

$W(t)$ IS THE WIENER PROCESS

WHENCE

$$x(t) - x(0) = \int_0^t dt'\, a(x(t'),t') + \int_0^t dt'\, b(x(t'),t')\eta(t')$$

WHERE THE SECOND IS AN INTEGRAL OVER A STOCHASTIC PROCESS.

WHAT IS $W(t)$? BY ITS DEFINITION, SETTING $W(0)=0$,

$$W(t) - W(s) = \int_s^t dt'\, \eta(t')$$

$\underbrace{W(t>s) - W(s)}_{\text{INCREMENT}}$ IS INDEPENDENT OF $W(s)$

THUS $W(t)$ IS A MARKOVIAN PROCESS.

WE CAN THEREFORE COMPUTE ITS K-M EXPANSION:

$$\alpha_1(x) = \lim_{\tau\to 0} \frac{1}{\tau}\langle W(t+\tau) - W(t)\rangle = \lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau}\langle\eta(t')\rangle\, dt' = 0$$

$$\alpha_2(x) = \lim_{\tau\to 0}\frac{1}{\tau}\langle [W(t+\tau)-W(t)]^2\rangle$$

$$= \lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau} dt' \int_t^{t+\tau} dt'' \langle \eta(t')\eta(t'')\rangle = 1$$

$$\alpha_{k>2}(x) = 0$$

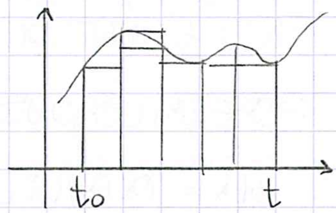
NOTE: EITHER THERE'S AN ODD NUMBER OF η'S, OR THEY DECOMPOSE BY WICK THM AND GIVE $O(\tau^2)$ OR HIGHER.

AS YOU CAN EASILY CHECK. WE CONCLUDE THAT $P(W,t)$ SATISFIES THE FOKKER-PLANCK EQUATION LEADING TO THE WIENER PROCESS.

STILL, WE ARE LEFT WITH

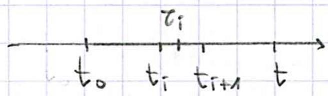
$$\int_0^t dW(t')\, b(x(t'),t') = ?$$

STOCHASTIC INTEGRATION



$$I = \int_{t_0}^t dW(t')\, G(t')$$

WE CONSTRUCT A PARTITION (AS FOR RIEMANN)



$$t_0 < t_1 < \dots < t_m < t .$$

USUALLY, YOU CAN CHOOSE ANY INTERMEDIATE POINT

$$t_i \leq \tau_i \leq t_{i+1}$$

AT WHICH TO EVALUATE THE FUNCTION,

$$f(\tau_i)(t_{i+1} - t_i)$$

WHENCE

$$S_m \equiv \sum_{i=1}^{m} G(\tau_i)\left[W(t_{i+1}) - W(t_i)\right] \qquad \Rightarrow \qquad I = \lim_{m\to\infty} S_m .$$

BUT THESE ARE STOCHASTIC PROCESSES: WHAT KIND OF LIMIT IS THIS? A GOOD CHOICE IS THE MEAN SQUARE LIMIT

$$I \equiv \text{ms-}\lim_{m\to\infty} S_m \qquad \Leftrightarrow \qquad \lim_{m\to\infty} \langle (S_m - I)^2 \rangle = 0$$

SO WE UPDATE THE RECIPE FOR "I" THIS WAY. BUT STILL WE HAVE A PROBLEM: THE RESULT DEPENDS ON THE CHOICE OF THE PARTITION $\{\tau_i\}$.

- EXAMPLE

$$G(t) \equiv W(t)$$

$$\int_{t_0}^t dW(t')\, W(t') \overset{?}{=} \frac{1}{2}\left[W^2(t) - W^2(t_0)\right]$$

LET'S CHECK IT.

WE HAVE

$$S_m = \sum_{i=1}^{m} W(\tau_i)\left[W(t_{i+1}) - W(t_i)\right] .$$

WE NOTICE

$$\langle W(t_1) W(t_2) \rangle = \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \underbrace{\langle \eta(t_1')\eta(t_2')\rangle}_{\delta(t_1'-t_2')} = \min\{t_1, t_2\}$$

WHENCE

$$\langle S_m \rangle = \sum_{i=1}^{m} \Big\{ \underbrace{\langle W(\tau_i) W(t_{i+1})\rangle}_{\tau_i} - \underbrace{\langle W(\tau_i) W(t_i)\rangle}_{t_i} \Big\} .$$

TAKING

$$\tau_i = t_i + \alpha(t_{i+1} - t_i) \qquad 0 \le \alpha \le 1$$

(A REGULAR CHOICE OF τ_i), WE GET

$$\langle S_m \rangle = \alpha(t - t_0) .$$

HOW DO WE SOLVE THIS PROBLEM? WE DECLARE OUR CHOICE!

· DEFINITIONS

ÎTO: $\tau_i = t_i$

AS A CONSEQUENCE (PROVE IT BY EXERCISE),

$$\int_{t_0}^{t} dW(t') W(t') = \frac{1}{2}\left[W^2(t) - W^2(t_0) - (t - t_0)\right]$$

STRATONOVICH: $\tau_i = \frac{1}{2}(t_{i+1} + t_i)$ AND $W(\tau_i) \to \frac{1}{2}\left(W(t_{i+1}) + W(t_i)\right)$

NOW YOU CAN PROVE (BY EXERCISE) THAT

$$S\int_{t_0}^{t} dW(t') W(t') = \frac{1}{2}\left[W^2(t) - W^2(t_0)\right]$$

i.e. THE STANDARD RULES OF CALCULUS APPLY.

* A SIMILAR REASONING APPLIES TO THE DISCRETIZATION OF

$$dx = a\,dt + b\,\eta\,dt$$

$$x(t_{i+1}) - x(t_i) = a(x(t_i), t_i)(t_{i+1} - t_i) + b(x(t_i), t_i)\eta(t_i)(t_{i+1} - t_i) \quad \text{(ITO!)}$$

· CONSEQUENCE (ÎTO)

GIVEN A FUNCTION f, IN GENERAL

$$S_m \equiv \sum_{i=1}^{m} \underbrace{(t_{i+1}-t_i)^2}_{\sim (t-t_0)/m} f(\tau_i) = 0$$

OR EQUIVALENTLY

$$\int_{t_0}^{t} (dt')^m f(t') = 0 \qquad \forall m > 1.$$

THIS IS NOT TRUE FOR A STOCHASTIC INTEGRAL. ASSUMING $G(t')$ TO BE A NON-ANTICIPATING FUNCTION (INDEPENDENT OF $W(s)$ IF $s > t'$), THEN FOR $m > 1$

$$\int_{t_0}^{t} [dW(t')]^m G(t') = \text{ms-}\lim_{n\to\infty} \sum_i G(t_i)\left[W(t_{i+1}) - W(t_i)\right]^m$$

$$= \begin{cases} 0 & m > 2 \\ \int_{t_0}^{t} dt'\, G(t') & m = 2 \end{cases} \qquad \Rightarrow \qquad \underline{\text{"}dW^2 = dt\text{"}}$$

WHICH IS SHORTHAND FOR $dW = O(dt^{1/2})$.

CONSIDER NOW $f(W(t),t)$. ITS DIFFERENTIAL IS

$$df(W(t),t) = \frac{\partial f}{\partial t} dt + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2} + \frac{\partial f}{\partial W} dW + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\underbrace{(dW)^2}_{\sim dt}$$

$$+ \frac{\partial^2 f}{\partial W \partial t}\underbrace{dW\,dt}_{\sim O(dt^{3/2})}$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}\right)dt + \frac{\partial f}{\partial W} dW.$$

LINEAR FUNCTIONS OF W BEHAVE AS USUAL, BUT A NEW TERM APPEARS IF f IS A NONLINEAR FUNCTION OF W.

BROWNIAN MOTION

$$\dot{v} = -\gamma v + \eta \qquad\qquad m = 1$$

LET $\gamma = 0$ (i.e. $\gamma \to +\infty$, $v \to \dot{x}$). WE WANT TO SOLVE

$$\dot{v} = \eta \,, \quad v(0) = 0 \qquad \Rightarrow \qquad v(t) = \int_0^t dt' \, \eta(t') \,.$$

WE MAY WANT TO EVALUATE, FOR $t_1 > 0$,

$$\langle \dot{v}(t_1) v(t_2) \rangle = \langle \eta(t_1) \int_0^t dt' \, \eta(t') \rangle$$

$$= \int_0^{t_2} dt' \, \delta(t' - t_1) = \theta(t_2 - t_1) \,.$$

BUT WHAT HAPPENS FOR $t_1 \to t_2$?

$$\langle \dot{v}(t_1) v(t_1) \rangle = \theta(0) = ? \,.$$

LET'S TRY TO GET AROUND IT:

$$\langle v^2(t) \rangle = \langle \int_0^t dt' \int_0^t dt'' \, \eta(t') \eta(t'') \rangle = t$$

AND

$$\langle \dot{v}(t) v(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle v^2(t) \rangle = \frac{1}{2} \,.$$

BUT WE ALSO KNOW $\dot{v}(t)$ IS PROBABLY ILL-DEFINED IN THE FIRST PLACE. LET'S TRY TO REGULARIZE IT THROUGH THE AVERAGE

$$\dot{v}(t) v(t) \longrightarrow \frac{1}{\tau} \int_t^{t+\tau} d\tau' \, \dot{v}(\tau') v(\tau')$$

$$\langle \dot{v}(t) v(t) \rangle = \lim_{\tau \to 0} \frac{1}{\tau} \langle \int_t^{t+\tau} dw(t') \, w(\tau') \rangle$$

WHICH CLEARLY DEPENDS ON THE WAY WE DEFINE THE INTEGRAL. USING ITO,

$$\int_{t_0}^t dw(t') \, w(t') = \frac{1}{2} \left(w^2(t) - w^2(t_0) - (t - t_0) \right) \,.$$

THE LAST TERM IS ABSENT WITH STRATONOVICH, WHENCE

$$\langle \dot{v}(t) v(t) \rangle = \lim_{\tau \to 0} \frac{1}{\tau} \begin{cases} \frac{1}{2}[(t+\tau)-t] \\ \cdot \\ \frac{1}{2}[(t+\tau)-t-\tau] \end{cases} = \begin{cases} \frac{1}{2} & \text{STRATONOVICH} \\ \\ 0 & \text{ÎTO} \end{cases}.$$

BUT $\theta(t)$ IS THE INVERSE OF $\frac{d}{dt}$ (AS THE UPPER TRIANGULAR MATRIX IS THE DERIVATIVE OF THE MATRIX WITH ALL $1, -1$). DEFINE

$$\dot{v}(t) = \lim_{\tau \to 0^+} \frac{1}{\tau} \left\{ v(t+\alpha\tau) - v(t-(1-\alpha)\tau) \right\}, \qquad 0 \le \alpha \le 1$$

WHICH REDUCES TO

$\alpha = 1$: FORWARD DERIVATIVE (ÎTO)

$\alpha = 0$: BACKWARD DERIVATIVE

$\alpha = \frac{1}{2}$: SYMMETRIC DERIVATIVE (STRATONOVICH)

$$\langle \dot{v}(t) v(t) \rangle = \lim_{\tau \to 0^+} \frac{1}{\tau} \left\{ \langle v(t+\alpha\tau) v(t) \rangle - \langle v(t-(1-\alpha)\tau) v(t) \rangle \right\}$$

$$= \lim_{\tau \to 0^+} \frac{1}{\tau} \left\{ t - (t-(1-\alpha)\tau) \right\} = (1-\alpha).$$

FROM LANGEVIN EQUATION TO FOKKER-PLANCK

22.11.19

CONSIDER THE ÎTO SDE

$$dx(t) = a(x(t),t)\,dt + b(x(t),t)\,dW(t) \rightsquigarrow x(t) . \qquad (I)$$

THE DIFFERENTIAL OF ANY FUNCTION $f(x(t))$ IS

$$df(x(t)) \equiv f(x(t)+dx(t)) - f(x(t))$$

$$f(x(t)+dx(t)) = f(x(t)) + f'(x(t))\,dx(t) + \frac{1}{2} f''(x(t))(dx(t))^2 + \text{h.o.}$$

$$df(x(t)) = f'(x(t))\left[a(x(t),t)\,dt + b(x(t),t)\,dW(t)\right] + \frac{1}{2} f''(x(t))\, b^2(x(t),t)\underbrace{dW^2}_{dt} + \text{h}$$

WE FOUND

$$df(x(t)) = \left[a(x(t),t) f'(x(t)) + \frac{1}{2} f''(x(t))\, b^2(x(t),t)\right] dt + f'(x(t))\, b(x(t),t)\, dW(t).$$

A PRIORI, THE SOLUTION OF A DIFFERENTIAL EQUATION LIKE (I) INVOLVING NON-ANTICIPATING FUNCTIONS (AND NO INTEGRAL DEPENDENC ON THE PAST) IS A MARKOVIAN PROCESS, CHARACTERIZED BY $P_{1|1}(x,t\,|\,x_0,0)$.

WE AVERAGE OVER

$$\langle df(x(t))\rangle = \langle a f' + \frac{1}{2} f'' b^2 \rangle dt + \underbrace{\langle f'(x(t))\, b(x(t),t)\, dW(t)\rangle}_{=0} .$$

IN FACT, WE ARE USING THE ÎTO PRESCRIPTION, SO THAT

$$\langle f'(x(t))\, b(x(t),t)\, dW(t)\rangle = \langle f'(x(t))\, b(x(t),t)\rangle \cdot \underbrace{\langle dW(t)\rangle}_{=0}$$

FOR NON-ANTICIPATING f AND b. WE CONCLUDE THAT

$$\frac{d}{dt}\langle f(x(t))\rangle = \langle a(x(t),t) f'(x(t)) + \frac{1}{2} b^2(x(t),t)\, f''(x(t))\rangle .$$

BUT

$$\langle a(x(t),t) f'(x(t))\rangle = \int dx\, P_{1|1}(x,t\,|\,x_0,0)\, a(x,t) f'(x)$$

$$\underset{\text{BY PARTS}}{\overset{\uparrow}{=}} - \int dx\, f(x)\, \partial_x \left[a(x,t)\, P_{1|1}(x,t\,|\,x_0,0)\right] .$$

SIMILARLY,

$$\langle b^2(x(t),t) f''(x(t)) \rangle = \int dx \, P_{1|1}(x,t|x_0,0)\, b^2(x,t) f''(x)$$

$$\overset{\text{BY PARTS}}{=} \int dx \, f(x)\, \partial_x^2 \left[b^2(x,t) P_{1|1}(x,t|x_0,0) \right]$$

AND

$$\langle f(x(t)) \rangle = \int dx \, f(x) P_{1|1}(x,t|x_0,0).$$

GIVEN THAT f IS ARBITRARY, THIS IMPLIES

$$\frac{d}{dt} P_{1|1}(x,t|x_0,0) = -\partial_x \left[a(x,t) P_{1|1}(x,t|x_0,0) \right] + \frac{1}{2} \partial_x^2 \left[b^2(x,t) P_{1|1}(x,t|x_0,0) \right]$$

WHICH IS THE <u>FORWARD FOKKER-PLANCK EQUATION</u> (WHICH IS RELATED TO ÎTO).

* BUT WHAT IF WE STARTED FROM STRATONOVICH?

$$dx(t) = \alpha(x(t),t)dt + \beta(x(t),t) dW.$$

THIS MEANS THAT THE FORMAL SOLUTION READS

$$x(t) = x(0) + \int_0^t dt' \, \alpha(x(t'),t') + \underbrace{S\int_0^t dW(t') \beta(x(t'),t')}_{\text{INTERPRETED À LA STRATONOVICH}}.$$

RECALL

$$S\int_0^t dW(t') \beta(x(t'),t') \underset{m.s.l.}{\simeq} \sum_i \beta\left(\frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right) \underbrace{\left[W(t_{i+1}) - W(t_i)\right]}_{\equiv \Delta W_i}$$

INSTEAD OF

$$\beta(x_i,t_i) \leftrightarrow \text{ÎTO}.$$

LET'S CALL

$$x_{i+1} = x_i + \Delta x_i \qquad t_{i+1} = t_i + \Delta t_i$$

$$\frac{x_{i+1}+x_i}{2} = x_i + \frac{\Delta x_i}{2} \qquad \frac{t_{i+1}+t_i}{2} = t_i + \frac{\Delta t_i}{2}$$

SO THAT

$$\beta\left(\frac{x_{i+1}+x_i}{2}, \frac{t_{i+1}+t_i}{2}\right) = \beta\left(x_i+\frac{1}{2}\Delta x_i, t_i+\frac{1}{2}\Delta t_i\right)$$

$$\simeq \beta(x_i,t_i) + \partial_x\beta(x_i,t_i)\frac{\Delta x_i}{2} + \partial_t\beta(x_i,t_i)\frac{\Delta t_i}{2} + h.o. \equiv (*)$$

WHERE THE LAST TERM CAN BE DISCARDED, SINCE IT MULTIPLIES

$\Delta W_i \sim (\Delta t_i)^{1/2}$

NOTE: WE INSERTED $b(x,t)$ TO MAKE CONTACT WITH THE ÎTO CASE

HENCE

$$(*) \simeq \beta(x_i,t_i) + \partial_x\beta(x_i,t_i)\frac{1}{2}b(x_i,t_i)\Delta W_i + h.o.$$

PUTTING EVERYTHING BACK TOGETHER,

$$S\int_0^t dW(t')\,\beta(x(t'),t') \simeq \sum_i\left[\beta(x_i,t_i)\Delta W_i + \partial_x\beta(x_i,t_i)\frac{1}{2}b(x_i,t_i)(\Delta W_i)^2\right]$$

$$= \int_0^t dW(t')\,\beta(x(t'),t') + \frac{1}{2}\int_0^t dt'\, b(x(t'),t')\,\partial_x\beta(x(t'),t')$$

WHERE WE SEE WE HAVE GENERATED A DRIFT TERM.

NOTICE THIS ONLY APPEARS IF WE HAVE A MULTIPLICATIVE NOISE: IF

$b(x(t),t) = \text{CONST.}$

THIS DOESN'T OCCUR.

★ THE TWO EQUATIONS

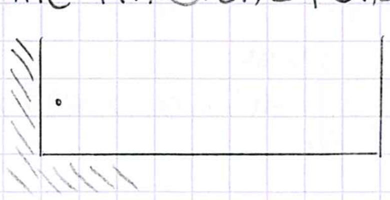
$$dx(t) = a(x(t),t)dt + b(x(t),t)dW(t) \qquad \text{(ÎTO)}$$

$$dx(t) = \left[a(x(t),t) - \frac{1}{2}b(x(t),t)\partial_x b(x(t),t)\right]dt + b(x(t),t)dW(t) \qquad \text{(STRAT.)}$$

ARE EQUIVALENT.

BUT WHICH OF THE TWO TERMS CORRESPOND TO THE PHYSICAL FORCE?

IMAGINE A COLLOID CLOSE TO A WALL. IN THE OVERDAMPED LIMIT,



$$\gamma(z)\,\dot{z} = F(z) + \sqrt{2k_BT\,D_\perp(z)\,\gamma^2(z)}\;\eta(t).$$

BUT THE FRICTION $F(z)$ IS DIFFERENT, ADOPTING ONE OF THE TWO INTERPRETATIONS...

REF : PRL 104, 170602 (2010)

PRL 107, 078901 (2011)

BUT THIS IS LARGELY NONSENSE. WHAT YOU WRITE IN THE LANGEVIN EQUATION IS NOT NECESSARILY THE "REAL" PHYSICAL FORCE; IT CAN DEPEND ON THE CHOICE OF INTERPRETATION. WHAT REALLY MATTERS IS THAT THEY LEAD TO THE SAME FOKKER-PLANCK EQUATION: THAT'S THE REAL DRIFT.

· EXERCISE: DERIVE THE FP EQUATION FROM STRATONOVICH AND SEE IF YOU FIND AGREEMENT.

NOTE: DONE IN THE FINAL EXERCISES.

a, b ÎTO $\rightarrow$ F-P (ÎTO) a, b

a, b STRAT. $\rightarrow$ F-P (STRAT.) $a + \frac{1}{2} b \partial_x b, b$

i.e. YOU HAVE TO CHOOSE

$a \rightarrow a - \frac{1}{2} b \partial_x b$

IN THE LANGEVIN EQUATION.

* AN ALTERNATIVE DERIVATION IS THE FOLLOWING. STARTING FROM THE LANGEVIN EQUATION,

$$\dot{x} = a(x) + b\,\eta \qquad \langle \eta(t)\eta(t') \rangle = \delta(t-t')$$

$$\Delta x(t) = \int_t^{t+\Delta t} dt'\, a(x(t')) + b \int_t^{t+\Delta t} dt'\, \eta(t')$$

AND THEN YOU COMPUTE THE COEFFICIENTS IN K-M EXPANSION:

$$\alpha_1(x) = \lim_{\Delta t \to 0} \frac{\langle \Delta x(t) \rangle_{x(t)}}{\Delta t} = a(x)$$

$$\int_t^{t+\Delta t} dt' \langle a(x(t')) \rangle \simeq \Delta t\, a(x(t))$$

WHILE

$$\alpha_2(x) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle (\Delta x(t))^2 \rangle_{x(t)}$$

$$\langle (\Delta x(t))^2 \rangle \underset{\substack{\uparrow \\ \text{UP TO } O(\Delta t)}}{=} 2 \langle \int_t^{t+\Delta t} dt' \, a(x(t'))\, b \int_t^{t+\Delta t} dt'' \, \eta(t'') \rangle + \int_t^{t+\Delta t} dt' \int_t^{t+\Delta t} dt'' \langle \eta(t') \eta(t'') \rangle b^2$$

$$= \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 \, 2b \, \langle a(x(t_1)) \eta(t_2) \rangle + b^2 \Delta t \, .$$

BUT

$$a(x(t_1)) = a(x(t)) + a'(x(t)) [x(t_1) - x(t)] + h.o.$$

THUS

$$\langle a(x(t_1)) \eta(t_2) \rangle = a(x(t)) \underbrace{\langle \eta(t_2) \rangle}_{= 0} + a'(x(t)) \langle [x(t_1) - x(t)] \eta(t_2) \rangle + h.o.$$

HENCE, BY SUBSTITUTING ITERATIVELY, WE CONCLUDE THAT

$$\alpha_2(x) = b^2$$

$$\alpha_{k>2}(x) = 0 \, .$$

- PURE RELAXATION

$$\dot{y} = -\gamma H'(y) + \eta \qquad \langle \eta \eta \rangle = \Gamma \delta(\dots)$$

WHERE $H(y)$ IS SOME ENERGY FUNCTION.

THIS CORRESPONDS TO THE F-P EQUATION

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y} \left[-\gamma H'(y) p \right] + \frac{\Gamma}{2} \frac{\partial^2 p}{\partial y^2} \, .$$

FIRST OF ALL, THE STATIONARY p_{ST} IS s.t.

$$-\gamma H'(y) p_{ST} + \frac{\Gamma}{2} \frac{\partial}{\partial y} p_{ST} = \text{CONST. IN } y \overset{y \to \pm\infty}{=} 0$$

HENCE

$$p_{\substack{ST \\ (eq)}}(y) \propto e^{-\frac{2\gamma}{\Gamma} H(y)} \qquad \Rightarrow \qquad \Gamma = 2 k_B T \gamma$$

TO MATCH WITH EQUILIBRIUM (FDT, OR EINSTEIN'S RELATION).

FLUCTUATION-DISSIPATION THEOREM

25.11.19

LAST TIME WE INTRODUCED

$$\dot{Y} = -\gamma H'(Y) + \xi \qquad\qquad \langle \xi(t)\xi(t')\rangle = \Gamma\,\delta(t-t')$$

$$P_{\substack{ST\\eq}}(Y) \propto e^{-\frac{2\gamma}{\Gamma}H(Y)} \quad \overset{\text{BOLTZMANN}}{\Longleftrightarrow} \quad \Gamma = 2k_BT\gamma$$

WHERE THE LAST IS CALLED FLUCTUATION-DISSIPATION RELATION.

· FDT

LINEAR RESPONSE FUNCTION (TO AN EXTERNAL PERTURBATION) $\overset{\text{AT EQUILIBRIUM}}{\longleftrightarrow}$ CORRELATION OF FLUCTUATIONS OCCURRING IN THE UNPERTURBED STATE.

LET H CONTAIN AN EXTERNAL FORCE f COUPLED LINEARLY,

$$H(Y;f) = H(Y) - fY.$$

DEFINE THE RESPONSE FUNCTION

$$R(t,s) \equiv \left.\frac{\delta\langle Y(t)\rangle_f}{\delta f(s)}\right|_{f=0}.$$

CAUSALITY IMPLIES

$$R(t,s) \propto \theta(t-s).$$

· OBSERVATION

$$\langle Y(t)\xi(s)\rangle = \int \mathcal{D}\xi\; Y_\xi(t)\,\xi(s)\, e^{-\frac{1}{2}\int_0^{+\infty}dt'\,\frac{\xi^2(t')}{\Gamma}}$$

$$= \int \mathcal{D}\xi\; Y_\xi(t)\left(-\Gamma\frac{\delta}{\delta\xi(s)}\right)e^{-\frac{1}{2}\int_0^{+\infty}dt'\,\frac{\xi^2(t')}{\Gamma}}$$

$$\overset{\text{BY PARTS}}{=} \Gamma\left\langle \frac{\delta Y_\xi(t)}{\delta\xi(s)}\right\rangle$$

WHERE Y_ξ SATISFIES

$$\dot{Y}(t) = -\gamma H'(Y) + \gamma f(t) + \xi(t).$$

NOTE: ONCE AND FOR ALL, WE PROVED THAT

$\dot{x} = \sqrt{2D}\,\eta \qquad \langle\eta(t)\eta(t')\rangle = \delta(t-t')$

$dx = \sqrt{2D}\,dw$

GIVES THE F-P EQUATION OF A WIENER PROCESS

$\partial_t P = D\,\partial^2_{xx}P \;\rightarrow\; \exp\left\{-\frac{x^2}{2\cdot 2Dt}\right\}$

WHOSE WIENER MEASURE IS WEIGHTED BY

$\exp\left\{-\frac{1}{2}\int_0^t d\tau\,\frac{\dot{x}(\tau)^2}{2D}\right\}.$

MORALLY, HERE $\xi(t)$ COMES FROM

$d\xi = \sqrt{\Gamma}\,dw.$

BUT NOW NOTICE A CHANGE

$$\xi(s) \to \xi(s) + \delta\xi(s) \qquad \text{FIXED } f$$

IS EQUIVALENT TO

$$f(s) \to f(s) + \underbrace{\delta f(s)}_{\equiv \frac{1}{\gamma}\delta\xi(s)} \qquad \text{FIXED } \xi$$

SO THAT

$$\frac{\delta y(t)}{\delta\xi(s)} = \frac{1}{\gamma}\frac{\delta y(t)}{\delta f(s)}$$

BECAUSE OF LINEARITY IN THE COUPLING. THEN

$$\left\langle \frac{\delta y(t)}{\delta\xi(s)} \right\rangle = \frac{1}{\gamma}\left\langle \frac{\delta y(t)}{\delta f(s)} \right\rangle = \frac{1}{\gamma}\frac{\delta}{\delta f(s)}\langle y(t)\rangle$$

AND WE FOUND

$$\langle y(t)\xi(s)\rangle = \frac{\Gamma}{\gamma}\frac{\delta}{\delta f(s)}\langle y(t)\rangle$$

WHICH GIVES THE ALTERNATIVE PRESCRIPTION

$$R(t,s) = \frac{\gamma}{\Gamma}\langle y(t)\xi(s)\rangle_{f=0}$$

WHERE WE ONLY USED THE FACT THAT THE NOISE ξ IS GAUSSIAN

* LET'S START FROM

$$\dot{y}(s) = -\gamma H'(y) + \xi(s).$$

MULTIPLYING BY $y(t)$ AND AVERAGING,

$$\underbrace{\langle y(t)\dot{y}(s)\rangle}_{\partial_s\langle y(t)y(s)\rangle = \partial_s C(t,s)} = -\gamma\langle y(t)H'(y(s))\rangle + \underbrace{\langle y(t)\xi(s)\rangle}_{=\frac{\Gamma}{\gamma}R(t,s)}$$

THAT IS

$$\partial_s C(t,s) = -\gamma\langle y(t)H'(y(s))\rangle + \frac{\Gamma}{\gamma}R(t,s) \qquad \text{(I)}$$

OR EQUIVALENTLY

$$\partial_t C(t,s) = -\gamma\langle y(s)H'(y(t))\rangle + \frac{\Gamma}{\gamma}R(s,t). \qquad \text{(II)}$$

BUT IF $t > s$, BY CAUSALITY

$R(s,t) = 0$.

IN GENERAL, IF THERE IS TIME REVERSAL (ABOUT $\frac{t+s}{2}$),

$$A(t)B(s) \underset{TR}{\leftrightarrow} A(s)B(t).$$

NOTE: $R(t,s) = R(t-s)$ IS NOT ALWAYS SATISFIED. IT IS IF THE LANGEVIN EQUATION IS TRANSLATIONALLY INVARIANT, OR IN THE STATIONARY LIMIT.

AT EQUILIBRIUM,

i) STATIONARITY $\Rightarrow$ $C(t,s) = C(t-s)$, $R(t,s) = R(t-s)$

ii) INVARIANCE UNDER T.R. $\Rightarrow$ $\langle A(t)B(s)\rangle = \langle A(s)B(t)\rangle$

HENCE, CALLING $\tau = t-s > 0$ AND TAKING THE DIFFERENCE (I) - (II),

$$-2\,\partial_\tau C^{eq}(\tau) = \frac{\Gamma}{\gamma} R^{eq}(\tau) \quad \Rightarrow \quad \underline{-\partial_\tau C^{eq}(\tau) = k_B T R^{eq}(\tau)}$$

WHICH IS THE FLUCTUATION - DISSIPATION THEOREM.

* WE SEE THAT THE DETAILS OF $H(\psi)$ HAVE DISAPPEARED: WE JUST HAD TO ASSUME TIME TRANSLATIONAL INVARIANCE AND TIME REVERSAL.

ONE COULD CONSIDER, INSTEAD OF THE SIMPLE

$$-\partial_\tau C^{eq}_{\psi\psi}(\tau) = k_B T R^{eq}_{\psi f}(\tau),$$

MORE COMPLICATED QUANTITIES LIKE

$$C^{eq}_{AB} \leftrightarrow R^{eq}_{A, f_B}.$$

IN Risken (P. 166), THE SAME RESULT IS DERIVED FROM THE F-P EQUATION.

NOTE THE FDT GIVES AN OPERATIVE WAY TO MEASURE THE TEMPERATURE T BY MEASURING CORRELATIONS AND RESPONSES. THIS IS ALSO A WAY TO CHECK IF WE ARE AT EQUILIBRIUM.

- <u>EXERCISE</u>

$\dot{v} = -\gamma v + \xi$ $\qquad v(0) = 0$

FROM HERE YOU CAN CALCULATE $C(t,s)$, $R(t,s)$. CHECK WHEN THE FDT IS SATISFIED ($\rightarrow t, s \gg \gamma^{-1}$).

BUT WHAT HAPPENS IF $\gamma = 0$? WHAT HAPPENS TO THE EFFECTIVE TEMPERATURE, i.e. TO THE RATIO OF $\partial_\tau C$ AND R?

· RELAXATION TO A STATIONARY STATE

$$\dot{y} = A(y) + \xi \qquad \langle \xi\xi \rangle = \Gamma \delta(\dots)$$

NOTE: THE SPIRIT OF THIS SECTION IS THE FOLLOWING: WITH MASTER EQUATIONS, GIVEN A TARGET P_{ST} WE HAD INFINITE WAYS TO CHOOSE THE TRANSITION RATES. DO WE HAVE SOMETHING SIMILAR WITH FOKKER-PLANCK AND LANGEVIN EQUATIONS?

WHENCE

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial y}\left[AP\right] + \frac{\Gamma}{2}\frac{\partial^2}{\partial y^2}P$$

$$= -\frac{\partial}{\partial y}\left[AP - \frac{\Gamma}{2}\frac{\partial P}{\partial y}\right].$$

WE MAY REQUIRE THAT THE TARGET DISTRIBUTION IS

$$P_{eq}(y) \propto e^{-E(y)}.$$

PLUGGING THIS ANSATZ IN, WE GET

$$\left(\frac{\partial}{\partial y} - \frac{\partial E}{\partial y}\right)\left(A - \frac{\Gamma}{2}\frac{\partial E}{\partial y}\right) = 0$$

SO THAT ONE POSSIBILITY IS ALWAYS

$$A = \frac{\Gamma}{2}\cdot\frac{\partial E}{\partial y} \qquad \text{(PURE RELAXATION)}.$$

HOWEVER, THE OTHER ONE IS

$$A = \underbrace{\frac{\Gamma}{2}\frac{\partial E}{\partial y}}_{\text{RELAXATIONAL FORCES}} + \underbrace{R(y)e^{E(y)}}_{\text{"REVERSIBLE" FORCES}} \qquad \frac{\partial R}{\partial y} = 0 \qquad (\underline{\nabla}\cdot\underline{R} = 0).$$

· SMALL DETOUR

THE F-P EQUATION CAN BE SEEN AS

$$\partial_t P = -HP \qquad H \propto -\frac{\partial}{\partial y}$$

WHERE P_{ST} IS THE RIGHT EIGENVECTOR ($\lambda = 0$). BUT H NEEDS NOT, IN GENERAL, BE HERMITIAN: THERE IS A LEFT EIGENVECTOR WITH $\lambda = 0$ WHICH IS CONSTANT.

THEN

$$\langle 0|H = 0 \qquad\qquad H|0\rangle = 0$$

$$\langle 0|\Psi\rangle = 1 \quad \Rightarrow \quad P_0(\Psi) = \langle \Psi|0\rangle$$

$$\underbrace{\int d\Psi\, P_0(\Psi)}_{\langle 0|0\rangle} < \infty .$$

· DISCRETE STOCHASTIC INTERACTING PARTICLE SYSTEMS

(FROM M.E. TO FIELD THEORY, Doi & Peliti, > 1976)

THESE INCLUDE REACTION - DIFFUSION SYSTEMS (LIKE CHEMICAL REACTIONS, BIOLOGICAL SYSTEMS, POPULATION DYNAMICS...) AND DIRECTED PERCOLATION, TO NAME A FEW.
THEY ARE ALL CHARACTERIZED BY A CONFIGURATION SPACE WHICH IS SPECIFIED BY A STRING OF INTEGERS:

$\{m_1, m_2, \dots\}$ $\qquad$ $m_i \in \mathbb{N}$, # OF INDIVIDUALS (PARTICLES)

PARTICLES CAN
- REACT
- DIFFUSE

AND BOTH ARE A SOURCE OF STOCHASTICITY.
WE ASSUME, FOR SIMPLICITY, THAT DIFFUSION OCCURS WITH TIME - INDEPENDENT RATES. THE STOCHASTIC DYNAMICS WILL LOOK LIKE

$$\frac{\partial}{\partial t} P(\{m\},t) = \sum_{\{m'\}}{}' \Big[W(\{m'\}\to\{m\}) P(\{m'\},t) - W(\{m\}\to\{m'\}) P(\{m\},t\Big]$$

$$= -\sum_{\{m'\}}{}' \mathcal{L}_{\{m\},\{m'\}} P(\{m'\},t) .$$

NOTE: "Eeeh, Lindblad!".

THIS REMINDS US OF QUANTUM MECHANICS, CONSIDER

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi$$

$\{\Psi_1, \Psi_2 \dots\}$ BASIS OF H.S.

THEN WE CAN EXPRESS

$$\Psi(t) = \sum_m c_m(t)\, \Psi_m$$

SO THAT THE SCHRÖDINGER EQUATION CAN BE PROJECTED OVER

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m H_{n,m}\, c_m(t) \qquad H_{n,m} = (\Psi_n, H\Psi_m).$$

THERE IS THUS A CLEAR ANALOGY BETWEEN

$$P(\{n\},t) \Leftrightarrow c_n(t).$$

★ BUT WHAT IS A NATURAL BASIS?

WE INTRODUCE A <u>FOCK SPACE</u>

$$|0\rangle \underset{a}{\overset{a^+}{\rightleftarrows}} |1\rangle \underset{a}{\overset{a^+}{\rightleftarrows}} |2\rangle \ldots \qquad \begin{cases} a^+|n\rangle = |n+1\rangle \\ a|n\rangle = n|n-1\rangle \end{cases}$$

WHENCE

$$[a, a^+] = 1 \qquad \leftrightarrow \qquad \begin{cases} [a_i, a_j^+] = \delta_{ij} \\ [a_i, a_j] = 0 \\ [a_i^+, a_j^+] = 0 \end{cases}$$

$$|\{n_1, n_2 \ldots\}\rangle = (a_1^+)^{n_1} (a_2^+)^{n_2} \ldots |0\rangle$$

LATTICE SITES

THESE ARE EIGENSTATES OF THE NUMBER OPERATOR

$$N|n\rangle = n|n\rangle.$$

ASSUMING THE VACUUM IS NORMALIZED,

$$\langle 0|0\rangle = 1$$

THEN YOU CAN SHOW THAT

$$\langle n'|n\rangle = \langle 0|(a)^{n'}(a^+)^n|0\rangle = \delta_{nn'}\, n!.$$

IN ANALOGY WITH QM, WE CAN NOW CONSTRUCT

$$|\Phi(t)\rangle = \sum_{\{n\}} P(\{n\},t)\, |\{n\}\rangle$$

$$\frac{\partial}{\partial t} |\Phi(t)\rangle = -H|\Phi(t)\rangle.$$

SUMMING UP: DISCRETE PARTICLE SYSTEMS

26.11.19

WE STARTED FROM THE MASTER EQUATION

$$\frac{\partial}{\partial t} P(\{m\},t) = \sum_{\{m'\}} \left[W(\{m'\}\to\{m\}) P(\{m'\},t) - W(\{m\}\to\{m'\}) P(\{m\},t) \right]$$

$$= -\sum_{\{m'\}} \mathcal{L}_{\{m\},\{m'\}} P(\{m'\},t)$$

WHICH RESEMBLES

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$$

$$\Psi(t) = \sum_m c_m(t) \Psi_m \qquad \{\Psi_1, \Psi_2, \dots\} \text{ BASIS}$$

$$i\hbar \frac{\partial}{\partial t} c_m(t) = \sum_{m'} H_{m,m'} c_{m'}(t) \qquad H_{m,m'} = (\Psi_m, H\Psi_{m'}).$$

IN ANALOGY WITH QM, WE CONSTRUCT THE BASIS

$|\{m\}\rangle$

USING THE BOSONIC LADDER OPERATORS

$a, a^+, |0\rangle \qquad [a_i, a_j^+] = \delta_{ij}.$

WE CAN THEN REWRITE, IN TERMS OF

$$|\Phi(t)\rangle = \sum_{\{m\}} P(\{m\},t)\, |\{m\}\rangle$$

THE MASTER EQUATION

$$\partial_t |\Phi(t)\rangle = -H|\Phi(t)\rangle \qquad \langle\{m\}|H|\{m'\}\rangle = \mathcal{L}_{\{m\},\{m'\}}$$

WHERE H CAN BE EXPRESSED IN TERMS OF a, a^+ AND THEN CAST IN NORMAL FORM, WITH THE a^+ TO THE LEFT:

$H(a^+, a).$

WE ALSO FOUND

$$\begin{cases} a^+|m\rangle = |m+1\rangle \\ a|m\rangle = m|m-1\rangle \end{cases} \qquad \begin{aligned} N &\equiv a^+ a \\ N|m\rangle &= m|m\rangle. \end{aligned}$$

NOTE THERE IS NO NOTION OF $(\cdot,\cdot)$: a^+ DIDN'T NEED TO BE $(a)^+$.

· EXAMPLE: FOCK-SPACE REPRESENTATION OF M.E.

CONSIDER THE IRREVERSIBLE BINARY ANNIHILATION

PART A: $A + A \xrightarrow{\lambda} A$ (ON-SITE REACTION)

+ DIFFUSION.

FOCUS ON A SINGLE SITE,

$m \equiv$ # OF INDIVIDUALS

$$\frac{\partial}{\partial t} P(m,t) = W(m+1 \to m) P(m+1,t) - W(m \to m-1) P(m,t).$$

SINCE

$$W(m \to m-1) = \lambda m(m-1)$$

(WE PUT THE ℓ INTO λ), WE HAVE

$$\partial_t P(m,t) = \lambda(m+1)m P(m+1,t) - \lambda m(m-1) P(m,t).$$

WHAT IS THE ASSOCIATED H? IN THIS CASE

$$|\Phi(t)\rangle = \sum_{m=0}^{+\infty} P(m,t)|m\rangle \qquad |m\rangle = (a^+)^m|0\rangle$$

WHENCE

$$\frac{\partial}{\partial t}|\Phi(t)\rangle = \sum_{m=0}^{+\infty} \lambda(m+1)m P(m+1,t)|m\rangle - \sum_{m=0}^{+\infty} \lambda m(m-1) P(m,t)|m\rangle$$

$$= \sum_{m=-1}^{+\infty} \lambda P(m+1,t) a^+ a^2 |m+1\rangle - \sum_{m=0}^{+\infty} \lambda P(m,t)(a^+)^2 a^2|m\rangle$$

$$= \left[\lambda a^+ a^2 - \lambda (a^+)^2 a^2\right]|\Phi(t)\rangle$$

AND

$$H = -\lambda a^+(1-a^+)a^2$$

WHICH INDEED IS NOT HERMITIAN.

IT IS USEFUL TO REMEMBER

$$(a^+)^\ell a^k |m+k-\ell\rangle = (m+k-\ell)(m+k-\ell-1)\dots(m-\ell+1)|m\rangle.$$

· EXERCISE

DETERMINE THE HAMILTONIAN ASSOCIATED WITH

a) THE IRREVERSIBLE REACTION

$kA \xrightarrow{\lambda} lA$ $\qquad\qquad W(m \to m-k+l) = \lambda m(m-1)\dots(m-k+1)$

b) THE SPECIAL CASE

$A + A \xrightarrow{\lambda} 0$.

* BUT WHAT ABOUT DIFFUSION?

WE START BY CONSIDERING DIRECTIONAL DIFFUSION WITH 2 SITES.

THE NEW M.E. IS MADE OF $\quad i \xrightarrow{D} j \qquad \{m_i, m_j\}$

GAIN: $\{m_i+1, m_j-1\} \to \{m_i, m_j\}$

LOSS: $\{m_i, m_j\} \to \{m_i-1, m_j+1\}$

SO THAT

$$\frac{\partial}{\partial t} P(m_i, m_j, t) = D\cdot(m_i+1)P(m_{i+1}, m_j-1, t) - Dm_i P(m_i, m_j, t)$$

WHERE

$$|\Phi(t)\rangle = \sum_{m_i, m_j=0}^{+\infty} P(m_i, m_j, t)|m_i, m_j\rangle$$

$$|m_i, m_j\rangle = (a_i^+)^{m_i}(a_j^+)^{m_j}\underbrace{|0_i, 0_j\rangle}_{\equiv |0\rangle}$$

AND WE GET

$$\frac{\partial}{\partial t}|\Phi(t)\rangle = \sum_{\substack{m_i=0 \to -1\\ m_j=1}}^{+\infty} D(m_i+1)P(m_i+1, m_j-1, t)|m_i, m_j\rangle$$

$$- \sum_{m_i, m_j=0}^{+\infty} Dm_i P(m_i, m_j, t)|m_i, m_j\rangle$$

$$= \sum_{\substack{m_i=-1\\ m_j=0}}^{+\infty} D P(m_i+1, m_j-1, t)\, a_j^+ a_i |m_i+1, m_j-1\rangle - \sum_{m_i, m_j=0}^{+\infty} D P(m_i, m_j, t) a_i^+ a_i |m_i, m_j\rangle$$

$$= D(a_j^+ - a_i^+)a_i |\Phi(t)\rangle$$

(FOR $i \to j$).

BUT ONE COULD REPEAT FOR $i \leftarrow j$, SO THAT ALLOWING DIFFUSION ON BOTH SIDES WE GET

$$\frac{\partial}{\partial t} |\Phi(t)\rangle = -D(a_j^+ - a_i^+)(a_j - a_i)|\Phi(t)\rangle \qquad (i \rightleftarrows j)$$

WHICH IS QUADRATIC (READ: "EASY"). NOTICE THE H WE HAD FOUND IN THE PREVIOUS CASE (INTERACTION AND NO DIFFUSION) WAS NOT QUADRATIC.

ALL IN ALL, FOR

$$A + A \xrightarrow{\lambda} A + (\text{DIFF.})$$

WE FOUND

$$H = D\sum_{\langle ij\rangle} (a_j^+ - a_i^+)(a_j - a_i) - \lambda \sum_i a_i^+ (1 - a_i^+) a_i^2 .$$

* SIMILARLY,

$$A + A \xrightarrow{\lambda} 0 + (\text{DIFF.})$$

WOULD GIVE

$$H = D\sum_{\langle ij\rangle} (a_j^+ - a_i^+)(a_j - a_i) - \lambda \sum_i (1 - (a_i^+)^2) a_i^2 .$$

- <u>EXERCISE</u>: FIND H FOR THE LOTKA-VOLTERRA MODEL.

* H IS IN GENERAL NOT HERMITIAN. HOWEVER, IF YOUR RATES SATISFY DETAILED BALANCE, THERE EXISTS A TRANSFORMATION WHICH MAKES H HERMITIAN (INDEED, H^+ GOVERNS THE BACKWARD EVOLUTION).

NOTE: SEE EX. 1 IN PROBLEM SET 4.

* ONCE H IS KNOWN, THEN FROM

$$\partial_t |\Phi(t)\rangle = -H|\Phi(t)\rangle$$

WE HAVE FORMALLY

$$|\Phi(t)\rangle = e^{-Ht}|\Phi(0)\rangle .$$

BUT WE ARE USUALLY INTERESTED IN THE MEAN VALUES (QM)

$$\langle O \rangle_{\psi(t)} = \langle \psi(t) | O | \psi(t) \rangle = \sum_{m,m} C_m^*(t)\, \underbrace{O_{mm}}_{= \langle \psi_m | O | \psi_m \rangle} C_m(t) .$$

HOWEVER, IN (SD) WE WANT

$$\langle O \rangle_{SD} = \sum_{\{m\}} O(\{m\})\, P(\{m\}, t)$$

$$\neq \langle \Phi(t) | O | \Phi(t) \rangle .$$

INDEED, THE FIRST EXPRESSION IS LINEAR IN P, WHILE THE SECOND IS QUADRATIC. WHAT BRA SHOULD WE CHOOSE? WE NOTICE

$$O(\{a^+ a\}) | \Phi(t) \rangle = \sum_{\{m\}} O(\{m\})\, P(\{m\}, t)\, | \{m\} \rangle$$

SO WE DEFINE A <u>PROJECTION STATE</u> SUCH THAT

$$\langle P | O(\{a^+ a\}) | \Phi(t) \rangle = \sum_{\{m\}} O(\{m\})\, P(\{m\}, t).$$

IN QM, IT WOULD BE

$$\langle P | = \sum_{m=0}^{+\infty} \alpha_m \langle m | \qquad \text{s.t.} \qquad \langle P | m \rangle = 1 \quad \forall m$$

BUT HERE

$$\langle m | m \rangle = \delta_{mm}\, m! \qquad \Rightarrow \qquad \langle P | m \rangle = \alpha_m\, m!$$

SO WE CHOOSE

$$\alpha_m = 1/m!$$

NOTE: WITH OUR CHOICE OF NOTATION, $|m\rangle = (a^+)^m |0\rangle$, $\langle m | = \langle 0 | a^m$ WITHOUT ANY NORMALIZATION CONSTANT.

AND WE FIND

$$\langle P | = \sum_{m=0}^{+\infty} \frac{1}{m!} \langle 0 | a^m = \langle 0 | e^{a}$$

OR, WITH MANY SITES,

$$\langle P | = \langle 0 | \prod_i e^{a_i} .$$

- PROPERTIES

$$\langle P| a_i^+ = \langle P|$$

THIS CAN BE SEEN BY ANALOGY WITH $\left[\frac{d}{dx}, e^x\right] = e^x$, BECAUSE

$$[q,p] = i\hbar \leftrightarrow [a,a^+] = 1 \quad \rightsquigarrow \quad a^+ \to -\frac{\partial}{\partial a}$$

SO MORALLY a^+ ACTS ON a AS A DERIVATIVE. INDEED,

$$[e^{a_i}, a_j^+] = \delta_{ij}\, e^{a_i} \qquad \text{(I)}$$

COMING FROM

$$[a_i^k, a_j^+] = \delta_{ij}\, k\, a_i^{k-1}.$$

ACTUALLY, (I) IMPLIES

$$\langle P| a_i^+ = \langle P|.$$

★ ANOTHER PROPERTY IS THIS:

NOTE: COMPARE WITH $f(q+\alpha) = e^{ip\alpha} f(q) e^{-ip\alpha}$.

$$e^{\lambda a} a^+ = (a^+ + \lambda) e^{\lambda a}$$

$$e^{\lambda a} f(a^+) = f(a^+ + \lambda) e^{\lambda a}$$

AND THE OPPOSITE IS ALSO TRUE:

$$f(a)\, e^{\lambda a^+} = e^{\lambda a^+} f(a+\lambda).$$

AS A RESULT,

$$e^{\lambda_1 a} \underset{\substack{\uparrow \\ \text{NORMAL ORDERED}}}{h(a^+, a)}\, e^{\lambda_2 a^+} = h(a^+ + \lambda_1, a)\, e^{\lambda_2 (a^+ + \lambda_1)} e^{\lambda_1 a}$$

$$= e^{\lambda_2 a^+} h(a^+ + \lambda_1, a + \lambda_2)\, e^{\lambda_2 \lambda_1} e^{\lambda_1 a}$$

$$\langle 0| e^{\lambda_1 a} h(a^+, a) e^{\lambda_2 a^+} |0\rangle = h(\lambda_1, \lambda_2)\, e^{\lambda_2 \lambda_1}.$$

★ NOW CONSIDER

$$1 = \langle 1 \rangle_{SD} = \langle P| 1 | \Phi(t)\rangle = \langle P| e^{-Ht} |\Phi(0)\rangle \quad \forall t, \forall |\Phi(0)\rangle$$

i.e.

$\forall |\{m\}\rangle,$

THIS IMPLIES

$$\langle P | e^{-Ht} = \langle P | \quad \forall t \qquad \Rightarrow \qquad \langle P | H = 0.$$

WRITING H IN NORMAL FORM,

$$0 = \langle P | H(a^+, a) \underset{\uparrow}{=} \langle P | H(a^+ \to 1, a)$$

$\langle P | a^+ = \langle P |$ NOTE: $\langle P | a^+ = \langle P | 1$.

ONE CAN CONCLUDE THAT, AS A CONSEQUENCE OF PROBABILITY CONSERVATION,

$$\underline{H(a^+ \to 1, a) = 0}.$$

THIS IS INDEED TRUE IN THE EXAMPLE WE HAVE SEEN.

DOI - PELITI FORMALISM

04.12.19

$$|\Phi(t)\rangle = \sum_{\{m\}} P(\{m\},t)\,|\{m\}\rangle$$

AND WE INTRODUCED $\langle P|$ SUCH THAT

$$\langle P|\{m\}\rangle = 1 \qquad \Rightarrow \qquad \langle P| = \langle 0|\prod_i e^{a_i}$$

WHICH ALSO IMPLIES

$$\langle P|a_i^+ = \langle P|.$$

THEN

$$\langle \mathcal{O}\rangle_t = \langle P|\mathcal{O}|\Phi(t)\rangle$$

AND IN PARTICULAR

$$\langle 1\rangle_t = 1 = \langle P|e^{-Ht}|\Phi(0)\rangle \qquad \forall|\Phi(0)\rangle;\ \text{e.g. } |\{m\}\rangle$$

WHENCE

$$\langle P|e^{-Ht} = \langle P| \qquad \Rightarrow \qquad \langle P|H(a^+,a) = 0.$$

BUT THIS IMPLIES THAT THE FUNCTION ITSELF HAS TO VANISH:

$$H(a^+\to 1, a) = 0.$$

★ SUPPOSE NOW I WANT TO KNOW THE AVERAGE NUMBER

$$m_i(t) = \langle P|a_i^+ a_i|\Phi(t)\rangle = \langle P|a_i|\Phi(t)\rangle = \langle a_i\rangle.$$

THIS DOESN'T ALWAYS WORK, THOUGH:

$$m_i^2(t) = \langle P|a_i^+ a_i a_i^+ a_i|\Phi(t)\rangle = \langle a_i a_i^+ a_i\rangle = \langle a_i^2\rangle + \langle a_i\rangle.$$

★ WE HAVE NOW ALL THE TOOLS TO BUILD A 2$^{\text{ND}}$ QUANTIZATION OF OUR DYNAMICS. BUT IN ORDER TO GO TOWARDS A PATH INTEGRAL FORMULATION, WE NEED TO GET RID OF OPERATORS FIRST:

$$K(t,q,q') = \langle q'|e^{-iHt}|q\rangle.$$

$$\uparrow$$

$$H(\hat{p},\hat{q})$$

· COHERENT-STATE PATH INTEGRAL

COHERENT-STATES ARE EIGENSTATES OF a:

$$a|\phi\rangle = \phi|\phi\rangle \qquad \phi\in\mathbb{C}.$$

DOES THIS $|\phi\rangle$ BELONG TO OUR FOCK SPACE? OMITTING SITE INDICES, WE APPLY a TO

$$|\phi\rangle = \sum_{m=0}^{+\infty} c_m|m\rangle$$

$$\sum_{m=0}^{+\infty} c_m\, m\,|m-1\rangle = \sum_{m=0}^{+\infty} \phi c_m|m\rangle \quad \Rightarrow \quad c_{m+1}(m+1) = \phi c_m.$$

THIS GIVES THE RECURSIVE RELATION

$$c_{m+1} = \frac{\phi}{m+1}c_m \quad \Rightarrow \quad c_m = \frac{\phi^m}{m!}c_0.$$

THEN

$$|\phi\rangle = \sum_{m=0}^{+\infty}\frac{\phi^m}{m!}c_0|m\rangle = c_0 e^{\phi a^+}|0\rangle.$$

c_0 IS FIXED BY REQUIRING NORMALIZATION:

$$\langle\phi|\phi\rangle = |c_0|^2\langle 0|e^{\phi^* a}e^{\phi a^+}|0\rangle = |c_0|^2\langle 0|e^{\phi(a^+ + \phi^*)}e^{\phi^* a}|0\rangle$$

$$= |c_0|^2 e^{|\phi|^2} \equiv 1 \quad \Rightarrow \quad c_0 = e^{-\frac{1}{2}|\phi|^2}$$

$$\Rightarrow \underline{|\phi\rangle = e^{-\frac{1}{2}|\phi|^2+\phi a^+}|0\rangle}.$$

(a) DIFFERENT $|\phi\rangle$'S ARE NOT ORTHOGONAL: IN FACT,

$$\langle\phi|\phi'\rangle = \exp\left\{-\frac{1}{2}|\phi|^2-\frac{1}{2}|\phi'|^2+\phi'\phi^*\right\} = \exp\left\{-\frac{1}{2}\phi^*(\phi-\phi')+\frac{1}{2}\phi'(\phi^*-\phi^{*\prime})\right\}$$

BECAUSE

$$\langle 0|e^{\phi^* a}e^{\phi' a^+}|0\rangle = e^{\phi'\phi^*}.$$

NOTE: WE ARE BASICALLY USING

$$e^{\lambda_1 a}h(a^+,a)e^{\lambda_2 a^+} = e^{\lambda_2 a^+}h(a^++\lambda_1, a+\lambda_2)e^{\lambda_1 a}\cdot e^{\lambda_1\lambda_2}.$$

(b) WE WANT TO VERIFY THAT

$$\int\frac{d\phi\, d\phi^*}{\pi}|\phi\rangle\langle\phi| = \mathbb{1}.$$

THIS, TOGETHER WITH PROPERTY (a), MEANS THAT $\{|\phi\rangle\}$ IS AN OVERCOMPLETE SET OF STATES.

LET'S PROVE IT:

$$\int \frac{d\phi\, d\phi^*}{\pi} |\phi\rangle\langle\phi| = \sum_{n,m=0}^{+\infty} \int \frac{d\phi\, d\phi^*}{\pi} e^{-\frac{1}{2}|\phi|^2} \frac{\phi^n}{n!} |n\rangle e^{-\frac{1}{2}|\phi|^2} \frac{(\phi^*)^m}{m!} \langle m|$$

$$= \sum_{n,m=0}^{+\infty} \frac{1}{n!}\frac{1}{m!} |n\rangle\langle m| \int \frac{d\phi\, d\phi^*}{\pi} e^{-|\phi|^2} \phi^n (\phi^*)^m = \sum_{n=0}^{+\infty} \frac{1}{n!} |n\rangle\langle n| = \mathbb{1}.$$

THE MEASURE IS ROTATIONALLY INVARIANT, AND U(1) IS BROKEN IF $n \neq m$: THIS IS WHY

$$\int \frac{d\phi\, d\phi^*}{\pi} e^{-\frac{1}{2}|\phi|^2} \phi^n (\phi^*)^m \propto \delta_{n,m}.$$

NOTE: TO EVALUATE IT, CONSIDER

$$d\phi\, d\phi^* = 2\, d\phi_R\, d\phi_I$$

SO THAT

$$\int \frac{d\phi\, d\phi^*}{\pi} e^{-|\phi|^2} |\phi|^{2m} \qquad r = |\phi|$$

$$= \frac{2}{\pi}\cdot 2\pi \int_0^\infty r\,dr\, e^{-r^2} r^{2m}$$

$$= 4\int_0^\infty r^{2m+1} e^{-r^2} dr$$

$$= 4 \cdot \frac{m!}{2} = 2m!$$

THERE'S ONE EXTRA 2 ... MAYBE $\frac{d\phi\, d\phi^*}{2\pi}$

(c) FINALLY, NOTICE THAT

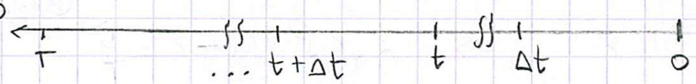
$$\langle\phi| h(a^+, a) |\phi'\rangle = h(\phi^*, \phi')\langle\phi|\phi'\rangle.$$

NORMAL ORDERED

* IN ORDER NOW TO GET OUR PATH INTEGRAL, LET'S EXPAND

$$e^{-HT} \underset{\Delta t \to 0}{=} (1 - H\Delta t)\cdot\ldots\cdot(1 - H\Delta t)\cdot\ldots\cdot(1 - H\Delta t),$$

T/Δt TIMES



TROTTER DECOMPOSITION

LET'S INTRODUCE SOME DECOMPOSITIONS OF $\mathbb{1}$:

$$e^{-HT} = (1 - H\Delta t)\cdot\ldots\cdot \mathbb{1}(1 - H\Delta t)\mathbb{1}\ldots\mathbb{1}(1 - H\Delta t)\mathbb{1} \longrightarrow \int \frac{d\phi_0\, d\phi_0^*}{\pi} |\phi_0\rangle\langle\phi_0|$$

$$\int \frac{d\phi_{t+\Delta t}\, d\phi^*_{t+\Delta t}}{\pi} |\phi_{t+\Delta t}\rangle\langle\phi_{t+\Delta t}| \qquad \int \frac{d\phi_{\Delta t}\, d\phi^*_{\Delta t}}{\pi} |\phi_{\Delta t}\rangle\langle\phi_{\Delta t}| \qquad \int \frac{d\phi_t\, d\phi_t^*}{\pi} |\phi_t\rangle\langle\phi_t|$$

BUT

$$\langle\phi_{t+\Delta t}| 1 - H(a^+, a)\Delta t |\phi_t\rangle = \left[1 - H(\phi^*_{t+\Delta t}, \phi_t)\Delta t\right]\langle\phi_{t+\Delta t}|\phi_t\rangle$$

$$\simeq e^{-\Delta t H(\phi^*_{t+\Delta t}, \phi_t) - \frac{1}{2}\phi^*_{t+\Delta t}(\phi_{t+\Delta t} - \phi_t) + \frac{1}{2}\phi_t(\phi^*_{t+\Delta t} - \phi^*_t)} + O(\Delta t^2).$$

THIS GIVES

$$e^{-HT} = \int \frac{d\phi_T\, d\phi_T^*}{\pi} \ldots \int \frac{d\phi_0\, d\phi_0^*}{\pi} |\phi_T\rangle\langle\phi_0| \cdot \exp\left\{\sum_{t=0}^{T-\Delta t}\left[-\Delta t\, H(\phi^*_{t+\Delta t}, \phi_t) - \frac{1}{2}\phi^*_{t+\Delta t}(\phi_{t+\Delta t} - \phi_t) + \frac{1}{2}\phi_t(\phi^*_{t+\Delta t} - \phi^*_t)\right]\right\}.$$

AS $\Delta t \to 0$,

$$\phi^*_{t+\Delta t} \sim \phi^*_t + O(\Delta t)$$

$$(\phi_{t+\Delta t} - \phi_t) \sim \Delta t \cdot \partial_t \phi_t \qquad\qquad (\phi^*_{t+\Delta t} - \phi^*_t) \sim \Delta t\, \partial_t \phi^*_t$$

AND THE SUM BECOMES AN INTEGRAL, SO THAT THE TERM AT THE EXPONENTIAL BECOMES

$$-\int_0^T dt \Big[\underbrace{\frac{1}{2}\phi^*_t \partial_t \phi_t - \frac{1}{2}\phi_t \partial_t \phi^*_t}_{\phi^*_t \partial_t \phi_t \text{ + BOUNDARY TERMS}} + H(\phi^*_t, \phi_t) \Big].$$

* HOW DO WE EXPRESS EXPECTATION VALUES OF $\mathcal{O}(a^+, a)$ IN THIS FORM?

$$\langle \mathcal{O} \rangle_T = \langle \mathcal{P} | \mathcal{O}(a^+, a)\, e^{-HT} | \Phi(0) \rangle .$$

USING THE DECOMPOSITION ABOVE, WE FIND THE MATRIX ELEMENT

$$\langle \mathcal{P} | \mathcal{O}(a^+, a) | \phi_T \rangle = \mathcal{O}(1, \phi_T) \langle \mathcal{P} | \phi_T \rangle = \mathcal{O}(1, \phi_T)\, e^{\phi_T - \frac{1}{2}|\phi_T|^2}.$$

IN FACT

$$|\phi_T\rangle = e^{-\frac{1}{2}|\phi_T|^2} \sum_{m=0}^{+\infty} \frac{\phi_T^m}{m!} |m\rangle \quad \Rightarrow \quad \langle \mathcal{P} | \phi_T \rangle = e^{-\frac{1}{2}|\phi_T|^2} e^{\phi_T}.$$

WE ALSO FIND

$$\langle \phi_0 | \Phi(0) \rangle = \sum_{m=0}^{+\infty} P_0(m) \langle \phi_0 | m \rangle = e^{-\frac{1}{2}|\phi_0|^2} \sum_{m=0}^{+\infty} P_0(m) (\phi_0^*)^m$$

BECAUSE

$$|\Phi(0)\rangle = \sum_{m=0}^{+\infty} P_0(m) |m\rangle, \qquad\qquad \langle \phi_0 | m \rangle = (\phi_0^*)^m e^{-\frac{1}{2}|\phi_0|^2}.$$

THE ONE ABOVE LOOKS LIKE A CUMULANT GENERATING FUNCTION: GIVEN $P_0(m)$,

$$e^{-A_0(\lambda)} = \sum_{m=0}^{+\infty} P_0(m) \lambda^m.$$

THEN

$$\langle \phi_0 | \Phi(0) \rangle = e^{-\frac{1}{2}|\phi_0|^2 - A_0(\phi_0^*)}.$$

WE FOUND

$$\langle \theta \rangle_T = \int \mathcal{D}\phi\mathcal{D}\phi^* \, \theta(1,\phi_T) \, e^{\phi_T - \frac{1}{2}|\phi_T|^2 - \frac{1}{2}|\phi_0|^2 - A_0(\phi_0^*) - \int_0^T dt \, \{\cdot\}}$$

$$\{\cdot\} = \frac{1}{2}\phi_t^* \partial_t \phi_t - \frac{1}{2}\phi_t \partial_t \phi_t^* + H(\phi_t^*, \phi_t)$$

WHICH INTEGRATED BY PARTS GIVES

$$-\int_0^T \{\cdot\} dt = \frac{1}{2}|\phi_T|^2 - \frac{1}{2}|\phi_0|^2 - \int_0^T dt \left[\phi_t^* \partial_t \phi_t + H(\phi_t^*, \phi_t) \right].$$

THEN WE GET

$$\langle \theta \rangle_T = \int \mathcal{D}\phi\mathcal{D}\phi^* \, \theta(1,\phi_T) \, e^{-|\phi_0|^2 - A_0(\phi_0^*) - A_T[\phi,\phi^*]}$$

WHERE

$$A_T[\phi,\phi^*] \equiv -\phi_T + \underbrace{\int_0^T dt \left[\phi_t^* \partial_t \phi_t + H(\phi_t^*, \phi_t) \right]}_{\text{BULK TERM}} . \qquad (I)$$

NOTICE THE APPEARANCE OF BOUNDARY TERMS, WHICH ARE NOT EXTENSIVE IN T (AND ARE GENERALLY ABSENT IN FIELD THEORIES, APART FROM THOSE WITH A BOUNDARY).

* GIVEN THAT

$$H(1, \phi_t) \equiv 0$$

IT IS CONVENIENT TO APPLY THE SO CALLED "DOI - SHIFT":

$$\phi_t^* \equiv 1 + \tilde{\phi}_t .$$

WE WILL THEN TREAT ϕ_t AND $\tilde{\phi}_t$ AS INDEPENDENT FIELDS.

* IMAGINE NOW WE ARE INTERESTED IN THE AVERAGE QUANTITY

$$\langle m_i(T) m_j(T' < T) \rangle = \langle P | m_i e^{-H(T-T')} m_j e^{-HT'} | \Phi(0) \rangle .$$

· <u>EXERCISE</u>: WRITE DOWN THE CORRESPONDENT PATH INTEGRAL, AND IN GENERAL THAT OF

$$\langle \theta_1(\dots)_{(T)} \, \theta_2(\dots)_{(T')} \rangle .$$

★ NOTICE WHAT WE ARE CALLING T IS NOTHING BUT THE INDEX OF THE COHERENT STATE, AND NOT NECESSARILY TIME.

· REMOVING THE LATTICE $(A+A \xrightarrow{\lambda} \emptyset)$

LAST TIME WE SPLIT H INTO ITS DIFFUSION AND REACTION PARTS,

$$H = H_D + H_R$$

$$= \sum_{\langle ij \rangle}' D(a_i^+ - a_j^+)(a_i - a_j) - \lambda \sum_i' (1 - a_i^{+2}) a_i^2 .$$

SENDING THE LATTICE SPACING* $\ell \to 0$,

$$H(\phi_t^*, \phi_t) = \int d^d x \left\{ \underbrace{\ell^{2-d} D}_{\equiv \bar{D}} \underbrace{(\nabla \phi_t^*)(\nabla \phi_t)}_{\sim -\phi_t^* \nabla^2 \phi_t} - \underbrace{\lambda \ell^{-d}}_{\equiv \bar{\lambda}} (1 - (\phi_t^*)^2) \phi_t^2 \right]$$

SO THAT

$$A_T\,(\text{BULK}) = \int_0^T dt \int d^d x \left\{ \underbrace{\phi_t^*}_{\substack{\text{DOI} \\ \tilde{\phi}_t \ (+\text{CONST.})}} (\partial_t - \bar{D} \nabla^2) \phi_t - \bar{\lambda} \phi_t^2 \underbrace{(1 - \phi_t^{*2})}_{(1+\tilde{\phi}_t)^2} \right\} .$$

THE "GAUSSIAN" PART CONTAINS THE KERNEL OF DIFFUSION AND IS NON-DIAGONAL: IT CONNECTS $\tilde{\phi}_t$ TO ϕ_t,

$$\tilde{\phi}(\partial_t - \bar{D} \nabla^2) \phi = (\tilde{\phi}, \phi) \begin{pmatrix} 0 & \partial_t - \bar{D} \nabla \\ +\partial_t - \bar{D} \nabla & 0 \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \phi \end{pmatrix} \cdot \frac{1}{2} .$$

*NOTE: AS USUAL,

$$a_i^+ - a_j^+ \rightsquigarrow (\phi_i^* - \phi_j^*) \cdot \frac{\ell^2}{\ell^2} \xrightarrow[\ell \to 0]{} \ell^2 \nabla \phi^* .$$

NOTICE ALSO THIS $H(\phi_t^*, \phi_t)$ IS NOT SIMPLY TO BE SUBSTITUTED INTO (I), BECAUSE IN THAT CASE ϕ WAS REALLY ONLY ONE OF THE POSSIBLE ϕ_i's (WE JUST SIMPLIFIED THE NOTATION). ALL IN ALL, A SPACE INTEGRATION IS TO BE INTENDED ALSO ON THE TERM

$$\int d^d x \ \phi_t^* \partial_t \phi_t .$$

DOI-PELITI AGAIN

06.12.19

LAST TIME WE DERIVED FOR

$A + A \xrightarrow{\lambda} \emptyset$ + diff.

$$A_T[\phi, \tilde{\phi}] = \int_0^T dt \int d^d x \left\{ \tilde{\phi}(\partial_t - D\nabla^2)\phi - \lambda \phi^2 \left[1 - (1+\tilde{\phi})^2 \right] \right\}$$

$$= \int_0^T dt \int d^d x \left\{ \tilde{\phi}(\partial_t - D\nabla^2)\phi + 2\lambda \phi^2 \tilde{\phi} + \lambda \phi^2 \tilde{\phi}^2 \right\}$$

$$\langle \theta \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \, \theta \, e^{-A_T}$$

(THE BOUNDARY TERM SHOULDN'T COUNT IN THE STATIONARY STATE).

LET'S TRY TO GET A LINEAR TERM OUT OF $\lambda \phi^2 \tilde{\phi}^2$:

$$e^{-\lambda \int dt \, d^d x \, \tilde{\phi}^2 \phi^2} = \int \mathcal{D}\xi \, e^{\int dt \, d^d x \, \xi \tilde{\phi}} P[\xi].$$

NOTE: I MAY BE WRONG, BUT $\int e^{-\alpha x^2 \pm \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$. SO IT SEEMS TO ME THAT, EVEN AT THIS LEVEL, $e^{-\lambda A \tilde{\phi}^2}$ APPEARS ONLY IF WE TAKE $\xi = ix$, $x \in \mathbb{R}$, SO THAT $P[\xi] = \exp\{\xi^2/4\lambda A\}$.

WHAT IS $P[\xi]$? IN GENERAL,

$$e^{-\lambda A \tilde{\phi}^2} = \int d\xi \, e^{\xi \tilde{\phi}} P[\xi]$$

IS AUTOMATICALLY SATISFIED IF $P[\xi]$ IS A GAUSSIAN.

THEN WE GET

$$\langle \theta \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{D}\xi \, P[\xi] \, e^{-\int dt \, d^d x \left[\tilde{\phi}(\partial_t \phi - D\nabla^2 \phi + 2\lambda \phi^2 - \xi) \right]} \cdot \theta$$

$$= \int \mathcal{D}\phi \mathcal{D}\xi \, \theta \cdot \delta(\partial_t \phi - D\nabla^2 \phi + 2\lambda \phi^2 - \xi) \cdot P[\xi] \cdot \underbrace{J}_{\text{JACOBIAN}}$$

$$= \int \mathcal{D}\xi \, P[\xi] \, \theta(\phi_\xi) \cdot J \qquad (\mathrm{I})$$

WHERE ϕ_ξ IS THE FIELD WHICH SATISFIES THE LANGEVIN EQUATION

$$\partial_t \phi = D\nabla^2 \phi - 2\lambda \phi^2 + \xi \qquad (\mathrm{II})$$

WHERE ξ IS A GAUSSIAN NOISE.

MAY ϕ BE THE DENSITY OF PARTICLES?

U WE TOLD US THAT, IF

$m \equiv$ AVERAGE DENSITY OF PARTICLES

THEN

$$\partial_t m = D\nabla^2 m - 2\lambda \underbrace{m^{(2)}}_{\text{NUMBER OF PAIRS}}$$

$$\simeq D\nabla^2 m - 2\lambda m^2. \qquad \text{MEAN FIELD.}$$

THE DETERMINISTIC PART OF THIS EQUATION IS ANALOGOUS TO (II).

ONE MIGHT AS WELL FORGET ABOUT EVERYTHING, START FROM THE RATE EQUATION AND ADD A GAUSSIAN NOISE. BUT HOW DO WE CHOOSE ITS VARIANCE? GIVEN

$$\partial_t m = D\nabla^2 m - 2\lambda m^2 + \xi$$

WE MIGHT EXPECT (SINCE WE EXPECT $\xi \propto$ NUMBER OF PAIRS)

$$\langle \xi\xi \rangle = \delta^d(x-x')\delta(t-t')\, m^2(x,t).$$

IN THE FORMULATION OF (I), ON THE OTHER HAND, SETTING

$$P[\xi] = e^{-\int dt\, d^d x\, \frac{1}{2}\frac{\xi^2}{\Gamma}}$$

WE FIND

$$\Gamma = -2\lambda\phi^2 \qquad \Rightarrow \qquad \langle \xi(x,t)\xi(x',t')\rangle = \Gamma\,\delta(x-x')\delta(t-t')$$

WHICH IS TERRIBLE, BECAUSE IT'S TELLING US THAT THE AUTOCORRELATION FUNCTION OF ξ IS NEGATIVE: ξ IS IMAGINARY! THIS IS THE PRICE TO PAY IF WE JUST WANT TO WRITE A LANGEVIN EQUATION.

HOWEVER, IT MAKES SENSE: NEIGHBOURING DENSITIES ARE INDEED ANTI-CORRELATED (INCREASE IN A SITE MEANS DEPLETION IN THE NEIGHBORHOOD). MORAL: DO IT, BUT BE CAREFUL.

ANOTHER DISCREPANCY IS THIS: IF IT'S TRUE THAT

$$\langle m \rangle = \langle a^\dagger a \rangle = \langle \phi \rangle$$

ON THE OTHER HAND IT'S NOT TRUE THAT $m = \phi$ $\quad (\langle m^2 \rangle \neq \langle \phi^2 \rangle)$.

QUANTUM DETOUR: RELATIONSHIP WITH QUANTUM SYSTEMS (OUT OF EQ.) (SCHWINGER - KELDISH FORMALISM)

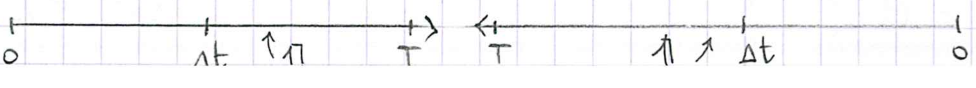
STOCHASTIC DYNAMICS	$\leftrightarrow$	QUANTUM MECHANICS
$\rho(\{m\},t) \to \lvert\Phi(t)\rangle$		$\lvert\psi(t)\rangle$
$\partial_t\lvert\Phi(t)\rangle = -\underset{\text{NON HERMITIAN}}{H}\lvert\Phi(t)\rangle$	EVOLUTION	$i\hbar\,\partial_t\lvert\psi(t)\rangle = \overset{\text{HERMITIAN}}{H}\lvert\psi(t)\rangle$
$U(t) = e^{-Ht}$		$U(t) = e^{-\frac{i}{\hbar}Ht}$ (UNITARY)
$\lvert 0\rangle \to \lvert 1\rangle \to \lvert 2\rangle \ldots$	FOCK SPACE	$\lvert 0\rangle \to \lvert 1\rangle \to \lvert 2\rangle \ldots$
$\lvert m\rangle = (a^+)^m\lvert 0\rangle$		$\lvert m\rangle = 1/\sqrt{m!}\,(a^+)^m\lvert 0\rangle$
$\begin{cases} a^+\lvert m\rangle = \lvert m+1\rangle \\ a\lvert m\rangle = m\lvert m-1\rangle \end{cases}$		$\begin{cases} a^+\lvert m\rangle = \sqrt{m+1}\,\lvert m+1\rangle \\ a\lvert m\rangle = \sqrt{m}\,\lvert m-1\rangle \end{cases}$
$[a,a^+] = 1$	$e^{\lambda a}f(a^+) = f(a^++\lambda)e^{\lambda a}$	$[a,a^+] = 1$
(i) $\langle\phi\lvert\phi\rangle = 1$	COHERENT STATES $\downarrow$ $a\lvert\phi\rangle = \phi\lvert\phi\rangle$	(i) $\langle\phi\lvert\phi\rangle = e^{\lvert\phi\rvert^2}$
(ii) $\lvert\phi\rangle = \exp\{-\frac{1}{2}\lvert\phi\rvert^2 + \phi a^+\}\lvert 0\rangle$		(ii) $\lvert\phi\rangle \equiv e^{\phi a^+}\lvert 0\rangle$
(iii) $\int\frac{d\phi\, d\phi^*}{\pi}\lvert\phi\rangle\langle\phi\rvert = \mathbb{1}$		(iii) $\int\frac{d\phi\, d\phi^*}{\pi}\lvert\phi\rangle e^{-\lvert\phi\rvert^2}\langle\phi\rvert = \mathbb{1}$
(iv) $\langle\phi\lvert\phi'\rangle = \exp\{-\frac{\lvert\phi\rvert^2}{2} - \frac{\lvert\phi'\rvert^2}{2} + \phi^*\phi'\}$		(iv) $\langle\phi\lvert\phi'\rangle = e^{\phi^*\phi'}$
$\lvert\phi_{t+\Delta t}\rangle\langle\phi_{t+\Delta t}\rvert(1-\Delta t H)\lvert\phi_t\rangle\langle\phi_t\rvert$	NORMAL ORDERING OF H, TROTTER DECOMPOSITION	$\underbrace{\lvert\phi_{t+\Delta t}\rangle e^{-\lvert\phi_{t+\Delta t}\rvert^2}\langle\phi_{t+\Delta t}\rvert}_{1\ \text{SPLIT}}\left(1-\frac{i\Delta t}{\hbar}H\right)\underbrace{\lvert\phi_t\rangle e^{-\lvert\phi_t\rvert^2}\langle\phi_t\rvert}_{1}$

SO FAR, IT'S EXACTLY THE SAME UP TO $\frac{i}{\hbar}$. BUT:

$\bar{\Theta}(T) = \langle\beta\lvert\Theta\lvert\Phi(T)\rangle = \langle\beta\lvert\Theta e^{-HT}\lvert\Phi(0)\rangle$	EXPECTATION VALUES	$\Theta(T) = \langle\psi(T)\lvert\Theta\lvert\psi(T)\rangle = \langle\psi(0)\lvert e^{i/\hbar HT}\Theta e^{-i/\hbar HT}\lvert\psi(0)\rangle$

AGAIN, WE CAN TROTTER DECOMPOSE $U(T)$: BUT IN THE QUANTUM CASE THIS HAS TO BE DONE TWICE. WE FIND

$$\langle\psi(0)\lvert\left(1+\frac{i}{\hbar}\Delta t H\right)\ldots\left(1+\frac{i}{\hbar}\Delta t H\right)\cdot\Theta\cdot\left(1-\frac{i}{\hbar}\Delta t H\right)\ldots\left(1-\frac{i}{\hbar}\Delta t H\right)\lvert\psi(0)\rangle$$



WE INTRODUCE TWO FAMILIES OF COHERENT STATES, IN TERMS OF WHICH WE WRITE THE IDENTITY: $\mathbb{1}$:

$$\mathbb{1} = \int \frac{d\phi_t^{(-)}\, d\bar\phi_t^{(-)}}{\pi}\, |\phi_t^{(-)}\rangle\, e^{-|\phi_t^{(-)}|^2} \langle \phi_t^{(-)}| \qquad \text{(LEFT)}$$

$$\mathbb{1} = \int \frac{d\phi_t^{(+)}\, d\bar\phi_t^{(+)}}{\pi}\, |\phi_t^{(+)}\rangle\, e^{-|\phi_t^{(+)}|^2} \langle \phi_t^{(+)}| \qquad \text{(RIGHT)}.$$

HENCE, PROCEEDING AS IN THE CLASSICAL CASE,

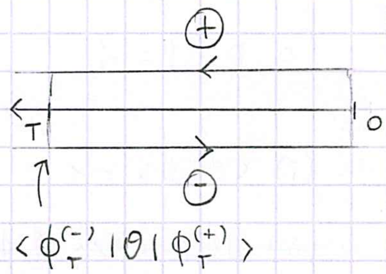
$$\Theta(T) = \int \mathcal{D}\phi^{(+)} \mathcal{D}\bar\phi^{(+)} \mathcal{D}\phi^{(-)} \mathcal{D}\bar\phi^{(-)} \langle \Psi(0)|\phi_0^{(-)}\rangle \langle \phi_T^{(-)}|\Theta|\phi_T^{(+)}\rangle \langle \phi_0^{(+)}|\Psi(0)\rangle \cdot$$

$$\cdot \exp\left\{ -|\phi_T^{(-)}|^2 - |\phi_0^{(+)}|^2 + \frac{i}{\hbar}\int_0^T dt \left[\bar\phi_t^{(+)} i\partial_t \phi_t^{(+)} - H(\bar\phi_t^{(+)}, \phi_t^{(+)})\right] - \frac{i}{\hbar}\int_0^T dt \left[\bar\phi_t^{(-)} i\partial_t \phi_t^{(-)} - H(\bar\phi_t^{(-)}, \phi_t^{(-)})\right] \right\}$$

DEFINE

$$\mathcal{L}(\phi_t, \bar\phi_t) \equiv \bar\phi_t\, i\partial_t \phi_t - H(\bar\phi_t, \phi_t) \qquad \text{(III)}$$

SO THAT AT THE EXPONENT WE GET



$$\frac{i}{\hbar}\int_0^T dt\, \mathcal{L}(\bar\phi_t^{(+)}, \phi_t^{(+)}) - \frac{i}{\hbar}\int_0^T dt\, \mathcal{L}(\bar\phi_t^{(-)}, \phi_t^{(-)})$$

$$= \frac{i}{\hbar}\int_0^T dt\, \mathcal{L}(\bar\phi_t^{(+)}, \phi_t^{(+)}) + \frac{i}{\hbar}\int_T^0 \mathcal{L}(\bar\phi_t^{(-)}, \phi_t^{(-)})$$

$$= \frac{i}{\hbar}\int_\gamma dt\, \mathcal{L}(\bar\phi_t, \phi_t) \qquad \text{CLOSED PATH TIME INTEGRAL}$$

WITH A SUBTLETY: ON THE TWO BRANCHES,

$$\phi_t^{(+)} \neq \phi_t^{(-)}$$

(EVEN THOUGH THEIR ACTION IS INDEED THE SAME, WHICH IS WHY WE CAN PUT THEM TOGETHER). BUT ARE THEM INDEPENDENT?

DEFINE THE GENERATING FUNCTIONAL

$$\mathcal{Z}[J^{(+)}, J^{(-)}] \equiv \left\langle e^{\int dt \left(J_t^{(+)}\phi_t^{(+)} - J_t^{(-)}\phi_t^{(-)}\right)} \right\rangle \qquad \left(\mathcal{L} \to \mathcal{L} + \int J\phi\right)$$

SO THAT WE CAN CHECK THE CORRELATION FUNCTION:

?

$$\langle \phi_T^{(-)}|\phi_T^{(+)}\rangle = e^{\bar\phi_T^{(-)}\phi_T^{(+)}} \qquad \left(\text{SO THAT } \langle 1\rangle_T = 1\right).$$

WHICH MEANS THEY ARE UNCORRELATED EVERYWHERE BUT AT THE BOUNDARY, WHERE THEY INTERACT.

WHAT KIND OF CORRELATION DO WE GET? FIRST OF ALL,

$$\langle \phi^{(+)}(t)\bar{\phi}^{(+)}(t') \rangle \equiv i G^{T}(t,t') \qquad \underline{\text{ORDER}}.$$

NOTICE INSTEAD

$$\langle \phi^{(+)}(t)\phi^{(+)}(t') \rangle = 0.$$

IN FACT, LOOK AT (III): IF THESE ARE BOSONS, THEN H IS A FUNCTION OF

$$n \sim a^{+}a \sim \bar{\phi}\phi$$

*NOTE: SEPARATELY IN $\phi^{(+)}$ AND $\phi^{(-)}$. THIS MEANS $\langle \phi^{m}\bar{\phi}^{m} \rangle \propto \delta_{mm}$ (SEE PROBLEM SET 5).

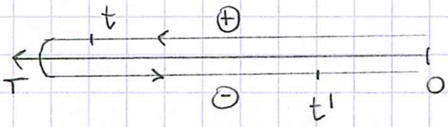
AND SO IS THE TERM $\bar{\phi}_t\, i\partial_t \phi$. THIS IMPLIES $U(1)$ SYMMETRY* OF H

SIMILARLY, DEFINE

$$\langle \phi^{(-)}(t)\bar{\phi}^{(-)}(t') \rangle = i G^{\tilde{T}}(t,t') \qquad \underline{\text{ANTI-ORDER}}$$

WHICH IS ANTI-TIME ORDERED. NOTICE ALONG THE CONTOUR

$$t' > t$$



SO WE CAN DEFINE

$$\langle \phi^{(+)}(t)\bar{\phi}^{(-)}(t') \rangle \equiv i G^{<}(t,t') \qquad \underline{\text{LESSER}}$$

$$\langle \phi^{(-)}(t)\bar{\phi}^{(+)}(t') \rangle \equiv i G^{>}(t,t') \qquad \underline{\text{GREATER}}.$$

ARE ALL 4 OF THEM INDEPENDENT? NOTICE IF I CHANGE $+ \leftrightarrow -$, NOTHING REALLY CHANGES:

$$\langle \phi^{(+)}\bar{\phi}^{(+)} \rangle \underset{t\to -t}{\longleftrightarrow} \langle \phi^{(-)}\bar{\phi}^{(-)} \rangle \quad \Rightarrow \quad (iG^{T})^{+} = iG^{\tilde{T}}$$

$$\langle \phi^{(+)}\bar{\phi}^{(-)} \rangle \underset{t\to -t}{\longleftrightarrow} \langle \phi^{(-)}\bar{\phi}^{(+)} \rangle \quad \Rightarrow \quad (iG^{>})^{+} = iG^{>}$$

(WHEN YOU TAKE THE DAGGER, YOU ALSO HAVE TO EXCHANGE THE ORDER OF t AND t').

· KELDYSH ROTATION

LET'S INTRODUCE THE TWO FIELDS

$$\begin{cases} \phi_c = \dfrac{\phi^{(+)} + \phi^{(-)}}{\sqrt{2}} & \text{CLASSICAL} \\[2ex] \phi_q = \dfrac{\phi^{(+)} - \phi^{(-)}}{\sqrt{2}}\,\hbar & \text{QUANTUM.} \end{cases}$$

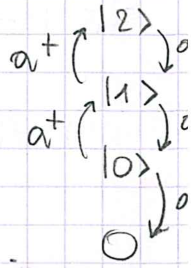
NOTICE WHAT WE DID IS VERY GENERAL : IT CAN BE USED TO DESCRIBE BOTH EQUILIBRIUM AND NON-EQUILIBRIUM (e.g. QUENCHES).

THIS CONCLUDES OUR CRASH-COURSE ON QUANTUM SYSTEMS.

FROM COLD BOSONS TO SPIN HAMILTONIANS

09.12.19

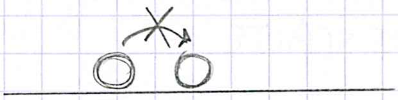
WE HAVE INTRODUCED THE BOSONIC OPERATORS



$$\begin{cases} a^+|m\rangle = |m+1\rangle \\ a|m\rangle = m|m-1\rangle \end{cases} \qquad m \in \mathbb{N}$$

$a^+a|m\rangle = m|m\rangle$ $\qquad$ a^+a : NUMBER OPERATOR.

BUT IN STATISTICAL PHYSICS THERE ARE MANY PROBLEMS WITH A SINGLE OCCUPANCY CONSTRAINT: THEY ARE THE EXCLUSION PROCESSES (EP THIS GIVES RISE (EVEN IN THE CASES WITH NO EXPLICIT INTERACTION) TO AN "ENTROPIC INTERACTION".



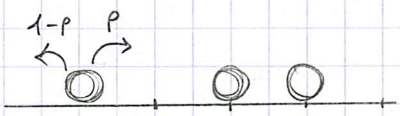
$m \in \{0, 1\}$

ACRONYMS:

S (EP) : SIMPLE

A : ASYMMETRIC $(p \neq \frac{1}{2})$

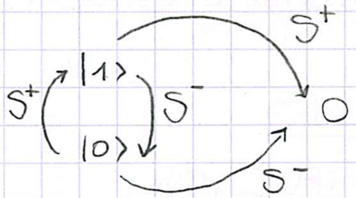
T : TOTALLY $(p = 1)$.



★ CAN WE STUDY THIS WITH A BOSONIC MODEL?

WE COULD PUT A VERY HIGH POTENTIAL ON SOME SITES, BUT IT'S UNHANDY. DEFINE INSTEAD

$$\begin{cases} S^+|m\rangle = (1-m)|m+1\rangle \\ S^-|m\rangle = m|m-1\rangle. \end{cases}$$



HOW DO WE RECOVER m? DEFINE

$$S^0|m\rangle = \left(m - \frac{1}{2}\right)|m\rangle.$$

IT'S EASY TO CHECK THEY SATISFY THE SU(2) ALGEBRA

$$\begin{cases} [S^0, S^\pm] = \pm S^\pm \\ [S^+, S^-] = 2S^0. \end{cases}$$

YOU'LL SEE IN PROBLEM SET 4 THAT

SSEP $\rightarrow$ HEISENBERG FERROMAGNET.

ALTERNATIVE PATH-INTEGRAL REPRESENTATION OF M.E.

THE MASTER EQUATION CAN BE WRITTEN AS

$$\partial_t P(m,t) = \sum_r \{ W(m-r \to m) P(m-r,t) - W(m \to m+r) P(m,t) \} .$$

(r: POSSIBLE JUMPS.)

BUT THE SECOND TERM IS EQUAL TO THE FIRST UP TO A TRANSLATION

$m \to m - r$.

SINCE

$$e^{-r\partial_m} f(m) = f(m-r)$$

WE MAY REWRITE

$$\partial_t P(m,t) = \sum_r \left[e^{-r\partial_m} - 1 \right] W(m \to m+r) P(m,t) .$$

DEFINING

$\hat{p} = -i\partial_m \quad \to \quad \partial_m = i\hat{p}$

WE CAN EXPRESS

$$\partial_t P(m,t) = -H(\hat{p},m) P(m,t) \qquad H(\hat{p},m) = -\sum_r \left(e^{-ir\hat{p}} - 1 \right) W(m \to m+r) \qquad (\mathrm{I})$$

WHICH IS NORMAL ORDERED.

OBSERVATIONS

a) BY CONSTRUCTION,

$H(\hat{p}=0, m) = 0 \quad \Rightarrow \quad$ CONSERVATION OF PROBABILITY.

IN FACT, THIS IMPLIES

$H(\hat{p}, m) \propto \hat{p} \quad \leftrightarrow \quad \partial_t P = \partial_m (\dots)$

OR YOU CAN SEE IT BY TAKING THE SCALAR PRODUCT WITH THE CONSTANT FUNCTION* 1.

*NOTE: MORALLY,

$$\frac{d}{dt} P_{TOT} = \frac{d}{dt} \sum_m P(m,t) = \frac{d}{dt} (1\ 1\ 1 \dots) \begin{pmatrix} P_m(t) \end{pmatrix} .$$
$$= (1\ 1\ 1 \dots) \left(-H(\hat{p},m) \right) \begin{pmatrix} P_m(t) \end{pmatrix} .$$

H ACTS ON THE LEFT AS A DERIVATIVE ON THE CONSTANT VECTOR, SO WE GET ZERO.

b) BY EXPANDING UP TO II ORDER IN $\hat{p}$, YOU GET THE FOKKER-PLANCK EQUATION (i.e. DIFFUSION).

c) $[m, \hat{p}] = i$.

THIS IS CANONICAL, SO WE CAN DEFINE $|p\rangle$ s.t.

$$\hat{p}|p\rangle = p|p\rangle$$

WHICH BECOMES, IN m REPRESENTATION,

$$-i\partial_m |p\rangle = p|p\rangle \qquad \rightarrow \qquad \langle m|p\rangle \propto e^{ipm}$$

WHICH IS A PLANE WAVE. SIMILARLY,

$$\langle p|m\rangle \propto e^{-ipm}.$$

★ AGAIN, THE EVOLUTION EQUATION (I) IS SOLVED BY e^{-HT}, WHICH WE CAN TROTTER-DECOMPOSE AS

$$e^{-HT} = \underset{T \longleftarrow\!\!- 0}{(1-\Delta t H)(1-\Delta t H)\dots \dots(1-\Delta t H)}.$$

WHAT SHALL WE PUT IN BETWEEN? IT IS HANDY TO HAVE

$$\langle p| H(\hat{p}, m)|m\rangle$$

AND SO WE USE

$$1 = \int dp_t dm_t \, |m_t\rangle \overbrace{\langle m_t|p_t\rangle}^{e^{ip_t m_t}} \langle p_t|.$$

LET'S THEN COMPUTE THE ELEMENT

$$\langle p_t|[1-\Delta t H(\hat{p},m)]|m_{t-\Delta t}\rangle = \langle p_t|m_{t-\Delta t}\rangle[1-\Delta t H(p_t, m_{t-\Delta t})]$$

$$\simeq \exp\{-ip_t m_{t-\Delta t} - \Delta t H(p_t, m_{t-\Delta t})\}.$$

EACH OF THIS IS MULTIPLIED BY $\langle m_t|p_t\rangle$, SO THAT

$$\langle m_t|p_t\rangle\langle p_t|[1-\Delta t H(\hat{p},m)]|m_{t-\Delta t}\rangle$$

$$= \exp\{ip_t \overbrace{(m_t - m_{t-\Delta t})}^{\Delta t \cdot \dot{m}_t} - \Delta t H(p_t, \underbrace{m_{t-\Delta t}}_{\to\, m_t})\}$$

$$e^{-HT} = \int \mathcal{D}m_t \mathcal{D}p_t \, |m_T\rangle\langle p_0| \exp\left\{\int_0^T dt \, [ip_t \dot{m}_t - H(p_t, m_t)]\right\}.$$

WE CAN NOW USE THIS RESULT TO CALCULATE

$$P_{1|1}(m_f, t \mid m_0, 0) = \langle m_f | e^{-HT} | m_0 \rangle$$

$$= \int_{\substack{m(0)=m_i \\ m(t)=m_f}} \mathcal{D}m \, \mathcal{D}p \, \exp\left\{ \int_0^t d\tau \left[i p \dot{m} - H(p, m) \right] \right\}$$

OR, SENDING $p \to -ip$ (INDEED, THE i WAS ARTIFICIAL),

$$\int_{\substack{m(0)=m_i \\ m(t)=m_f}} \mathcal{D}m \, \mathcal{D}(ip) \, \exp\left\{ \int_0^t d\tau \left[p\dot{m} - H(-ip, m) \right] \right\}.$$

· <u>EXAMPLE</u>: BRANCHING & DECAY

(B) $A \xrightarrow{\sigma} 2A$ $\qquad W(m \to m+1) = \sigma m$

(D) $A \xrightarrow{\mu} \emptyset$ $\qquad W(m \to m-1) = \mu m$

(C) $2A \xrightarrow{\lambda} A$ $\qquad W(m \to m-1) = \lambda m(m-1)$.

THIS IS A WAY TO LIMIT THE INFINITE GROWTH OF THE POPULATION: WITH A RATE PROPORTIONAL TO m^2 WE KILL PARTICLES.

SUMMARIZING THE THREE RATES,

$$W(m \to m+r) = \sigma m \, \delta_{r,1} + \left[\mu m + \lambda m(m-1) \right] \delta_{r,-1}.$$

OUR HAMILTONIAN THEN READS*

$$H(\hat{p}, m) = -(e^{-i\hat{p}} - 1)\sigma m - (e^{i\hat{p}} - 1)\left[\mu m + \lambda m(m-1)\right]$$

$$H(-ip, m) = -(e^{-p} - 1)\sigma m - (e^{p} - 1)\left[\mu m + \lambda m(m-1)\right].$$

THIS IS THE REASON WHY WE NEVER PRECISELY MANAGED TO CAST THIS IN THE FORM OF A LANGEVIN EQUATION: INDEED,

LANGEVIN $\leftrightarrow$ FOKKER-PLANCK

WHILE HERE THERE ARE MANY MORE TERMS.

*NOTE: CHECK $\to$ $H(0, m) = 0$.

WE CAN REWRITE

$$H(-ip, m) = (e^{p} - 1)(e^{-p}\sigma - \lambda m + \lambda - \mu)\, m .$$

* WHAT IS THE RELATION BETWEEN THIS AND DOI-PELITI? i.e.,

$$\{a, a^+\} \overset{?}{\leftrightarrow} \{\hat{p}, m\}$$

REMEMBER

$$|\Phi(t)\rangle = \sum_{m=0}^{\infty} P(m,t)\,|m\rangle \qquad P(m,t) = \frac{1}{m!}\langle m|\Phi(t)\rangle$$

SO WE JUST HAVE TO CHECK

$$a^+ |\Phi(t)\rangle = ?$$

$$P'(m,t) = \frac{1}{m!}\langle m|\, a^+ |\Phi(t)\rangle = \frac{1}{m!}\, m \langle m-1|\Phi(t)\rangle = P(m-1, t)$$

WHICH SHOWS THAT

$$\underline{a^+ = e^{-\partial_m} = e^{-i\hat{p}}} .$$

SIMILARLY,

$$a|\Phi(t)\rangle = ?$$

$$P'(m,t) = \frac{1}{m!}\langle m|\, a\, |\Phi(t)\rangle = \frac{1}{m!}\langle m+1|\Phi(t)\rangle = (m+1)\, P(m+1, t)$$

WHENCE

$$\underline{a = e^{\partial_m} m = e^{i\hat{p}} m} .$$

YOU CAN CHECK THAT THE COMMUTATION RELATION IS STILL $[a, a^+] = 1$

NOTE ALSO THAT

$$\hat{p} = 0 \leftrightarrow a^+ = 1 .$$

THE RELATION BETWEEN (a, a^+) AND $(\hat{p}, m)$ WE DERIVED IS KNOWN AS <u>COLE-HOPF TRANSFORMATION</u>.

- RARE EVENTS

WE DERIVED

$$P_{1|1}(q_f, t | q_0, 0) = \int_{q(0)=q_0}^{q(t)=q_f} \mathcal{D}(ip)\mathcal{D}q \, e^{\overbrace{\int_0^t d\tau \, [p\dot{q} - H(-ip, q)]}^{\displaystyle S_t}}$$

WHERE AT THE exp WE RECOGNIZE THE ACTION OF A CLASSICAL SYSTEM: THE DYNAMICS WILL BE DOMINATED BY CLASSICAL TRAJECTORIES, WHICH MAKE THE ARGUMENT STATIONARY.
THE FIRST APPROXIMATION IS THEN THAT OF THE STATIONARY-PATH (WKB). LET

$$\left.\frac{\delta S_t}{\delta p(t)}\right|_{OP} \equiv 0 \qquad \Rightarrow \qquad \dot{q} = \partial_p H$$

$$\left.\frac{\delta S_t}{\delta q(t)}\right|_{OP} \equiv 0 \qquad \Rightarrow \qquad \dot{p} = -\partial_q H$$

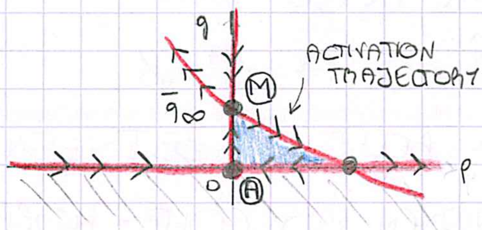
WHICH ARE HAMILTON EQUATIONS OF MOTION, SATISFIED ALONG THE OPTIMUM PATH (OP). YOU CAN SHOW THAT

$H(\bar{p}(t), \bar{q}(t))$ ↳ OP IS INDEPENDENT OF t, i.e. $\frac{d}{dt} H(\bar{p}(t), \bar{q}(t)) = 0$.

AT LONG TIMES, ASSUMING THE EXISTENCE OF A STATIONARY STATE,

$$S_t[\bar{p}(t), \bar{q}(t)] \simeq -\int_0^t d\tau \, H(\bar{p}(t), \bar{q}(t)) \simeq -\bar{H} \cdot t .$$

ARE ALL THE POSSIBLE VALUES $\bar{H}$ ALLOWED? FIRST OF ALL, $\bar{H} > 0$ (NORMALIZABILITY).
BUT ACTUALLY, "TYPICAL" TRAJECTORIES CORRESPOND TO $\bar{H} \equiv 0$.



WHAT DO THEY LOOK LIKE? NOTE THAT:
(a) $H(p=0, q) = 0$.
(b) IF $m=0$ IS ABSORBING, THEN SINCE

$$W(m \to m+r) \propto m \qquad \Rightarrow \qquad H(p, q=0) = 0 .$$

CONSIDER THE STATIONARY PATH $\bar q(t)$ WITH $\bar p(t) \equiv 0$.

ITS EQUATION OF MOTION READS

$$\dot{\bar q} = -\partial_p H(\bar p, \bar q)\big|_{\bar p = 0}$$

NOTE:
$H(-ip, m) = (e^{p}-1)(e^{p}\sigma - \lambda m + \lambda - \mu)m$.

$$= (\lambda - \mu + \sigma)m - \lambda m^2$$

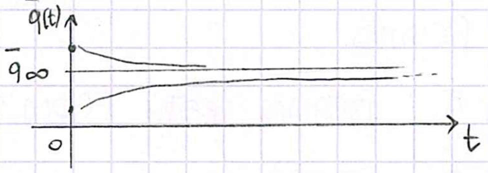
(BRANCHING & DECAY)

$$= (\lambda - \mu + \sigma)\bar q - \lambda \bar q^2 = (\sigma - \mu)\bar q - \lambda \bar q(\bar q - 1)$$

WHICH IS THE MEAN FIELD EQUATION (NOISELESS). THIS SHOWS THAT p IS SOMEHOW LINKED TO THE FLUCTUATIONS.

IF $\sigma \gg \mu$,

$$\bar q(t \to +\infty) = \bar q_\infty = \frac{\lambda - \mu + \sigma}{\lambda}.$$



THERE ARE 2 FIXED POINTS IN THE DYNAMICS: $\bar q_\infty$ AND $\bar q = 0$. BUT IN THE ABSENCE OF NOISE, THERE SEEMS TO BE NO WAY TO REACH $\bar q = 0$. ACTUALLY

$$H(-ip, m) = (e^p - 1)\underbrace{(e^{-p}\sigma - \lambda m + \lambda - \mu)}_{\equiv 0}m$$

GIVES RISE TO ANOTHER BRANCH: THIS INTERSECTS THE p AXIS AT A NEW STATIONARY POINT, AND THE FLOW ALONG THE p AXIS IS TOWARDS $\bar q = 0$.

$\bar q_\infty$ IS REALLY A METASTABLE STATE.

WHAT IS, THEN, THE PROBABILITY OF REACHING THE ABSORBING STATE "A" STARTING FROM "M" (RARE EVENTS)?

$$P(M \to A) \propto e^{-S_t(\gamma)}$$

$$S_t(\gamma) = \int d\tau \,[\, p\dot q - \underset{0}{\underset{\shortparallel}{H(\gamma)}} \,] = \int p\, dq$$

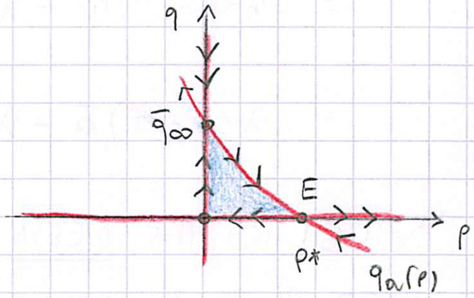
WHICH IS BASICALLY THE BLUE AREA (WE'LL SEE IT NEXT TIME).

RECAP: BRANCHING & DECAYING

LAST TIME WE DERIVED

$$P_{1,1}(q_f, t \mid q_0, 0) = \int_{\substack{q(0)=q_0 \\ q(t)=q_f}} \mathcal{D}q\, \mathcal{D}ip\; e^{\int_0^t d\tau \left[p\dot{q} - \underbrace{H(-ip, q)}_{\equiv \mathcal{H}(p,q)} \right]}$$

$q \leftrightarrow m$



FOR THE PROCESS

$$A \xrightarrow{2\sigma} 2A, \quad A \xrightarrow{\mu} \emptyset, \quad 2A \xrightarrow{\lambda} A \quad (m=0 \text{ ABSORBING})$$

$$\mathcal{H}(p,q) = \lambda (e^p - 1) \Big[\underbrace{\frac{1}{\lambda} \left(e^{-p}\sigma + \lambda - \mu \right)}_{q_a(p), \text{ ACTIVATION TRAJECTORY}} - q \Big] q .$$

WE FOUND

$\bar{p}, \bar{q}$: OPTIMAL PATH FROM SADDLE POINT EQUATION (WKB)

$\mathcal{H}(\bar{q}(t), \bar{p}(t))$ INDEPENDENT OF t

$\rightsquigarrow$ OPTIMAL PATHS: $\mathcal{H} \equiv 0$.

LET'S STUDY THE OPTIMAL PATH $q_a(p)$ FOR $\lambda \ll \mu$. WE SEE THAT

$$q_a(p^*) \equiv 0 \qquad \Rightarrow \qquad e^{-p^*} = \frac{\mu - \lambda}{\sigma}$$

SO THAT IF

$$\begin{cases} \mu - \lambda > \sigma \rightarrow p^* < 0 \\ \mu - \lambda < \sigma \rightarrow p^* > 0 . \end{cases}$$

FOCUSING ON THE SECOND CASE, $\mu - \lambda < \sigma$, THE PATH $q_a(p)$ INDIVIDUATES ON THE 2 AXES THE POINTS $(0, \bar{q}_\infty)$ AND $(p^*, 0)$.

FOR $p = 0$,

$$\dot{\bar{q}} = \partial_p \mathcal{H}(p, \bar{q}) \Big|_{p=0} = \lambda \left(q_a(0) - \bar{q} \right) \bar{q} = \lambda \left(\bar{q}_\infty - \bar{q} \right) \bar{q}$$

WHICH IS THE RATE EQUATION (MEAN FIELD, NOISELESS). YOU RECOVER IT FOR $p = 0$: THIS SHOWS HEURISTICALLY THAT p IS RELATED TO THE NOISE.

BUT WHEN YOU ARE FLUCTUATING AROUND $\bar{q}_\infty$, WHAT FIXES THE

PROBABILITY OF GETTING EXTINCT (IN FACT, $P \neq 0$, AND THEN THERE ARE FLUCTUATIONS)? FROM $P_{1,1}$ WE SEE THAT

$$P_{ex} \propto \exp\left\{ S_t[q_a, p_a] \right\}$$

BUT

$$S_t[q_a, p_a] = \int_0^t d\tau \left[p_a(\tau)\dot{q}_a(\tau) - \underbrace{H(p_a, q_a)}_{=0} \right] \xrightarrow[t\to\infty]{} \int_0^{\bar{q}_\infty} p_a \, dq_a$$

WHICH IS JUST THE BLUE AREA.

* WHAT HAPPENS IN THE OTHER CASE, $\mu - \lambda > \sigma$?
NOTHING SPECIAL, REALLY.

* WHAT WE HAVE DONE SO FAR CAME FROM REWRITING THE MASTER EQUATION AS A PATH INTEGRAL. CAN WE DO THE SAME FOR A LANGEVIN EQUATION?

· <u>FIELD THEORETICAL REPRESENTATION OF THE LANGEVIN EQUATION</u>

(RESPONSE FUNCTION FORMALISM, OR MARTIN-SIGGIA-ROSE, OR JANSSEN-DE DOMINICIS-PELITI, > '76).

WE'LL TREAT IT WITHOUT MULTIPLICATIVE NOISE:

$$\dot{\varphi}_i = F_i[\varphi] + \xi_i \qquad\qquad \langle \xi_i(t)\xi_j(t')\rangle = \Omega_{ij}\,\delta(t-t')$$

WHERE φ IS THE FIELD AND i IS THE LATTICE SITE, AND ξ IS A GAUSSIAN NOISE WITH ZERO AVERAGE:

$$P[\xi] \propto [d\xi]\, e^{-\frac{1}{2}\int dt\, \xi\, \Omega^{-1} \xi} .$$

NOTE: WE WILL OFTEN OMIT THE LATTICE SITE INDEX i BECAUSE ALL OF THIS IS VALID FOR A SINGLE SITE AS WELL, NOT ONLY FOR A FIELD THEORY

FOR EACH REALIZATION OF THE NOISE

$\{\xi_i(t)\}_{i,t}$

THE CORRESPONDING REALIZATION OF THE FIELD IS

$\varphi_i^{(\xi)}(t)$.

AVERAGING OVER THE NOISE, THEN, MEANS TAKING

$$\langle \mathcal{O} \rangle_\xi = \int [d\xi]\, \mathcal{O}(\varphi^{(\xi)})\, P[\xi]$$

$$= \int [d\xi][d\varphi]\, \delta(\varphi - \varphi^{(\xi)})\, \mathcal{O}(\varphi)\, P[\xi]$$

$$= \int [d\varphi]\, \mathcal{O}(\varphi) \int [d\xi]\, P[\xi]\, \delta(\varphi - \varphi^{(\xi)}).$$

THE FIELD φ IS TO SATISFY THE CONSTRAINT

$$E(\varphi, \xi) = \dot{\varphi} - F[\varphi] - \xi$$

$$E(\varphi^{(\xi)}, \xi) = 0.$$

THIS IS ANALOGOUS TO

$$\delta(x - x_0)$$

WHERE $f(x_0) = 0$. WE ARE USED TO THE REVERSE:

$$\delta(f(x)) = \sum_{(i)} \frac{\delta(x - x_0^{(i)})}{|f'(x_0^{(i)})|}.$$

SO HERE WE EXPRESS

$$\delta(\varphi - \varphi^{(\xi)}) = \delta(E(\varphi, \xi)) \cdot J(\varphi \to E)$$

WHERE THE JACOBIAN J IS FORMALLY

$$J = \det M \qquad\qquad M_{ij}(t, \tau) = \frac{\delta E_i(\varphi(t), \xi(t))}{\delta \varphi_j(\tau)}.$$

THEN

NOTE: IT LOOKS LIKE WE ARE SUPPOSING IT INVERTIBLE.

$$\langle \mathcal{O} \rangle_\xi = \int [d\varphi]\, \mathcal{O}(\varphi) \int [d\xi]\, P[\xi]\, \delta(E(\varphi, \xi))\, J(\varphi \to E) \qquad (I)$$

$$= \int [d\varphi]\, \mathcal{O}(\varphi) \int [d\, i\tilde{\varphi}] \int d[\xi]\, P[\xi]\, e^{-\int \tilde{\varphi}\, E(\varphi, \xi) dt} \cdot J(\varphi \to E)$$

$$= \int [d\varphi]\, \mathcal{O}(\varphi) \int [d i\tilde{\varphi}]\, e^{-\int dt\, \tilde{\varphi}[\dot{\varphi} - F(\varphi)]} \int [d\xi]\, P[\xi]\, e^{+\int dt\, \xi \tilde{\varphi}} \cdot J(\varphi \to E).$$

$\tilde{\varphi}$ IS KNOWN AS THE RESPONSE FIELD.

NOTICE THE JACOBIAN IS NOISE INDEPENDENT, SO THE INTEGRAL IN $[d\xi]$ CAN BE COMPUTED. SINCE IT'S A GAUSSIAN INTEGRAL,

$$\int [d\xi]\, e^{+\int dt\, \tilde{\varphi}\xi}\, e^{-\frac{1}{2}\int dt\, \xi(t)\Omega^{-1}\xi(t)} \propto e^{\int dt\, \frac{1}{2}\tilde{\varphi}\Omega\tilde{\varphi}}$$

AND

$$\langle \theta \rangle_\xi = \int [d\varphi][di\tilde{\varphi}]\, \theta(\varphi, \underset{\text{SEE LATER}}{\tilde{\varphi}})\, e^{-\overbrace{\left\{\int dt\left[\tilde{\varphi}(\dot{\varphi}-F[\varphi]) - \frac{1}{2}\Omega\tilde{\varphi}^2\right] - \ln J(\varphi\to E)\right\}}^{J[\varphi,\tilde{\varphi}]}} \qquad (II)$$

THE QUANTITY $J[\varphi, \hat{\varphi}]$ IS KNOWN AS DYNAMICAL FUNCTIONAL (OR ACTION).

WE MIGHT AS WELL INTEGRATE (I) IN $[d\xi]$, OR ELSE (II) IN $[di\tilde{\varphi}]$. WE WOULD GET THE ONSAGER-MACHLUP FUNCTIONAL

$$\frac{1}{2}\Omega^{-1}(\dot{\varphi} - F(\varphi))^2$$

NOTE: HE MEANS THAT $\langle \theta \rangle_\xi = \int [d\varphi]\, \theta(\varphi)\, e^{-\int dt \frac{1}{2}(\dot{\varphi}-F(\varphi))\Omega^{-1}(\dot{\varphi}-F(\varphi))} \cdot J(\varphi\to E)$

(WHICH, HOWEVER, WE SHALL NOT ADOPT, BECAUSE $\dot{\varphi}$ IS GENERALLY ILL-DEFINED).

★ NOW LET'S COMPUTE THE JACOBIAN:

$$E_i(\varphi, \xi) = \dot{\varphi}_i - F_i[\varphi] - \xi_i$$

$$M_{ij}(t,\tau) = \frac{\delta E_i(\varphi(t), \xi(t))}{\delta \varphi_j(\tau)}$$

$$= \frac{d}{dt}\delta(t-\tau)\delta_{ij} - \frac{\delta F_i[\varphi(\tau)]}{\delta\varphi_j(\tau)}\underbrace{\delta(t-\tau)}_{\frac{d}{dt}\theta(t-\tau)} \equiv \frac{d}{dt}\bar{M}_{ij}$$

WHERE

$$\bar{M}_{ij}(t,\tau) = \delta(t-\tau)\delta_{ij} - \underbrace{\frac{\delta F_i}{\delta \varphi_j}(\tau)}_{\equiv V_{ij}(\tau)}\theta(t-\tau).$$

WE ARE ONLY INTERESTED IN THE φ-DEPENDENT PART OF THE DETERMINANT, WHICH IS, BY BINET THEOREM,

$$J = \det M \propto \det \bar{M} \qquad \frac{d}{dt} \sim \begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & & \ddots \end{pmatrix}$$

WE'RE LEFT WITH

$$\det \tilde{M} = \det\left[\mathbb{1} + v(\tau)\Theta(t-\tau)\right] = \exp\left\{ \mathrm{Tr}\ln\left[\mathbb{1}+v(\tau)\Theta(t-\tau)\right]\right\}.$$

BUT

$$\ln\left[\mathbb{1}+\Theta(t-\tau)v(\tau)\right] = v(\tau)\Theta(t-\tau) - \frac{1}{2}\int d\tau_1\, \Theta(t-\tau_1)v(\tau_1)\Theta(\tau_1-\tau)v(\tau) + \text{h.o}$$

WHEN WE TAKE THE TRACE, WE HAVE TO SET $t=\tau$ AND INTEGRATE IN dt, SO WE GET

$$\int dt \int d\tau_1\, \Theta(t-\tau_1)\Theta(\tau_1-t) = 0.$$

THEN WE SIMPLY HAVE

$$\mathrm{Tr}\ln\left[\mathbb{1}+\Theta(t-\tau)v(\tau)\right] = \mathrm{Tr}\left[v(\tau)\Theta(t-\tau)\right] = \Theta(0)\int dt\, \mathrm{Tr}\, v(t)$$

$$= -\Theta(0)\int dt\, F'[\varphi(t)]$$

NOTE: THIS LAST ONE IS A TRACE OVER THE SPACIAL INDICES OF $v(t)$.

AND THE DYNAMICAL FUNCTIONAL BECOMES

$$J[\varphi,\hat{\varphi}] = \int dt \left\{ \hat{\varphi}(\dot{\varphi} - F[\varphi]) - \frac{1}{2}\Omega\hat{\varphi}^2 + \Theta(0)F'[\varphi] \right\}.$$

IT SEEMS LIKE WE HAVE A PROBLEM: HOW MUCH IS $\Theta(0)$?
ACTUALLY, THE RESULT CANNOT DEPEND ON OUR CHOICE OF $\Theta(0)$, BECAUSE THE ORIGINAL PROCESS WAS WELL-DEFINED.
WE WILL CHECK THIS EXPLICITLY NEXT TIME.

MARTIN-SIGGIA-ROSE

12.12.19

YESTERDAY WE DERIVED

$$\langle O \rangle_\xi = \int \mathcal{D}\varphi\, \mathcal{D}\tilde{\varphi}\; O(\varphi, \tilde{\varphi})\, e^{-J[\varphi,\tilde{\varphi}]}$$

WHERE

$$J[\varphi, \tilde{\varphi}] = \int dt \left(\int d^d x\right) \left\{ \tilde{\varphi}(\dot{\varphi} - F[\varphi]) - \frac{1}{2}\Omega \tilde{\varphi}^2 + \theta(0) F'[\varphi] \right\}$$

WHICH SEEMS TO BE ILL-DEFINED BECAUSE OF THE PRESENCE OF $\theta(0)$. NOTICE THIS IS NOT RELEVANT IF $F[\varphi]$ IS LINEAR, BECAUSE IN THAT CASE $\theta(0) F'[\varphi]$ IS ONLY A CONSTANT. LET'S THEN EXPAND

$$F[\varphi] = -r\varphi + \alpha \varphi^m \qquad (r > 0 \text{ MEANS RESTORING FORCE})$$

(A SINGLE ADDITIONAL TERM IS ENOUGH FOR OUR DISCUSSION). IF

$$\alpha = 0: \quad \tilde{\varphi}(\partial_t \varphi + r\varphi) - \frac{1}{2}\Omega\tilde{\varphi}^2 \equiv (*).$$

THIS IS A NON-DIAGONAL GAUSSIAN THEORY, WHICH WE CAN REWRITE AS

$$(*) = \frac{1}{2} (\varphi, \tilde{\varphi}) \overbrace{\begin{pmatrix} 0 & -\partial_t + r \\ +\partial_t + r & -\Omega \end{pmatrix}}^{\equiv G^{-1}} \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix}$$

BECAUSE WE CAN SYMMETRIZE (UP TO BOUNDARY TERMS)

$$\tilde{\varphi}\partial_t \varphi = \frac{1}{2}\left(\tilde{\varphi}\partial_t\varphi - \varphi\partial_t\tilde{\varphi}\right) + (\dots)$$

SO AS TO MAKE IT HERMITIAN ($\rightarrow G^{-1}$ IS A HERMITIAN MATRIX). THE PROPAGATOR IS ITS INVERSE: RECALL

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{pmatrix}.$$

IN FOURIER SPACE, THEN, WE FIND

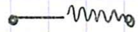
$$G^{-1} = \begin{pmatrix} 0 & -i\omega + r \\ +i\omega + r & -\Omega \end{pmatrix} \Rightarrow G = \begin{pmatrix} \frac{\Omega}{\omega^2 + r^2} & \frac{1}{+i\omega + r} \\ \frac{1}{-i\omega + r} & 0 \end{pmatrix}$$

SO THAT

$$\langle \varphi(\omega) \varphi(\omega') \rangle_0 = 2\pi \delta(\omega+\omega') \frac{\Omega}{\omega^2 + r^2}$$

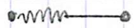
CORRELATOR

$$\langle \varphi(\omega) \tilde{\varphi}(\omega') \rangle_0 = 2\pi \delta(\omega+\omega') \frac{1}{+i\omega + r}$$

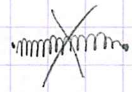


RESPONSE

$$\langle \tilde{\varphi}(\omega) \varphi(\omega') \rangle_0 = 2\pi \delta(\omega+\omega') \frac{1}{-i\omega + r}$$

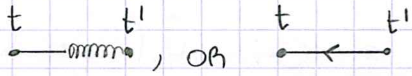


$$\langle \tilde{\varphi}(\omega) \tilde{\varphi}(\omega') \rangle_0 = 0$$



THIS LAST RESULT, WHICH WE DERIVED IN THE GAUSSIAN THEORY, WILL BE CARRIED OVER AT ALL ORDERS IN PERTURBATION THEORY.

LET'S CHECK THE MIXED PRODUCTS:



$$\langle \varphi(t) \tilde{\varphi}(t') \rangle = \int \frac{d\omega}{2\pi} e^{i\omega t} \int \frac{d\omega'}{2\pi} e^{i\omega' t'} \langle \varphi(\omega) \tilde{\varphi}(\omega') \rangle \qquad (\mathrm{I})$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} \frac{1}{+i\omega + r} = \begin{cases} 0 , & t-t' < 0 \\ e^{-r(t-t')} , & t-t' > 0 \end{cases} = \theta(t-t') e^{-r(t-t')}.$$

★ SO, WHAT IS THE MEANING OF $\tilde{\varphi}$? IS IT REALLY ONLY A LAGRANGE MULTIPLIER? THEN WHY IS IT CALLED "RESPONSE FIELD"?

FIRST, WHAT IS THE RESPONSE OF THE FIELD φ UNDER THE EFFECT OF AN EXTERNAL PERTURBATION? IMAGINE WE PERTURB THE DYNAMICS WITH A "FORCING" f:

$$\dot{\varphi} = F[\varphi] + f + \xi .$$

THEN THE DYNAMICAL FUNCTIONAL BECOMES

$$J[\varphi, \tilde{\varphi}; f] = J[\varphi, \tilde{\varphi}; 0] - \int dt \, \tilde{\varphi} f$$

AND THE MEAN VALUE OF AN OBSERVABLE BECOMES

$$\langle O \rangle_{\xi, f} = \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi}\, O(\varphi, \tilde{\varphi})\, e^{-J[\varphi, \tilde{\varphi}; f]}$$

$$= \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi}\, O\, e^{-J[\varphi,\tilde{\varphi};0] + \int dt\, \tilde{\varphi} f} .$$

THE RESPONSE IS THEN GIVEN BY

$$\frac{\delta}{\delta f(t')} \langle O \rangle_{\xi, f} \Big|_{f=0} = \langle O\, \tilde{\varphi}(t') \rangle_{\xi, f=0} .$$

IN PARTICULAR,

$$\frac{\delta}{\delta f(t')} \langle \varphi(t) \rangle_{\xi, f} \Big|_{f=0} = \langle \varphi(t)\, \tilde{\varphi}(t') \rangle .$$

IF EVERYTHING IS DONE PROPERLY, THEN,

$$\langle \tilde{\varphi} \rangle = \frac{\delta}{\delta f} \langle 1 \rangle = 0$$

$$\langle \tilde{\varphi} \tilde{\varphi} \dots \rangle = 0$$

AND THIS MUST BE TRUE AT ALL ORDERS IN PERTURBATION THEORY.
MOREOVER, CAUSALITY IMPLIES

$$\langle \varphi(t)\, \tilde{\varphi}(t') \rangle = 0 \qquad \text{IF } t' > t .$$

★ WHAT HAPPENS IF $\alpha \neq 0$? WE DO PERTURBATION THEORY STARTING FROM THE GAUSSIAN CASE:

$$J = \int dt \left(\int d^d x \right) \Big\{ \underbrace{\tilde{\varphi}(\dot{\varphi} + r\varphi) - \tfrac{1}{2}\Omega \tilde{\varphi}^2}_{\text{GAUSS}} - \alpha \tilde{\varphi} \varphi^m + \underbrace{\alpha m \theta(0) \varphi^{m-1}}_{\text{FORGET IT FOR THE MOMENT}} \Big\} \qquad (\mathrm{II})$$

$$\langle O \rangle = \langle O\, e^{\int dt\, \alpha \tilde{\varphi} \varphi^m} \rangle_0 \rightsquigarrow \langle \tilde{\varphi}(t) \varphi^m(t) \dots \text{OTHER FIELD}(t') \rangle_0 .$$

WICK'S THEOREM APPLIES ON GAUSSIAN EXPECTATION VALUES.
THE CONTRACTIONS YOU GET BELONG TO TWO CLASSES,

INDEED,

$$\langle \mathcal{O} \rangle \rightsquigarrow \begin{cases} (a) & \underline{\langle \tilde{\varphi}(t)\varphi^m(t) \ldots \text{OTHER FIELD}(t') \rangle}_0 \\ (b) & m \underbrace{\langle \tilde{\varphi}(t)\varphi(t) \rangle_0}_{= \theta(0) \text{ BY (I)}} \langle \varphi^{m-1} \ldots \text{OTHERS} \rangle_0 . \end{cases}$$

WE NOTICE THAT THE WHOLE CLASS (b) CAN BE GENERATED BY

$$\langle \mathcal{O} e^{-\int dt \, \alpha m \theta(0) \varphi^{m-1}} \rangle_0$$

AND THIS CANCELS EXACTLY THE TERM WE "FORGOT FOR THE MOMENT".

FOR INSTANCE,

NOTE: THIS WILL NOT BE PROVED HERE.

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} e^{\int dt \, \alpha \tilde{\varphi} \varphi^m} \odot \rangle_0$$

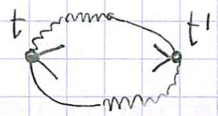
$$= \langle \mathcal{O} e^{\theta(0) \int dt \, \alpha m \varphi^{m-1}} \odot \rangle_0 + \langle \ldots \rangle_{(\text{NO SELF-CONTRACTIONS})}$$

$$\odot \equiv e^{-\theta(0) \int dt \, \alpha m \varphi^{m-1}}$$

SO THE DYNAMICAL FUNCTIONAL IS WELL-DEFINED.

NOTICE ALSO, CHOOSING $\theta(0) = 0$, VACUUM-VACUUM DIAGRAMS VANISH. THIS IMPLIES

$$\int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi} \, e^{-J[\varphi, \tilde{\varphi}]} = 1 .$$

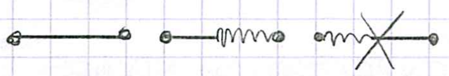


$\sim \theta(t-t')\theta(t'-t) = 0$

(IN THE GRAPH, WE CHOSE $\theta(0) = 0$; WITH OTHER CHOICES, YOU GET PAIRS OF OPPOSITE CONTRIBUTIONS WHICH CANCEL).

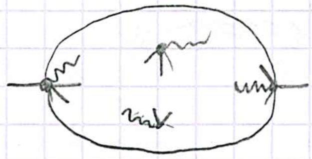
★ WHAT KIND OF DIAGRAMS DO WE HAVE?

AT ZERO ORDER,



NOTICE INTERACTION TERMS STEM FROM $\tilde{\varphi} F[\varphi]$, SO ALL VERTICES HAVE <u>AT LEAST</u> ONE $\tilde{\varphi}$ LEG.

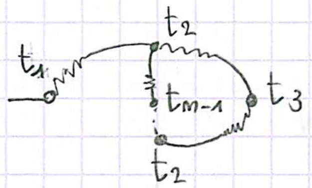
WHICH VERTICES DO WE FIND AT FIRST ORDER? NOTICE FIRST



$= 0.$

NOTE: THIS WORKS GENERALLY FOR m-POINTS AMPUTATED DIAGRAMS. HERE WE CONSIDER φ^2 WITH INSERTIONS IN BETWEEN $\varphi - \varphi$. SINCE A WHIRLY LINE CAN ONLY BE LINKED TO A SOLID LINE, IF I WANT TO VISIT ALL VERTICES I NECESSARILY CREATE A LOOP AT SOME POINT, THE ONLY SURVIVING CORRELATION FUNCTIONS ARE IN THE FORM $\varphi\tilde{\varphi}$

INDEED, IF I ENTER A LOOP



$$\theta(t_2 - t_{m-1}) \dots \theta(t_2 - t_3)\theta(t_3 - t_2) = \theta(0) = 0$$

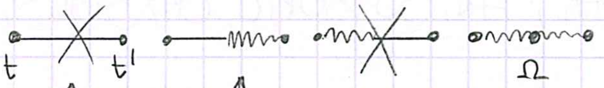
(LOOP OF TIME ORDERED POINTS).

NOTICE WE DID THIS HIDDENLY EVEN WITH THE DOI-PELITI FORMALISM: THE DISCRETIZATION WE CHOSE IMPLIES $\theta(0) = 0$.

IN TERMS OF EFFECTIVE ACTION, WE NOTICE THAT (II) DOES NOT CONTAIN φ^2.

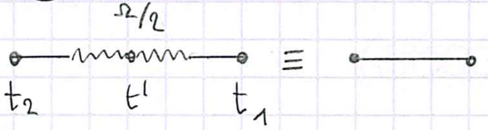
?

★ NOTICE INITIALLY THE MATRIX G^{-1} WOULD BE PERFECTLY WELL DEFINED EVEN IF $\Omega = 0$: I MAY THEN TREAT Ω PERTURBATIVELY. I WOULD GET



$C(t,t')$ $R(t,t')$

AND



$\equiv$

$$C(t,t') = \frac{\Omega}{2} \cdot 2 \int dt' \, R(t_1, t') R(t_2, t'). \qquad \text{(III)}$$

BUT WE HAVE SEEN THAT R IS TIME TRANSLATIONAL INVARIANT,

$$R(t,t') = \theta(t-t') e^{-r(t-t')}$$

SO (III) IS REALLY A CONVOLUTION AND IT BECOMES, IN FOURIER,

$$C(\omega) = \Omega \, R(\omega) R^*(\omega) \quad \leftrightarrow \quad \frac{1}{i\omega + r} \cdot \frac{1}{-i\omega + r}.$$

BEYOND MEAN FIELD

16.12.19

RENORMALIZABLE THEORY → IT HAS A WELL DEFINED EFFECTIVE THEORY (i.e. FINITE NUMBER OF COUPLINGS).

HOW DO WE KNOW THAT OUR THEORY IS RENORMALIZABLE?

· **BRANCHING AND DECAY WITH LETHAL COMPETITION**

$$\left.\begin{aligned} &(B)\quad A \xrightarrow{\sigma} 2A \\ &(D)\quad A \xrightarrow{\mu} \emptyset \\ &(C)\quad 2A \xrightarrow{\lambda} A \end{aligned}\right\} + \text{DIFFUSION}$$

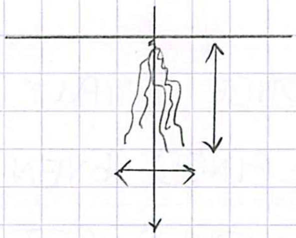
WHY DO WE FOCUS ON THIS PROBLEM? ACTUALLY, IT IS REPRESENTATIVE OF A UNIVERSALITY CLASS TO WHICH MANY OTHER PROBLEMS BELONG, SUCH AS

i) DIRECTED PERCOLATION

$$\xi_{\parallel} \sim |p - p_c|^{-\nu_{\parallel}}$$

$$\xi_{\perp} \sim |p - p_c|^{-\nu_{\perp}}$$

$$\frac{\nu_{\parallel}}{\nu_{\perp}} = 1 + \underset{\substack{\text{ANISOTROPY}\\ \text{COEFFICIENT}}}{\Delta}$$



NOTICE THIS IS DIFFERENT FROM, SAY, ANISOTROPIC CRYSTALS, WHERE

$$\xi_{\perp} \sim A_{\perp} |T - T_c|^{-\nu} \qquad \xi_{\parallel} \sim A_{\parallel} |T - T_c|^{-\nu}.$$

THIS IS WHY WE SPEAK, IN THE FIRST CASE, OF STRONG ANISOTROPY.

ii) INFECTION SPREADING (CONTACT PROCESS)

AND MANY OTHERS.

WE HAVE ALREADY SEEN (IT WAS PARTLY LEFT AS AN EXERCISE) THAT THIS BECOMES, IN THE DOI - PELITI FORMALISM,

$$A_T[\phi, \phi^*] = \int_0^T dt \int d^d x \left\{ \phi^*(\partial_t - D\nabla^2)\phi + (1 - \phi^*)\left[\sigma\phi\phi^* - \mu\phi - \lambda\phi^*\phi^2\right]\right\}$$

(IN THE BULK).

THE FIRST THING TO DO IS TO CHECK ITS MINIMUM, WHICH

MEANS ITS STATIONARY SOLUTION: BY SADDLE POINT (MEAN FIELD!).

$$\frac{\delta A_T}{\delta \phi^*}\Big|_{\bar\phi,\bar\phi^*} = 0 = (\partial_t - D\nabla^2)\bar\phi - (\sigma\bar\phi\bar\phi^* - \mu\bar\phi - \lambda\bar\phi^*\bar\phi^2) + (1-\bar\phi^*)(\dots)'$$

$$\frac{\delta A_T}{\delta \phi}\Big|_{\bar\phi,\bar\phi^*} = 0 \underset{\text{BY PARTS}}{=} (-\partial_t - D\nabla^2)\bar\phi^* + (1-\bar\phi^*)[\sigma\bar\phi^* - \mu - 2\lambda\bar\phi^*\bar\phi].$$

A POSSIBLE SOLUTION FOR THE SECOND EQUATION IS

$\bar\phi^* \equiv 1$ (CONST. IN SPACE & TIME)

(BEWARE OF ACTIVATION TRAJECTORIES! BUT THIS IS FINE, IF WE ARE NOT INTERESTED IN RARE EVENTS). THEN THE FIRST EQUATION GIVES

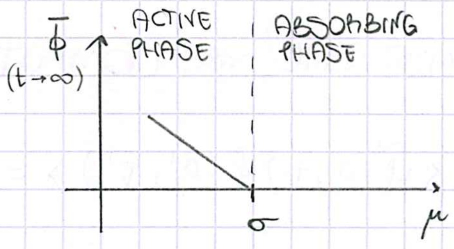
$$(\partial_t - D\nabla^2)\bar\phi - \sigma\bar\phi + \mu\bar\phi + \lambda\bar\phi^2 = 0$$

WHICH IS THE <u>RATE EQUATION</u>

$$\partial_t\bar\phi = D\nabla^2\bar\phi + (\sigma-\mu)\bar\phi - \lambda\bar\phi^2.$$

AS WE EXPECTED, WE RECOVERED IT IN MEAN FIELD. THIS CAN BE EXACTLY SOLVED AND WE GET

$$\bar\phi(t\to\infty) \sim \begin{cases} e^{-(\mu-\sigma)t} & \mu > \sigma \\ 1/\lambda t & \mu = \sigma \\ \bar\phi_S = \frac{\sigma-\mu}{\lambda} & \mu < \sigma \end{cases}$$



SO THAT $\bar\phi(t\to+\infty)$ CAN BE SEEN AS AN ORDER PARAMETER.

* IF WE GO BEYOND MEAN FIELD, WHAT IS THE EFFECT OF FLUCTUATIONS? IT DEPENDS ON THE DIMENSIONALITY.
FOR INSTANCE, IN $d=0$ ONCE YOU DIE, YOU ARE DEAD; IN $d=1$, DIFFUSION ALLOWS MIGRATION, WHICH ELIMINATES THE ABSORBING STATE.
IN LANDAU THEORY, THE INTRODUCTION OF FLUCTUATIONS DOES 2 THINGS
i) IT MODIFIES THE PHASE DIAGRAM (e.g. T_c) IN A NON-UNIVERSAL WAY
ii) IT MODIFIES THE CRITICAL SINGULARITIES IN A UNIVERSAL WAY.

THIS IS WHY WE CAN USE FIELD THEORY FOR THE SECOND.
BUT, FIRST OF ALL, WHAT IS THE LOWER CRITICAL DIMENSION FOR OUR MODEL?

* BEFORE WE PROCEED, LET'S CHANGE THE PARAMETRIZATION OF OUR FIELDS TO

$$\begin{cases} \phi^* = 1 + \sqrt{\frac{\lambda}{\sigma}}\,\tilde{\varphi} \\ \phi = \sqrt{\frac{\sigma}{\lambda}}\,\varphi . \end{cases}$$

THIS DOESN'T AFFECT AVERAGE VALUES CALCULATED VIA FIELD INTEGRATION; THE NEW ACTION BECOMES

$$A_T = \int dt\, d^dx \left\{ \tilde{\varphi}(\partial_t - D\nabla^2 + r)\varphi - \frac{u}{2}(\tilde{\varphi} - \varphi)\tilde{\varphi}\varphi + \lambda \tilde{\varphi}^2\varphi^2 \right\}$$

WHERE

$$\begin{cases} r \equiv \mu - \sigma \\ u \equiv 2\sqrt{\lambda\sigma} . \end{cases}$$

WE HAVE TO APPLY RENORMALIZATION GROUP TO THIS ACTION.
LAST TIME WE COMPUTED THE GAUSSIAN PROPAGATORS

$$\langle \tilde{\varphi}(q,t)\varphi(q',t')\rangle = (2\pi)^d \delta^d(q+q')\,\theta(t'-t)\, e^{-(Dq^2+r)(t'-t)} .$$

ADD A TIME SCALE τ, SO THAT

$$\tilde{\varphi}(\partial_t - D\nabla^2 + r)\varphi \rightarrow \tilde{\varphi}(\tau\partial_t - D\nabla^2 + r)\varphi$$

$$\langle \hat{\varphi}(q,t)\varphi(q',t')\rangle = (2\pi)^d \delta^d(q+q')\,\frac{1}{\tau}\theta(t'-t)\, e^{-(Dq^2+r)\frac{t'-t}{\tau}} .$$

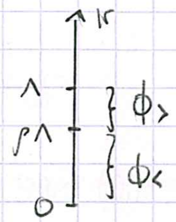
- <u>WILSON'S RG</u>

i) THE THEORY (A_T) IS DEFINED WITH A LARGE MOMENTUM CUTOFF $\Lambda \sim a^{-1}$. YOUR MOMENTUM SPACE IS LIMITED TO $|k| < \Lambda$.
WE CAN SPLIT ANY FIELD IN OUR THEORY INTO

$$\phi(k) = \phi_>(k) + \phi_<(k)$$

WHERE $\phi_>(k)$ FLUCTUATES A LOT AND $\phi_<(k)$ DOES NOT:

$$\phi_>(k) \equiv \begin{cases} 0 & \text{FOR } |k| < \rho\Lambda \\ \phi(k) & \text{FOR } \rho\Lambda < |k| < \Lambda \end{cases} \qquad \rho < 1$$



$$\phi_<(k) \equiv \begin{cases} \phi(k) & \text{FOR } |k| < \rho\Lambda \\ 0 & \text{FOR } \rho\Lambda < |k| < \Lambda . \end{cases}$$

AT THE GAUSSIAN LEVEL, $\phi_>$ AND $\phi_<$ ARE DECOUPLED ($\phi(k)$ ONLY TALKS TO $\phi(-k)$).

IMAGINE I'M INTERESTED IN A COARSE-GRAINED DESCRIPTION OF MY SYSTEM (i.e. THINGS THAT VARY ON A LONG LENGTH SCALE). THEN I WILL LOOK FOR

$$\langle O(\phi_<) \rangle = \int \mathcal{D}\phi \, O(\phi_<) \, e^{-S[\phi_> + \phi_<]} .$$

BUT IN FOURIER SPACE

$$\mathcal{D}\phi = \mathcal{D}\phi_< \mathcal{D}\phi_>$$

SO THAT ACTUALLY I CAN CALCULATE IN TWO STEPS

$$\langle O(\phi_<) \rangle = \int \mathcal{D}\phi_< \, O(\phi_<) \underbrace{\int \mathcal{D}\phi_> \, e^{-S[\phi_> + \phi_<]}}_{\equiv e^{-S_{eff}[\phi_<]}} .$$

WHAT WE DID IS

$$S \to S_{eff}$$

$$\Lambda \to \rho\Lambda < \Lambda$$

AND THE NEW THEORY WILL BE DIFFERENT FROM THE ORIGINAL.

ii) WE INTEGRATE OUT THE LARGE MOMENTUM COMPONENT $\phi_>$ TO DETERMINE $S_{eff}[\phi_<]$.

IMAGINE

$$S[\phi] = S_0[\phi] + S_{INT}[\phi]$$

WHERE S_0 IS GAUSSIAN, WHICH MEANS

$$S_0[\phi_> + \phi_<] = S_0[\phi_>] + S_0[\phi_<] .$$

THEN YOU CAN CHECK THAT, UP TO h.o. IN S_{INT},

$$S_{eff}[\phi_<] = S_0[\phi_<] + \langle S_{INT}[\phi_> + \phi_<] \rangle_{0,>} - \frac{1}{2} \langle S^2_{INT}[\phi_> + \phi_<] \rangle_{0,>} + h.o.$$

iii) WE RESCALE $\rho\Lambda \to \Lambda$.

TO THIS AIM, WE INTRODUCE

$$x' = \rho x$$

$$t' = \rho^z t$$

AND

$$\phi_<(x,t) = \rho^{\frac{d+\eta}{2}} \phi'(x',t')$$

NOTE: I THINK WHAT WE ARE DOING IS SENDING K IN $k' = k/\rho$.

NOTE: WE ARE USED TO $\rho^{\frac{1}{2}(d \pm 2 + \eta)}$. HERE AS WELL η DENOTES THE ANOMALOUS DIMENSION. THE FACT IS WE HAVE $D\nabla^2\varphi$, AND $[D] = \frac{l^2}{t}$.

TAILORED FOR OUR CASE (IN GENERAL IT'S THE GAUSSIAN DIMENSION).

ONE COULD ALLOW $\tilde{\eta} \neq \eta$ FOR THE FIELDS $\tilde{\varphi}$ AND φ.

THEN, AFTER RESCALING,

$$(\phi', x', t') \xrightarrow{\text{RENAMING}} (\phi, x, t) \quad \Rightarrow \quad \begin{cases} S \to S_{eff} \to S'_{eff} \\ \Lambda \to \rho\Lambda \to \Lambda. \end{cases}$$

☆ LET'S DO IT. WE START FROM THE <u>GAUSSIAN THEORY</u>

$$u = \lambda = 0.$$

SET (STEP (i))

$$\begin{cases} \tilde{\varphi} = \tilde{\varphi}_< + \tilde{\varphi}_> \\ \varphi = \varphi_< + \varphi_>. \end{cases}$$

NOTE: I THINK THE SUFFIX $\rho\Lambda$ IN THE INTEGRAL IS SHORT FOR "UP TO $\rho\Lambda$", BUT WE'RE NOT INTEGRATING IN K.

THEN WE HAVE SIMPLY

$$S_{eff}[\varphi_<, \tilde{\varphi}_<] \underset{(ii)}{=} S_0[\varphi_<, \tilde{\varphi}_<] = \int_{\rho\Lambda} dt\, d^dx\, \tilde{\varphi}_< (\tau\partial_t - D\nabla^2 + r)\varphi_<$$

$$\underset{(iii)}{=} \rho^{-z-d} \int_\Lambda dt'\, d^dx'\, \rho^{\frac{d+\eta}{2}\cdot 2}\, \tilde{\varphi}'(\tau\rho^z \partial_{t'} - D\rho^2 \nabla'^2 + r)\varphi'$$

IN GENERAL, $d + \frac{\eta + \eta'}{2}$

$$\overset{\text{RENAMING}}{=} \int_\Lambda dt\, d^dx\, \rho^\eta\, \tilde{\varphi}(\tau\partial_t - D\rho^{2-z}\nabla^2 + r\rho^{-z})\varphi$$

WHICH HAS INDEED THE SAME FORM AS BEFORE, BUT THE PARAMETERS CHANGED.

THE NEW ONES READ

$$\begin{cases} \tau' = \rho^{\eta} \tau \\ D' = \rho^{2-z+\eta} D \\ r' = \rho^{\eta - z} r \end{cases} .$$

SCALE INVARIANCE IS ACHIEVED FOR $r=0$ WITH $\eta=0$, $z=2$.
ANY $r \neq 0$ WOULD GROW UNBOUNDED UNDER RG.

* LET'S ADD INTERACTION. WE WILL FIND, IN OUR S_{eff}, A TERM

$$\int_{\rho\Lambda} dt\, d^d x\, u\, \tilde{\varphi}_<^2 \varphi_< \underset{\substack{\uparrow \\ (iii) \\ \text{AROUND THE GAUSSIAN FP} \\ z=2,\ \eta=0}}{=} \rho^{-2-d} \cdot u \cdot \rho^{\frac{d}{2}\cdot 3} \int_\Lambda dt' d^d x' \tilde{\varphi}'^2 \varphi'$$

WHENCE

$$\underline{u \to u' = \rho^{\frac{d}{2}-2} u .}$$

THIS MEANS THAT $u \to 0$ FOR $d > 4$. HENCE,

$d_{UCD} = 4$ (UPPER CRITICAL DIMENSION).

THE OTHER TERM, $u \varphi \tilde{\varphi} \varphi$, IS MACROSCOPICALLY THE SAME AS THE FIRST (INDEED, WE CHOSE $\tilde{\eta} = \eta$ FROM THE ONSET IN ORDER NOT TO SPOIL THIS SYMMETRY). FINALLY, THE λ TERM GIVES

$$\int_{\rho\Lambda} dt\, d^d x\, \lambda\, \tilde{\varphi}_<^2 \varphi_<^2 \underset{\substack{\text{AROUND} \\ \text{GAUSS}}}{=} \rho^{-2-d} \lambda\, \rho^{\frac{d}{2}\cdot 4} \int_\Lambda dt' d^d x' \tilde{\varphi}'^2 \varphi'^2$$

WHENCE

$$\underline{\lambda \to \lambda' = \rho^{d-2} \lambda}$$

i.e. λ IS IRRELEVANT FOR $d > 2$. FOR $2 < d < 4$, WE CAN SET $\lambda = 0$ (AT THE LEADING ORDER). THIS IS NOT BAD, BECAUSE WE LIVE IN $d = 3$. WE GET THE NEW ACTION

$$A_T = \int_\Lambda dt\, d^d x \left\{ \tilde{\varphi} (\tau \partial_t - D \nabla^2 + r) \varphi - \frac{u}{2} (\tilde{\varphi} - \varphi) \tilde{\varphi} \varphi \right\}$$

WHICH IS SYMMETRIC UNDER

$$\begin{cases} \varphi(x,t) \to -\tilde{\varphi}(x,-t) \\ \tilde{\varphi}(x,t) \to -\varphi(x,-t) \end{cases}$$

WHICH IS CALLED RAPIDITY REVERSAL.

<u>CONJECTURE</u>: ANY CONTINUOUS NONEQUILIBRIUM PHASE TRANSITION TOWARDS ONE ABSORBING STATE WHICH IS BUILT ON A MARKOVIAN DYNAMICS (AND NOTHING ELSE) SHOULD BE DESCRIBED BY THIS ACTION.

· 1 - LOOP RENORMALIZATION

18.12.19

LAST TIME WE CONSIDERED THE UNIVERSALITY CLASS OF CP/DP (CONTACT PROCESS / DIRECT PERCOLATION) WHOSE ACTION IS

$$A = \int dt\, d^d x \left\{ \underbrace{\tilde{\varphi}(\tau\partial_t - D\nabla^2 + r)\varphi}_{S_0} \underbrace{- \frac{u}{2}(\tilde{\varphi} - \varphi)\tilde{\varphi}\varphi}_{S_{INT}} \right\}$$

$$\begin{cases} r = \mu - \sigma \\ u = 2\sqrt{\lambda\sigma}. \end{cases} \qquad d > 2$$

WE USED THE FACT THAT THE INTEGRATION OVER THE FAST MODES $\phi_>$ GIVES

$$\int \mathcal{D}\phi_>\, e^{-S[\phi_> + \phi_<]} = e^{-S_{eff}[\phi_<]}$$

WHERE

$$S[\phi_> + \phi_<] = S_0 + S_{INT}$$

$$S_{eff}[\phi_<] = S_0[\phi_<] + \langle S_{INT}[\phi_> + \phi_<] \rangle_{0,>} - \frac{1}{2}\langle S_{INT}^2[\phi_> + \phi_<] \rangle_{0,>} + h.o. \ldots$$

* WE WANT TO CONVINCE OURSELVES THAT, WITH OUR CONVENTION, $\theta(0) = 0$.

RECALL WE DEFINED, IN THE DISCRETE, THE ADVANCED DERIVATIVE

$$(\partial^A f_i) = f_{i+1} - f_i \qquad f = (f_1, f_2, \ldots)$$

SO THAT IN MATRIX FORM

$$\partial^A = \begin{pmatrix} -1 & 1 & & 0 \\ & -1 & 1 & \\ & & -1 & 1 \\ 0 & & & \ddots \end{pmatrix} \xrightarrow{\text{INVERSE}} \theta^A = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & & 0 \end{pmatrix}.$$

ON THE CONTINUUM,

$$\theta(i - j) = \theta_{ij} \qquad \rightarrow \qquad \theta^A(0) = 0.$$

ON THE OTHER HAND, WE COULD HAVE DEFINED THE RETARDED

$$(\partial^R f_i) = f_i - f_{i-1} \qquad \rightarrow \qquad \partial^R = \begin{pmatrix} 1 & & 0 \\ -1 & 1 & \\ & -1 & \ddots \\ 0 & & \end{pmatrix}$$

WHOSE INVERSE IS

$$\mathcal{O}^R = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 1 & & 1 \end{pmatrix} \qquad \to \qquad \mathcal{O}^R(0) = 1 .$$

☆ WE HAD DERIVED

$$G_0^{-1} = \begin{pmatrix} 0 & \tau\partial_t - D\nabla^2 + r \\ -\tau\partial_t - D\nabla^2 + r & 0 \end{pmatrix}$$

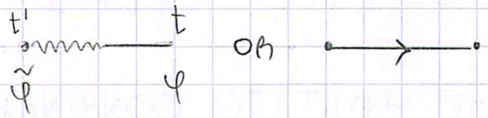
WHENCE

$$G_0 = \begin{pmatrix} 0 & \frac{1}{i\omega\tau + Dq^2 + r} \\ \frac{1}{-i\omega\tau + Dq^2 + r} & 0 \end{pmatrix} (2\pi)^d \delta^d(q+q') 2\pi \delta(\omega+\omega')$$

AND

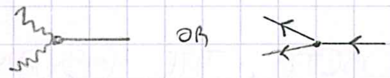
$$\langle \varphi(q,t)\underset{\substack{\uparrow \\ \text{PERTURBATION}}}{\tilde{\varphi}(q',t')} \rangle = \Theta(t-t') \frac{1}{\tau} e^{-\frac{1}{\tau}(Dq^2+r)(t-t')} (2\pi)^d \delta^d(q+q') \equiv R .$$

THIS IS THE ONLY FEYNMAN RULE WE HAVE AND IT IMPLIES A TRANSMUTATION $\tilde{\varphi} \to \varphi$.



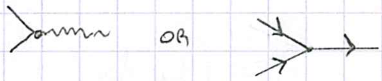
OUR VERTICES ARE

$\tilde{\varphi}^2\varphi$: $\frac{u}{2}$

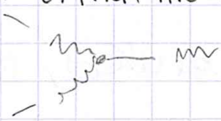


RR

$\tilde{\varphi}\,\varphi^2$: $-\frac{u}{2}$

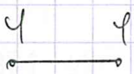


NOTE: FROM THE WIGGLE OUTWARDS, LIKE



'RR' STANDS FOR RAPIDITY REVERSAL.

(IN LANGEVIN THERE IS ALSO CORRELATION C:



, BUT HERE THERE IS NO AMBIGUITY).

CONSIDER THE FIRST:

$$-\frac{u}{2}\tilde{\varphi}^2\varphi \quad \to \quad -\frac{u}{2}\int dt\, d^dx\, \tilde{\varphi}^2(x,t)\varphi(x,t)$$

$$\to -\frac{u}{2}\langle(\tilde{\varphi}_< + \tilde{\varphi}_>)^2(\varphi_< + \varphi_>)\rangle_{0,>} = -\frac{u}{2}\tilde{\varphi}_<^2\varphi_< - \frac{u}{2} 2\tilde{\varphi}_< \langle \tilde{\varphi}_> (\varphi_< + \varphi_>)\rangle_{0,>}$$

NONE OF THESE CONTRIBUTES, BECAUSE FOR INSTANCE

$$\langle \tilde{\varphi}_> \varphi_> \rangle_{0,>} = \Theta(0) = 0 .$$

THE SAME HOLDS FOR THE OTHER, BECAUSE OF RAPIDITY REVERSAL. THEN

$$\langle S_{INT}[\phi_> + \phi_<] \rangle_{0,>} = -\frac{u}{2}(\tilde{\varphi}_< - \varphi_<)\tilde{\varphi}_< \varphi_<$$

$$\langle S^2_{INT}[\phi_> + \phi_<] \rangle_{0,>} = \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d q'}{(2\pi)^d} \int dt \int dt' \langle [\ \ ]_{x',t'} [\ \ ]_{x,t} \rangle_{0,>} .$$

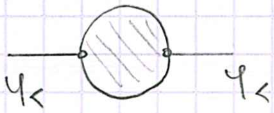
THIS IS USUALLY COMPUTED VIA FEYNMAN DIAGRAMS. HERE WE WRITE

$$\langle S^2_{INT}[\phi_> + \phi_<] \rangle_{0,>} = \tilde{\varphi}_< \Gamma^{(\overset{\#\tilde{\varphi}}{1},\overset{\#\varphi}{1})} \varphi_< + \underline{\tilde{\varphi}_<^2 \Gamma^{(2,0)}} + \underline{\Gamma^{(0,2)} \varphi_<^2}$$

$$+ \tilde{\varphi}_<^2 \Gamma^{(2,1)} \varphi_< + \tilde{\varphi}_< \Gamma^{(1,2)} \varphi_<^2 + \dots$$

NOTE: THESE Γ'S ARE VERTEX FUNCTIONS (SUM OF IRREDUCIBLE 1PI'S). THINK ALL OF THIS IN FOURIER SPACE.

WHERE THE LAST TWO TERMS ARE EQUAL BY RAPIDITY REVERSAL. THE RED OBJECT HAS A DIAGRAM LIKE



$= 0$ BY CAUSALITY.

THIS WAS ALSO TRUE FOR LANGEVIN: HENCE, IN GENERAL, $\Gamma^{(0,2)} = 0$.

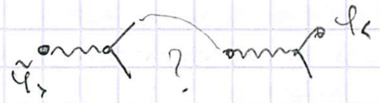
IN OUR SPECIFIC MODEL, RAPIDITY REVERSAL IMPLIES $\Gamma^{(2,0)} = 0$.

WE CONCLUDE THAT OUR EXPANSION IS STABLE UNDER RENORMALIZATION.

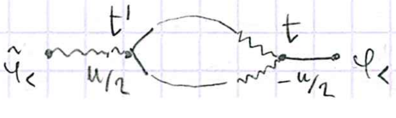
★ LET'S THEN EVALUATE

$\Gamma^{(1,1)} =$  .

WE TRY

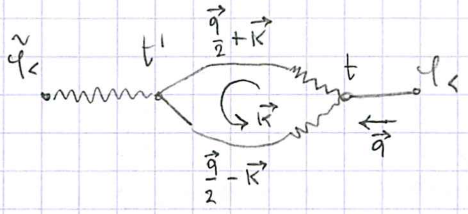


$\longrightarrow 0$



$\longrightarrow 2 \cdot \frac{u}{2} \cdot \left(-\frac{u}{2}\right)$

WE FOUND OUR FIRST 1-LOOP CONTRIBUTION TO $\Gamma^{(1,1)}$:



$$= -2\left(\frac{u}{2}\right)^2 \theta(t'-t)\,\frac{1}{\tau^2}\int_{\rho\Lambda<|k|<\Lambda} \frac{d^dk}{(2\pi)^d}\, e^{-\left[D\left(\frac{\vec{q}}{2}+\vec{k}\right)^2+r\right](t-t')/\tau}\, e^{-\left[D\left(\frac{\vec{q}}{2}-\vec{k}\right)^2+r\right]\frac{(t-t')}{\tau}}.$$

RECALL AT TREE LEVEL IN FOURIER SPACE

$$\Gamma^{(1,1)} = -i\omega\tau + Dq^2 + r + \dots$$

SO IT'S EASIER TO COMPUTE OUR LOOP INTEGRAL IN FOURIER:

$$-\frac{u^2}{2}\frac{1}{\tau}\int_{\rho\Lambda<|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\,\frac{1}{i\omega\tau + D\frac{q^2}{2} + 2Dk^2 + 2r}$$

$$= -\frac{u^2}{2}\frac{1}{\tau}\cdot\frac{\Omega_d}{(2\pi)^d}\int_{\rho\Lambda}^{\Lambda} dk\, k^{d-1}\,\frac{1}{i\omega\tau + Dq^2/2 + 2Dk^2 + 2r} = (*)$$

WHERE Ω_d IS THE SOLID ANGLE. CALLING

$$\rho = e^{-l},\quad l\to 0$$

WE GET

$$(*) = -\frac{u^2}{2}\frac{1}{\tau}\cdot\frac{\Omega_d}{(2\pi)^d}\, l\, \frac{\Lambda^d}{i\omega\tau + \frac{D}{2}q^2 + 2D\Lambda^2 + 2r}$$

AND RECALL THE SOLID ANGLE IS GIVEN BY

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}.$$

★ SINCE THIS IS THE ONLY NONZERO CONTRIBUTION, UP TO 1 LOOP

$$\Gamma^{(1,1)}(q,\omega) = i\omega\tau + Dq^2 + r + \frac{u^2}{2}K_d\frac{\Lambda^d}{\tau}\, l\, \frac{1}{i\omega\tau + D\frac{q^2}{2} + 2D\Lambda^2 + 2r} + O(l^2, u^4)$$

WHERE

$$K_d = \frac{\Omega_d}{(2\pi)^d} = \frac{2}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}\right)}.$$

DEFINE

$$r' \equiv \Gamma^{(1,1)}(q=0, \omega=0)$$

NOTE: WHY THE HELL DEFINE r' LIKE THIS?
SEE CAVAGNA FOR CLARITY, BUT:
1) ON-SHELL DIAGRAMS MAY STILL DEPEND ON INCOMING MOMENTUM.
2) IF Λ IS LARGE AND WE WORK WITH $q \ll \Lambda$, SETTING $q=0$ PRODUCES AN ERROR OF $O(\varepsilon = d_{ucd} -$

WHICH GIVES

$$r' = r + \underbrace{\frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} \cdot \frac{1}{2D\Lambda^2 + 2r}}_{\equiv r S_2} \, l \left(= \Gamma^{(1,1)}(q=0, \omega=0) \right).$$

SIMILARLY, DEFINE

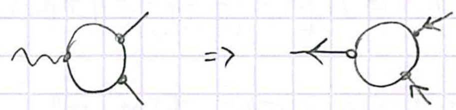
NOTE: AND NOW WTF IS HE DOING? ON-SHELL DIAGRAMS CORRECT S_{eff}; THIS IS JUST A WAY TO EXTRACT THE COUPLINGS.

$$D' = \frac{\partial}{\partial q^2} \Gamma^{(1,1)}(q,\omega)\Big|_{q,\omega=0} = D - \underbrace{\frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} \frac{D}{2} \frac{1}{(2D\Lambda^2 + 2r)^2}}_{\equiv S_1} \, l .$$

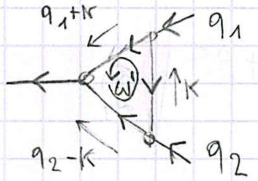
$$\tau' = \frac{\partial}{\partial (i\omega)} \Gamma^{(1,1)}(q,\omega)\Big|_{q,\omega=0} = \tau - 2 S_1 \tau \, l .$$

NOTE: THEY MUST COINCIDE, AT 0^{TH} ORDER, WITH THE BARE COUPLINGS.

* LET'S NOW CHECK THE LOOP CONTRIBUTION TO $\Gamma^{(1,2)}$, i.e.



YOU COULD CHECK THERE IS ONLY ONE:



$+ (q_1 \leftrightarrow q_2)$

FOR $q_i \to 0$, $\omega_i \to 0$ $(i = 1, 2)$ THIS AMOUNTS TO

$$\int \frac{d\omega'}{2\pi} \int_{>} \frac{d^d k}{(2\pi)^d} \frac{1}{i\omega'\tau + D(q_1+k)^2 + 2r} \cdot \frac{1}{-i\omega'\tau + D(q_2-k)^2 + 2r} \cdot \frac{1}{-i\omega'\tau + Dk^2 + 2r},$$

NOTICE THE PROPAGATOR IS DIFFERENT DEPENDING ON THE RELATIVE SIGN OF MOMENTA AND ω'.

THIS IS A COMPLEX INTEGRAL WHICH GIVES, AFTER WE HAVE SET $q_1 = q_2 = 0$,

$$\int_{>} \frac{d^d k}{(2\pi)^d} \frac{1}{\tau} \frac{1}{(2Dk^2 + 2r)^2} \simeq K_d \frac{\Lambda^d}{\tau} \, l \, \frac{1}{(2D\Lambda^2 + 2r)^2}.$$

RECALL

$$\langle S_{INT} \rangle_{0,>} = -\frac{u}{2} (\tilde{\varphi}_< - \varphi_<) \tilde{\varphi}_< \varphi_<$$

$$\Gamma^{(1,2)} = \frac{u}{2} - \frac{1}{2} \left(\frac{u}{2}\right) \left(-\frac{u}{2}\right)^2 \cdot 8 \cdot 2 \cdot K_d \frac{\Lambda^d}{\tau} \, l \, \frac{1}{(2D\Lambda^2 + 2r)^2}$$

SO WE UNDERSTAND THAT

$$u' = u - \underbrace{2u^3 K_d \frac{\Lambda^d}{\tau} \ell \frac{1}{(2D\Lambda^2+2r)^2}}_{= 8u S_1 \ell}$$

* NOW WE HAVE TO RESCALE

$$\tau' \to \tau'' = \rho^{\eta} \tau' \underset{\ell\to 0}{\simeq} (1-\eta\ell)\tau' \simeq \tau\left[1-(\eta+2S_1)\ell\right]$$

$$D' \to D'' = \rho^{2-z+\eta} D' \simeq \left[1-(2-z+\eta)\ell\right] D' \simeq D\left[1-(2-z+\eta+S_1)\ell\right]$$

$$r' \to r'' = \rho^{\eta-z} r' \simeq \left[1-(\eta-z)\ell\right] r' \simeq r\left[1-(\eta-z-S_2)\ell\right]$$

$$u' \to u'' = \rho^{\frac{d}{2}-z+\frac{3}{2}\eta} \simeq \left[1-\left(\frac{d}{2}-z+\frac{3}{2}\eta\right)\ell\right] u' \simeq u\left[1-\left(\frac{d}{2}-z+\frac{3}{2}\eta+8S_1\right)\ell\right]$$

USING THE EXPRESSIONS WE HAVE DERIVED UP HERE FOR τ'',
WE CAN INTERPRET AS A DIFFERENTIAL EQUATION

$$\underset{\tau(\ell)}{\underset{\uparrow}{\tau''}} \simeq \underset{\tau(\ell=0)}{\underset{\uparrow}{\tau}}\left[1-(\eta+2S_1)\ell\right] \quad \to \quad \partial_\ell \tau(\ell) = -(\eta+2S_1)\tau$$

AND SIMILARLY FOR THE OTHERS:

$$\partial_\ell D(\ell) = -(2-z+\eta+S_1) D(\ell)$$

$$\partial_\ell r(\ell) = -(\eta-z-S_2) r(\ell)$$

$$\partial_\ell u(\ell) = -\left(\frac{d}{2}-z+\frac{3}{2}\eta+8S_1\right) u(\ell)$$

THESE ARE <u>RG EQUATIONS</u> FOR THE FLOW OF OUR PARAMETERS.

· RECAP: RG FLOW 19.12.19

$$\partial_\ell \tau(\ell) = -\tau(\ell)(\eta + 2\mathcal{S}_1)$$
$$\partial_\ell D(\ell) = -D(\ell)(2 - z + \eta + \mathcal{S}_1)$$
FROM $\Gamma^{(1,1)}$
$$\partial_\ell r(\ell) = -r(\ell)(\eta - z - \mathcal{S}_2)$$
$$\partial_\ell u(\ell) = -u(\ell)\left(\frac{d}{2} - z + \frac{3}{2}\eta + 8\mathcal{S}_1\right) \rightarrow \text{FROM } \Gamma^{(2,1)}, \Gamma^{(1,2)}$$

WHERE

$$\mathcal{S}_1 \equiv \frac{u^2}{4} K_d \frac{\Lambda^d}{\tau} \frac{1}{(2D\Lambda^2 + 2r)^2}$$

$$\mathcal{S}_2 \equiv \frac{u^2}{2} K_d \frac{\Lambda^d}{\tau} \frac{1}{2D\Lambda^2 + 2r} \cdot \frac{1}{r}.$$

THE EFFECTIVE CUTOFF IS $\Lambda e^{-\ell}$, SO THAT FOR $\ell = 0$ WE HAVE $(\tau, D, r, u)(0)$.

WE CHOOSE TO REQUIRE

$$\partial_\ell \tau(\ell) = 0 \quad \Rightarrow \quad \eta = -2\mathcal{S}_1$$

$$\partial_\ell D(\ell) = 0 \quad \Rightarrow \quad z = 2 + \eta + \mathcal{S}_1 = 2 - \mathcal{S}_1.$$

IF YOU DON'T DO THIS, YOU TRIVIALIZE THE THEORY.

THERE ARE NO OTHER CHOICES TO MAKE: IT FOLLOWS THAT

$$\partial_\ell r(\ell) = r(\ell)(2 + \mathcal{S}_1 + \mathcal{S}_2)$$

$$\partial_\ell u(\ell) = -u(\ell)\left(\frac{\varepsilon}{2} - 6\mathcal{S}_1\right) \qquad \varepsilon \equiv 4 - d.$$

NOTICE WITH THIS CHOICE τ AND D REMAIN THE ONES WE INSERTED AT THE VERY BEGINNING (THEY WOULD PROBABLY HAVE FLOWN TO $\pm\infty$ OTHERWISE). WHAT STILL FLOWS ARE THE COUPLINGS

· DIMENSIONLESS PARAMETERS

$$\tilde{r} \equiv \frac{r}{D\Lambda^2} \qquad\qquad g \equiv K_d \frac{u^2}{4} \frac{\Lambda^d}{\tau} \frac{1}{(2D\Lambda^2)} = \frac{u^2}{6D^2\tau} K_d \Lambda^{d-4}$$

(NOTICE g LOSES ITS DEPENDENCE ON Λ FOR $d = 4$).

THEIR FLOW IS GIVEN BY

$$\begin{cases} \partial_\ell \tilde{r} = \tilde{r}(\ell)(2+\mathcal{S}_1+S_2) \\ \partial_\ell g = g(\ell)(\varepsilon - 12\mathcal{S}_1). \end{cases}$$

BUT NOTICE WITH THIS CHOICE

$$\mathcal{S}_1 = \frac{g}{(1+\tilde{r})^2}$$

$$S_2 = 4\frac{g}{(1+\tilde{r})\tilde{r}}$$

WHICH IS A CLOSED SYSTEM OF EQUATIONS.

<u>FIXED POINTS</u>:

$$\begin{cases} g^* = 0 \\ \tilde{r}^* = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \eta^* = 0 \\ z^* = 2 \end{cases} \quad \text{GAUSSIAN FP.}$$

THE OTHER POSSIBILITY IS

$$\begin{cases} \mathcal{S}_1^* = \varepsilon/12 \\ S_2^* = -2 - \mathcal{S}_1^* = -2 - \varepsilon/12 \end{cases} \quad \Rightarrow \quad \begin{cases} \eta^* = -\frac{\varepsilon}{6} \\ z^* = 2 - \frac{\varepsilon}{12} \end{cases} \quad \text{WILSON-FISHER}$$

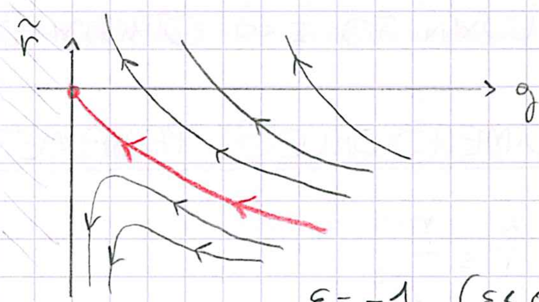
WHICH MEANS

$$\begin{cases} \dfrac{g^*}{(1+\tilde{r}^*)^2} = \dfrac{\varepsilon}{12} \\ 4\dfrac{g^*}{(1+\tilde{r}^*)\tilde{r}^*} = -2 - \dfrac{\varepsilon}{12}. \end{cases} \quad \Rightarrow \quad \begin{cases} g^* \simeq \dfrac{\varepsilon}{12} \\ \tilde{r}^* \simeq -2g^* = -\dfrac{\varepsilon}{6} \end{cases}$$

(AT ORDER ε, WE CAN DISCARD THE DENOMINATORS). NOTICE $\tilde{r}^*$ IS NEGATIVE (i.e. IT DESTABILIZES ORDER: $\tilde{r}^* = 0$ AT THE FIXED POINT).

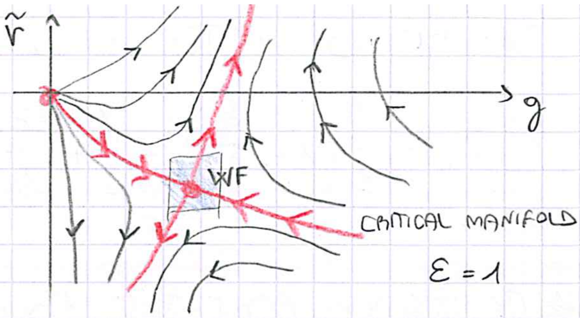
WE GET THE PLOT ON THE RIGHT FOR $d = 5$.

NOTICE THE 2^{ND} F.P. IS IN THE UNPHYSICAL REGION.



$\varepsilon = -1 \quad (\varepsilon < 0)$

IF INSTEAD $\varepsilon > 0$, WE HAVE BOTH FIXED POINTS IN THE PHYSICAL REGION. THIS TIME THE WF-FP IS STABLE, AND WE FOUND ITS CRITICAL EXPONENTS



$$\begin{cases} \eta^* = -\dfrac{\varepsilon}{6} \\ z^* = 2 - \dfrac{\varepsilon}{12} \end{cases}.$$

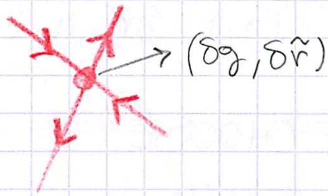
BUT THE CRITICAL EXPONENT ν IS STILL MISSING. INTRODUCE

$$\delta \tilde{r} \equiv \tilde{r} - \tilde{r}^*$$

$$\delta g \equiv g - g^*$$

AND LINEARIZE THE FLOW AROUND THE FP. YOU GET

$$\partial_\ell \begin{pmatrix} \delta \tilde{r} \\ \delta g \end{pmatrix} = \begin{pmatrix} 2 - \frac{\varepsilon}{4} & 4 + \frac{\varepsilon}{2} \\ 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} \delta \tilde{r} \\ \delta g \end{pmatrix}.$$



ITS EIGENVECTORS ARE

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ (UNSTABLE)} \quad \rightarrow \quad \partial_\ell u = Mu = \left(2 - \frac{\varepsilon}{4}\right) u \equiv \lambda_u u$$

$$s = \begin{pmatrix} a \\ b \end{pmatrix} \text{ (STABLE)} \quad \rightarrow \quad \partial_\ell s = Ms = -\varepsilon s.$$

ALONG THE UNSTABLE DIRECTION, HOW IS $\delta \tilde{r}$ MAGNIFIED?

$$\delta r(\ell) = \delta r(0)\, e^{\lambda_u \ell}$$

$$= \delta r(0)\, \rho^{-\lambda_u} \equiv \delta r' \qquad (\rho = e^{-\ell}).$$

BUT REMEMBER A CHANGE IN LENGTH WAS GIVEN BY

$$x' = \rho x$$

HENCE WE CAN REWRITE

$$(\delta r')^{-1/\lambda_u} = (\delta r)^{-1/\lambda_u} \rho$$

NOTE: THE NATURAL SCALE OF x AROUND A FIXED POINT IS ξ.

AND WE CONCLUDE THAT THE QUANTITY $\dfrac{x}{(\delta r)^{-1/\lambda_u}}$ IS INVARIANT.

WE IDENTIFY THIS WITH THE NATURAL LENGTH SCALE

$$\xi_\perp \equiv (\delta r)^{-1/\lambda_u} \quad \rightarrow \quad \nu_\perp \equiv \frac{1}{\lambda_u}.$$

WE FOUND, UP TO $O(\varepsilon)$,

$$\nu_\perp = \frac{1}{2}\left(1+\frac{\varepsilon}{8}\right) = \frac{1}{2}+\frac{\varepsilon}{16} .$$

★ CONSIDER NOW THE PROPAGATOR

$$\langle \varphi(x,t)\tilde{\varphi}(0,0)\rangle \equiv G_R^{(1,1)}(x,t;\delta r, \overset{\nearrow u^*}{u},\tau,D)$$

WHAT HAPPENS TO IT AFTER THE FLOW?

$$\rightarrow \rho^{\frac{d+\eta}{2}\cdot 2}\langle \varphi(\rho x,\rho^z t)\tilde{\varphi}(0,0)\rangle\Big|_{\substack{\delta r\to \rho^{-\lambda_u}\delta r\\ u^*,\tau,D}} .$$

BUT THEY HAVE TO BE THE SAME (CALCULATED IN DIFFERENT POINTS), SO THAT WE CAN WRITE

$$G_R^{(1,1)}(x,t;\delta r,u,\tau,D) = \rho^{d+\eta}\, G_R^{(1,1)}(\rho x,\rho^z t;\delta r\rho^{-\lambda_u}, u^*,\tau,D).$$

IF WE CHOOSE

$$\rho \equiv (\delta r)^{1/\lambda_u}$$

THIS BECOMES

$$G_R^{(1,1)}(x,t;\delta r,u,\tau,D) = (\delta r)^{\frac{d+\eta}{\lambda_u}}\, G_R^{(1,1)}\left((\delta r)^{1/\lambda_u}x,(\delta r)^{z/\lambda_u}t;1,u^*,\tau,D\right)$$

$$= (\xi_\perp)^{-(d+\eta)\nu_\perp}\, F\left(\frac{x}{\xi_\perp},\frac{t}{\xi_\perp^z}\right) \qquad \text{(SCALING)}$$

AND WE ALSO IDENTIFIED (ANISOTROPY)

$$\xi_\parallel = \xi_\perp^z \equiv |\delta r|^{-\nu_\parallel} \qquad \Rightarrow \qquad \nu_\parallel = 2\nu_\perp .$$

★ WHAT IS THE MEANING OF $G_R^{(1,1)}$? IMAGINE

$$|\Phi\rangle \xrightarrow[\text{PARTICLE IN } (x',t')]{\text{ADD A}} a^+(x',t')|\Phi\rangle$$

$$\xrightarrow[N(x,t)=a^+(x,t)a(x,t)]{\text{MEASURE}} m_\rho(x,t) = \langle P|N(x,t)|\Phi'\rangle$$

$$= \langle P|a^+(x,t)a(x,t)\,a^+(x',t')|\Phi\rangle = \langle P|a(x,t)a^+(x',t')|\Phi\rangle$$

WITHIN A FIELD THEORETICAL APPROACH, THIS SAME OBJECT IS

$$\langle \phi(x,t)\,\phi^*(x',t')\rangle \underset{\phi^* = 1+\tilde{\phi}}{=} \underbrace{\langle \phi(x,t)\rangle}_{= m(x,t) \text{ in } |\Phi\rangle} + \langle \phi(x,t)\,\tilde{\phi}(x',t')\rangle$$

SO THE RESPONSE FUNCTION IS INDEED THE RESPONSE OF THE SYSTEM IN TERMS OF PARTICLE NUMBER IF I ADD A PARTICLE.