Critical points of coupled vector-Ising systems. Exact results

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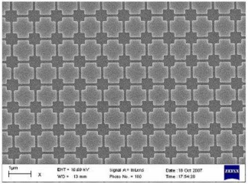
20 March 2020

Physical motivation

Josephson junction array in a transverse magnetic field $\mathbf{B}$:



$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j + A_{ij})$$

where θ_i is the phase of the superconducting order parameter, $\mathbf{B} = \nabla \times \mathbf{A}$ and

$$A_{ij} = \frac{2\pi}{\phi_0} \int_i^j \mathbf{A} \cdot \mathrm{d}\mathbf{l} \, .$$

Define the uniform frustration f as the number of flux quanta per plaquette:

$$f \equiv \frac{1}{2\pi} \sum_{\texttt{plaquette}} A_{ij} = \frac{Ba^2}{\phi_0}$$

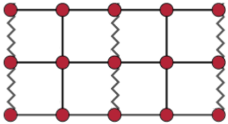
($a =$ lattice spacing). Full frustration: $f = \frac{1}{2}$.

FFXY model

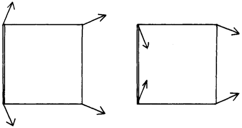
Choosing $A_{ij} \in \{0, \pi\}$ and identifying $J_{ij} \equiv J \cos(A_{ij}) = \pm J$,

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \cos(\theta_i - \theta_j) = -\sum_{\langle i,j \rangle} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j$$

fully frustrated when
$\prod J_{ij} < 0$, $\forall$ plaquette.



The model admits two ground states with opposite chirality: $O(2) \otimes Z_2$ symmetry.
We expect two kinds of transition (Ising, BKT).



$\rightarrow$ From simulations, it is not clear whether $T_{BKT} < T_c$ or $T_{BKT} = T_c$ and what are the exact values of critical exponents.

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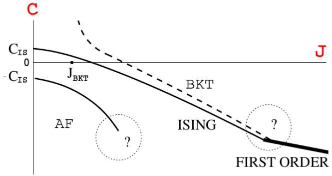
XY-Ising model

The same ground state symmetry is shared by the XY-Ising model

$$\mathcal{H} = -\sum_{\langle i,j \rangle} \left\{ \frac{J}{2} (1 + \sigma_i \sigma_j) \mathbf{s}_i \cdot \mathbf{s}_j + C \sigma_i \sigma_j \right\}$$

- Ising transition for $J = 0$, $C = C_{IS} = \frac{1}{2} \log(1 + \sqrt{2}) \simeq 0.44$
- BKT transition for $C = \infty$ and $J = J_{BKT} \simeq 1.12$
- A single first order transition for $C \lesssim -4$



We expect $T_{BKT} \leq T_c$ because, due to the term $(1 + \sigma_i \sigma_j)$, spins $\mathbf{s}_i$,$\mathbf{s}_j$ belonging to different Ising correlated clusters are decoupled: *Ising disorder induces XY disorder.*

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Coupled vector-Ising systems

On universality grounds, one can generically consider

$$\mathcal{H} = -\sum_{\langle i,j \rangle} \{ (A + B\sigma_i\sigma_j)\, \mathbf{s}_i \cdot \mathbf{s}_j + C\sigma_i\sigma_j \}$$

where $\sigma_i = \pm 1$ and $\mathbf{s}_i$ is a N-component unit vector.

- The models we have seen so far correspond to different initial points in its parameter space (A, B, C) and $N = 2$
- We look for the points of simultaneous $O(N)$ and Z_2 criticality, where

$$\langle \mathbf{s}_i \cdot \mathbf{s}_j \rangle \sim |i-j|^{-2X_s} \quad , \quad \langle \sigma_i \cdot \sigma_j \rangle \sim |i-j|^{-2X_\sigma} \quad .$$

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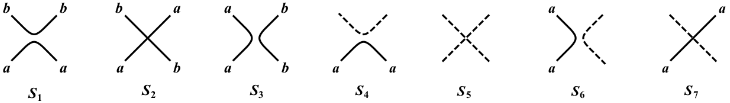
Relativistic QFT approach

- At fixed points, scale invariance allows us to adopt a continuum description corresponding to a Euclidean field theory ($d = 2$)
- This is the continuation in imaginary time of a (1+1) dimensional relativistic QFT, whose collective excitation modes are described in terms of massless particles
- Relativistic + scale invariance
  $\Rightarrow$ conformal invariance $\Rightarrow$ ∞ generators
  $\Rightarrow$ ∞ conserved quantities
  $\Rightarrow$ completely elastic particle scattering



Scattering amplitudes

Since the center of mass energy $\sqrt{s}$ is dimensionful, then the S-matrix is just a constant phase.
Crossing equations take the simplified form

$$S_{ab}^{cd} = (S_{\bar{d}a}^{\bar{b}c})^*$$

$$\begin{aligned} S_1 &= S_3^* \equiv \rho_1 e^{i\phi_1} \\ S_2 &= S_2^* \equiv \rho_2 \\ S_4 &= S_6^* \equiv \rho_4 e^{i\theta_4} \\ S_5 &= S_5^* \equiv \rho_5 \\ S_7 &= S_7^* \equiv \rho_7 \end{aligned}$$

while unitarity equations give

$$\sum_{ef} S_{ab}^{ef} (S_{cd}^{ef})^* = \delta_{ac}\delta_{bd}$$

$$\begin{aligned} &\rho_1^2 + \rho_2^2 = 1 \\ &\rho_1 \rho_2 \cos\phi_1 = 0 \\ &N\rho_1^2 + \rho_4^2 + 2\rho_1^2 \cos 2\phi_1 = 0 \\ &\rho_4^2 + \rho_7^2 = 1 \\ &N\rho_4^2 + \rho_5^2 = 1 \\ &\rho_4 \rho_7 \cos\theta_4 = 0 \\ &\rho_4 [\rho_2 e^{-i\theta_4} + \rho_1 e^{-i(\theta_4+\phi_1)} + \\ &+ N\rho_1 e^{i(\phi_1-\theta_4)} + \rho_5 e^{i\theta_4}] = 0 \end{aligned}$$

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Type D solutions

Solution	N	ρ_2	$\cos\phi_1$	ρ_4	$\cos\theta_4$	ρ_5
$D1_\pm$	$\mathbb{R}$	± 1	—	0	—	$(\pm)1$
$D2_\pm$	$[-2,2]$	0	$\pm\frac{1}{2}\sqrt{2-N}$	0	—	$(\pm)1$
$D3_\pm$	2	$\pm\sqrt{1-\rho_1^2}$	0	0	—	$(\pm)1$

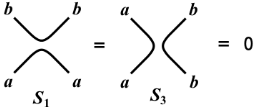
$\rho_1 = \sqrt{1-\rho_2^2}$, $\rho_7 = (\pm)\sqrt{1-\rho_4^2}$

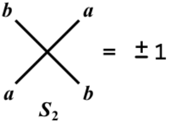
In this class of solutions, vector and scalar are decoupled:

Scattering in $d = 2$ involves position exchange and mixes statistics with interaction.
We recognize, for the scalar sector,

- $S_5 = -1$: Ising criticality (neutral free fermion)
- $S_5 = 1$: trivial fixed point (free boson)

Type $D1_\pm$ solutions: free bosons/fermions





- Focus on the vector part. We know $O(N > 2)$ ferromagnets exhibit in $d = 2$ a $T = 0$ critical point whose scaling properties are described by

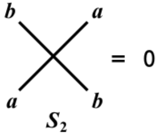
$$\mathcal{A}_{SM} = \frac{1}{T} \sum_{j=1}^{N} \int \mathrm{d}^2 x \, (\nabla \varphi_j)^2 \ , \ \sum_{j=1}^{N} (\varphi_j)^2 = 1$$

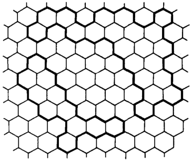
  This theory is asymptotically free: the short distance fixed point is a theory of free bosons.

- For $S_2 = 1$ (solution $D1_+$) and $S_5 = 1$, we retrieve the $T = 0$ critical point of the $O(N+1)$ model.

Type D2 solutions

In this class of solutions, the vector part corresponds to nonintersecting trajectories. Recall the partition function of the $O(N)$ model in the form





$$\mathcal{H} = -\sum_{\langle i,j \rangle} \log(1 + x\mathbf{s}_i \cdot \mathbf{s}_j)$$

can be expressed as a sum over loop configurations

$$\mathcal{Z} = \sum_{\text{configs.}} x^{\#\text{bonds}} N^{\#\text{loops}}$$

- SAW for $N \to 0$
- Critical lines of the gas of nonintersecting planar loops in its dense and dilute regimes for $N \in [-2, 2]$

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Type D3 solutions: BKT line

- This class of solutions is only defined for $N = 2$ and contains ρ_1 as a free parameter
- Gaussian field theory in $d = 2$:

$$\mathcal{A}_{Gauss} = \frac{1}{4\pi} \int \mathrm{d}^2 x \, (\boldsymbol{\nabla} \varphi)^2 \, .$$

  Choosing the energy density field

$$\epsilon(x) = \cos(2b\varphi) \quad , \quad X_\epsilon = 2b^2$$

  we know b^2 parameterizes a line of fixed points.
- Letting $z, \bar{z} = x_1 \pm i x_2$, the equations of motion give

$$\partial\bar{\partial}\varphi = 0 \quad \Rightarrow \quad \varphi(x) \equiv \phi(z) + \bar{\phi}(\bar{z})$$

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Type D3 solutions: BKT line

- Let's introduce, for integer m, the set of fields

$$U_m(x) = e^{i\frac{m}{2b}[\phi(z)-\bar{\phi}(\bar{z})]} \equiv e^{im\frac{\tilde{\varphi}}{2b}} \quad , \quad X_m = \frac{m^2}{8b^2}$$

  mutually local wrt $\epsilon(x)$; here $\frac{\tilde{\varphi}}{2b}$ is the $O(2)$ angular variable, so that

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad \Rightarrow \quad U_{\pm 1} = s_1 \pm i s_2 \quad \Rightarrow \quad X_s = X_{(m=1)} = \frac{1}{8b^2} \; .$$

- Studying the scattering process

$$\sum_a aa \to S \sum_a aa \quad \Rightarrow \quad 2S_1 + S_2 + S_3 \equiv e^{-2\pi i X_\eta}$$

  where η must be the most relevant chiral field local wrt ϵ, we have $\eta = e^{i\frac{\phi}{b}}$, $X_\eta = \frac{1}{4b^2}$.

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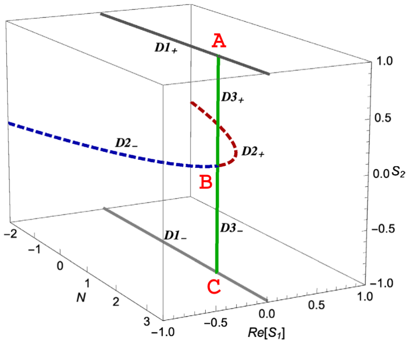
Type D solutions

This gives the mapping $\rho_2 = \cos\left(\frac{\pi}{2b^2}\right)$.
In particular:

- A: $b^2 = \infty$, $T = 0$ critical point
- B: $b^2 = 1$, BKT transition point
- C: $b^2 = \frac{1}{2}$, free fermion!



This is clear in the fermionic formulation of the Gaussian fixed point, where $g(b^2 = \frac{1}{2}) = 0$:

$$\mathcal{A}_{\mathcal{T}} = \int \mathrm{d}^2 x \left\{ \sum_{i=1}^{2} (\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i) + g(b^2) \psi_1 \bar{\psi}_1 \psi_2 \bar{\psi}_2 \right\}$$

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Type F solutions: Ashkin-Teller model

Solution	N	ρ_2	$\cos\phi_1$	ρ_4	$\cos\theta_4$	ρ_5
F1	1	0	$[-\frac{1}{2},\frac{1}{2}]$	$\sqrt{1-4\cos^2\phi_1}$	0	$2\cos\phi_1$
F2	1	$[-1,1]$	0	$\sqrt{1-\rho_2^2}$	0	ρ_2
F3	1	0	0	1	$[-1,1]$	0

$\rho_1=\sqrt{1-\rho_2^2}\ ,\quad \rho_7=(\pm)\sqrt{1-\rho_4^2}$

- These are all defined for $N=1$. Our Hamiltonian

$$\mathcal{H}=-\sum_{\langle i,j\rangle}\left\{\left(A+B\sigma_i\sigma_j\right)s_is_j+C\sigma_i\sigma_j\right\}$$

becomes that of the Ashkin-Teller model. Choosing $A=C\equiv J$ and $B\equiv J_4$ we get its isotropic version

$$\mathcal{H}_{AT}=-\sum_{\langle i,j\rangle}\left\{J(\sigma_i\sigma_j+s_is_j)+J_{\mathbf{4}}\sigma_i\sigma_js_is_j\right\}\ .$$

- A-T is a generalization of the Ising model where each site of a lattice is occupied by one of four kinds of atom $(\alpha,\beta,\gamma,\delta)$ with interaction energy ϵ_0 (equal atoms), ϵ_1 $(\alpha\beta,\gamma\delta)$, ϵ_2 $(\alpha\gamma,\beta\delta)$, ϵ_3 $(\alpha\delta,\beta\gamma)$.

Ashkin-Teller model

- We can associate to each site *two* spins $\binom{s_i}{\sigma_i}$:

$$\alpha \equiv \binom{+}{+}\ ,\ \beta \equiv \binom{+}{-}\ ,\ \gamma \equiv \binom{-}{+}\ ,\ \delta \equiv \binom{-}{-}$$

- The interaction energy for the edge (i, j) becomes

$$\epsilon(i,j) = -J s_i s_j - J' \sigma_i \sigma_j - J_4 s_i \sigma_i s_j \sigma_j - J_0$$

$$J = -\frac{1}{4}(\epsilon_0 + \epsilon_1 - \epsilon_2 - \epsilon_3) \quad J_4 = -\frac{1}{4}(\epsilon_0 - \epsilon_1 - \epsilon_2 + \epsilon_3)$$

$$J' = -\frac{1}{4}(\epsilon_0 - \epsilon_1 + \epsilon_2 - \epsilon_3) \quad J_0 = -\frac{1}{4}(\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3)$$

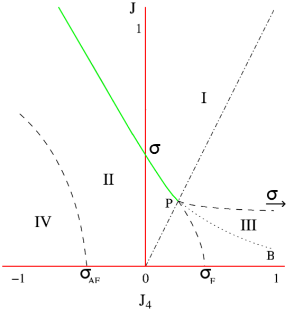
- If $J = J'$ ($\rightarrow \epsilon_1 = \epsilon_2$) we get the isotropic version
- If moreover $J = J' = J_4$, we have only two energies:

(a) ϵ_0 : equal atoms
(b) $\epsilon_1 = \epsilon_2 = \epsilon_3$: distinct atoms

$\rightarrow$ 4-states Potts

Ashkin-Teller model

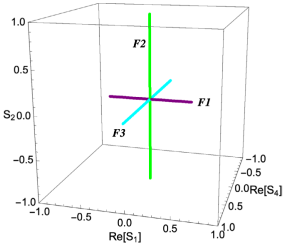
Indeed, the line $J = J_4$ in the phase diagram identifies the *Potts subspace*.
The green line is critical with varying critical exponents. It can be described by the scaling action



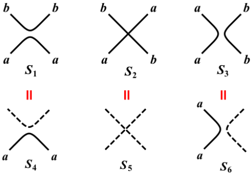
$$\mathcal{A} = \sum_{i=1}^{2} \left[\mathcal{A}_i^{\text{Ising}} + \tau \int d^2x \, \epsilon_i(x) \right] + g \int d^2x \, \epsilon_1 \epsilon_2(x)$$

- The energy density $\epsilon_i = \psi_i \bar{\psi}_i$ has $X_\epsilon = 2b^2$
- The operator $\epsilon_1 \epsilon_2$ is marginal ($X_\epsilon^{\text{Ising}} = 1$)
- For $\tau = 0$, $g(b^2)$ describes a line of fixed points along which $X_\sigma = X_s = \frac{1}{8}$

Type F solutions



Type F solutions have a free parameter each and they meet in a point.
Since $N = 1$, this basically reconstructs the vector part of $D3$ (XY model, BKT):



$$S_2 = 0 \quad \Rightarrow \quad b^2 = 1\,.$$

Consider the theory with action

$$\mathcal{A} = \mathcal{A}_{\text{Gauss}} + \int \mathrm{d}^2x \left\{ \lambda \epsilon(x) + \bar{\lambda} \left[U_4(x) + U_{-4}(x) \right] \right\}$$

$$X_\epsilon = 2b^2 \quad , \quad X_{(m=\pm 4)} = \frac{2}{b^2}$$

$\rightarrow$ These fields are marginal for $b^2 = 1$.

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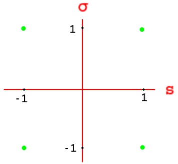
Type F solutions

The terms $U_{\pm 4}$ have Z_4 symmetry:

$$U_4 + U_{-4} \sim \cos\left[4\left(\frac{\tilde{\varphi}}{2b}\right)\right] \, .$$



- Indeed, the isotropic A-T model

$$\mathcal{H}_{AT} = -\sum_{\langle i,j \rangle} \{ J(\sigma_i\sigma_j + s_i s_j) + J_4 \sigma_i \sigma_j s_i s_j \}$$

  does present Z_4 symmetry in $\binom{s}{\sigma}$.

- Studying RG equations at first order around $b^2 = 1$, $\lambda = \bar{\lambda} = 0$ gives 3 lines of fixed points:

$$\begin{cases} \lambda = \bar{\lambda} = 0 & , \quad \text{varying } b^2 \\ b^2 = 1 & , \quad \lambda = \pm\bar{\lambda} \end{cases}$$

  On a square lattice, $b^2 \le \frac{3}{4}$: first line only.

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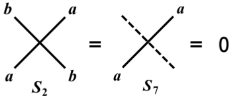
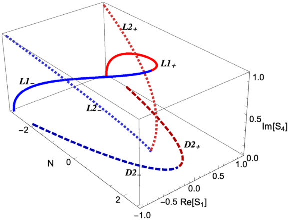
Type L/T solutions

Solution	N	ρ_2	$\cos\phi_1$	ρ_4	$\cos\theta_4$	ρ_5
$L1_\pm$	$[-3,1]$	0	$\pm\frac{1}{2}\sqrt{1-N}$	1	$(\pm)\frac{1}{2}\sqrt{1-N}$	$\pm\sqrt{1-N}$
$L2_\pm$	$[-3,1]$	0	$\pm\frac{1}{2}\sqrt{1-N}$	1	$(\pm)\frac{1}{2}\sqrt{3+N}$	$\mp\sqrt{1-N}$
$T1_\pm$	$(-\infty,1]$	$\pm\sqrt{\frac{1-N}{2-N}}$	0	1	$\frac{(\pm)1}{\sqrt{2}}\sqrt{1(\pm)\frac{1}{\sqrt{2-N}}}$	$(\pm)\sqrt{1-N}$
$T2_\pm$	$[-3,-2]$	0	± 1	$\sqrt{-2-N}$	0	$\pm(N+1)$

$\rho_1 = \sqrt{1-\rho_2^2}\ ,\quad \rho_7 = (\pm)\sqrt{1-\rho_4^2}$

- Type L: nonintersecting trajectories, $N \in [-3,1]$ $\rightarrow$ nonintersecting loop gas.
- Type T: cannot be traced back to the decoupled $O(N)$ case $\rightarrow$ new universality classes.





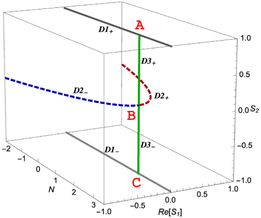
XY-Ising reloaded

The only solution defined for $N = 2$ is Type D.

Vector sector $\rightarrow$ Gaussian theory. Since $S_\pm = U_{\pm 1}$,

$$X_s = X_{(m=1)} = \frac{1}{8b^2}$$

where the only transition is BKT ($b^2 = 1$, $X_s = \frac{1}{8}$).



Scalar sector:

- $S_5 = -1$: free fermion, $b^2 = \frac{1}{2}$ (C), $X_\sigma = \frac{1}{8}$
- $S_5 = +1$: free boson, $b^2 = \infty$ (A), $X_\sigma = 0$ and

$$X_s \xrightarrow[b^2 \to \infty]{} 0$$

$\rightarrow$ $O(3)$ zero-temperature critical point.

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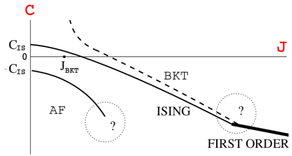
FFXY reloaded

The parameter space of XY-Ising will contain phase transition lines bifurcating from a multicritical point ($\rightarrow$ zero temperature $O(3)$ fixed point) and ending in Ising/BKT points.



FFXY has a single parameter (J), so it corresponds to a line in XY-Ising parameter space. Measured values

$$X_\sigma \in [0.1 - 0.2] \quad \leftrightarrow \quad X_\sigma = \frac{1}{8} = 0.125$$

$$\nu_\sigma \in [0.8 - 1] \quad \leftrightarrow \quad \nu_\sigma = \frac{1}{d - X_\epsilon} = \frac{1}{2 - 2b^2} \xrightarrow[b^2 \to \frac{1}{2}]{} 1$$

are compatible with Ising critical exponents.
We also predicted $X_s = \frac{1}{8}$ at the vector transition.

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Take home messages

- Scale-invariant scattering theory gives the exact solution for RG fixed points of systems in $d = 2$ with coupled $O(N)$ and Ising order parameters
- For $N = 2$, we identified the multicritical points allowed in XY-Ising and determined the critical exponents of FFXY
- For $N = 1$, three lines of fixed points intersect at the BKT transition point of the Gaussian theory. They are related to Z_4 symmetry of the isotropic AT model
- For $N \leq 1$ new universality classes appear (gas of nonintersecting loops).

Thanks for your attention!

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