Effective action methods for stochastic PDEs

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REFS:
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D. Hochberg, C. Molina-Paris, J. Perez-Mercader, M.Visser, Phys. Lett. A 278, 177 (2001)
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Outline

Outline

Stochastic PDEs

$$\mathcal{L}\phi(\mathrm{x}, t) = F[\phi] + \eta(\mathrm{x}, t)$$

- $\mathcal{L} = \partial_t - \nu\nabla^2$, $\mathcal{L} = \partial_t^2 - \nabla^2$, $\mathcal{L} = \partial_t$
- $F[\phi]$ forcing term
- Terms linear in ϕ like $-\gamma\phi$ or $-\nu m^2\phi$ may be moved from $F[\phi]$ into $\mathcal{L}$
- $\eta(\mathrm{x}, t)$ generic random noise
- We do *not* require $\mathcal{L}\phi = F[\phi]$ to be derived from an action principle.

Examples

- Reaction-diffusion-decay systems

$$\frac{\partial \phi}{\partial t} - \nu \nabla^2 \phi = P[\phi] + \eta$$

- KPZ equation

$$\frac{\partial \phi}{\partial t} - \nu \nabla^2 \phi = -\nu m^2 \phi + \frac{\lambda}{2} (\boldsymbol{\nabla} \phi)^2 + \eta$$

- Purely dissipative systems

$$\frac{\partial \phi}{\partial t} = -\frac{\delta \mathcal{H}[\phi]}{\delta \phi} + \eta$$

$$\left.\frac{\partial \mathcal{H}[\phi]}{\partial t}\right|_{\eta=0} = -\int \mathrm{d}^d x \left(\frac{\delta \mathcal{H}[\phi]}{\delta \phi(\mathrm{x})} \right)^2 \leq 0$$

Ensemble average

Assume a unique solution $\phi_{\mathsf{sol}}(x, t|\eta)$ exists. For any observable $\mathcal{O}$, introduce

$$\begin{aligned}\langle\mathcal{O}(\phi)\rangle &= \int \mathcal{D}\eta\, \mathcal{P}[\eta]\mathcal{O}\left(\phi_{\mathsf{sol}}(x, t|\eta)\right) \\ &= \int \mathcal{D}\eta\, \mathcal{P}[\eta] \int \mathcal{D}\phi\, \mathcal{O}\left(\phi\right) \delta\left[\phi - \phi_{\mathsf{sol}}(x, t|\eta)\right] \\ &= \int \mathcal{D}\eta\, \mathcal{P}[\eta] \int \mathcal{D}\phi\, \mathcal{O}\left(\phi\right) \delta\left[\mathcal{L}\phi - F[\phi] - \eta\right] \sqrt{\mathcal{J}\mathcal{J}^\dagger}\end{aligned}$$

where

$$\mathcal{J} = \det\left(\mathcal{L} - \frac{\delta F}{\delta\phi}\right)$$

Ensemble average

- MSR

$$\delta\left[\mathcal{L}\phi - F[\phi] - \eta\right] = \int \mathcal{D}i\tilde{\phi}\, e^{-\int \tilde{\phi}(\mathcal{L}\phi - F[\phi] - \eta)}$$

and integrate over the Gaussian distribution $\mathcal{P}[\eta]$ to get MSRJD dynamical functional.

- Integrating η straight away yields

$$\langle \mathcal{O}(\phi) \rangle = \int \mathcal{D}\phi\, \mathcal{P}\left[\mathcal{L}\phi - F[\phi]\right] \sqrt{\mathcal{J}\mathcal{J}^\dagger}\, \mathcal{O}\left(\phi\right)$$

where $\mathcal{P}\left[\mathcal{L}\phi - F[\phi]\right] \sim e^{-\mathcal{S}_{\mathrm{OM}}[\phi]}$ is the generalized OM functional.

Gaussian noise

$$\mathcal{P}[\eta] = \frac{1}{\sqrt{\det(2\pi G_\eta)}} e^{-\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, \eta(x) G_\eta^{-1}(x,y)\eta(y)}$$

- Can always choose $F[\phi]$ so that $\langle \eta \rangle = 0$
- Gaussian noise $\not\Rightarrow$ Gaussian fluctuations of ϕ!

Then

$$\langle \mathcal{O}(\phi) \rangle = \frac{1}{\sqrt{\det(2\pi G_\eta)}} \int \mathcal{D}\phi \sqrt{\mathcal{J}\mathcal{J}^\dagger} e^{-\mathcal{S}_{\mathrm{OM}}[\phi]}\, \mathcal{O}(\phi)$$

where

$$\mathcal{S}_{\mathrm{OM}}[\phi] \equiv \frac{1}{2} \int\int (\mathcal{L}\phi - F[\phi])\, G_\eta^{-1} (\mathcal{L}\phi - F[\phi])$$

Dynamical action

- Jacobian

$$\mathcal{J} = \det\left(\mathcal{L} - \frac{\delta F}{\delta \phi}\right) = \frac{1}{\det(2\pi\mathcal{I})} \int \mathcal{D}[c^\dagger, c] e^{-\frac{1}{2}\int c^\dagger \left[\mathcal{L} - \frac{\delta F}{\delta \phi}\right] c}$$

- Dynamical action

$$\mathcal{S}_{\text{free}} = \frac{1}{2} \int \int [\mathcal{L}\phi] G_\eta^{-1} [\mathcal{L}\phi] + \frac{1}{2} \int c^\dagger \mathcal{L} c$$

$\rightarrow$ propagators $G_{\phi\phi}$, $G_{c^\dagger c}$,

$$\mathcal{S}_{\text{int}} = \int \int \left\{ \frac{1}{2} F[\phi] G_\eta^{-1} F[\phi] - [\mathcal{L}\phi] G_\eta^{-1} F[\phi] \right\} - \frac{1}{2} \int c^\dagger \frac{\delta F}{\delta \phi} c$$

$\rightarrow$ vertices $\phi - F[\phi]$, $F[\phi] - F[\phi]$, $c^\dagger - c$.

Effective action

- Write noise as $G_\eta(x, y) = \mathcal{A} \cdot g(x, y)$.
- Construct the characteristic functional

$$\mathcal{Z}[J] \equiv \left\langle e^{\frac{1}{\mathcal{A}} \int J\phi} \right\rangle \equiv \mathcal{Z}[0] e^{\frac{1}{\mathcal{A}} \mathcal{W}[J]}$$

and cumulant generating functional

$$\mathcal{W}[J] = \mathcal{A} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$$

- Legendre transform,

$$\Gamma[\varphi; \varphi_0] = \sup_J \left\{ -\mathcal{W}[J] + \int J\varphi \right\} = -\mathcal{W}[\tilde{J}] + \int \tilde{J}\varphi$$

where $\tilde{J}$ is *s.t.*

$$\varphi = \left. \frac{\delta \mathcal{W}[J]}{\delta J} \right|_{J=\tilde{J}} = \langle \phi \rangle_{J=\tilde{J}} \quad , \quad \left. \frac{\delta \Gamma[\phi; \phi_0]}{\delta \phi} \right|_{\phi=\varphi} = \tilde{J} \, .$$

Effective action

Recalling

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{\frac{1}{\hbar}\left(\int J\phi - \mathcal{S}(\phi)\right)}$$

$$\to \Gamma[\varphi;\varphi_0] = \mathcal{S}(\varphi) + \frac{\hbar}{2} \ln\det\left(\frac{\delta^2 \mathcal{S}}{\delta\varphi^2}\right) - [\varphi \leftrightarrow \varphi_0] + \mathcal{O}(\hbar^2)$$

we get

$$\Gamma[\varphi;\varphi_0] = \frac{1}{2}\int\int \left(\mathcal{L}\varphi - F[\varphi]\right) g^{-1} \left(\mathcal{L}\varphi - F[\varphi]\right)$$

$$+ \frac{\mathcal{A}}{2} \ln\det\left\{ \mathbb{1} - \left[\left(\mathcal{L}^\dagger - \frac{\delta F}{\delta\varphi}^\dagger\right)^{-1} g \left(\mathcal{L} - \frac{\delta F}{\delta\varphi}\right)^{-1}\right] \left[(\mathcal{L}\varphi - F[\varphi]) g \frac{\delta^2 F}{\delta\varphi^2}\right] \right\}$$

$$- [\varphi \leftrightarrow \varphi_0] + \mathcal{O}(\mathcal{A}^2)$$

Effective potential

For homogeneous and static field configurations ($\mathcal{L}\varphi = 0$),

$$\mathcal{V}[\varphi;\varphi_0] \equiv \frac{\Gamma[\varphi;\varphi_0]}{\Omega}$$

Translational invariant noise, $\int \mathrm{d}x\, g^{-1}(x) = 1$:

$$\mathcal{V}[\varphi;\varphi_0] = \frac{1}{2}F^2[\varphi] + \frac{\mathcal{A}}{2}\int \frac{\mathrm{d}^d k\,\mathrm{d}\omega}{(2\pi)^{d+1}} \ln\left\{1 + \frac{g(\boldsymbol{k},\omega)F[\varphi]\frac{\delta^2 F}{\delta\varphi^2}}{\left[\mathcal{L}^\dagger(\boldsymbol{k},\omega) - \frac{\delta F}{\delta\varphi}^\dagger\right]\left[\mathcal{L}(\boldsymbol{k},\omega) - \frac{\delta F}{\delta\varphi}\right]}\right\} - [\varphi \leftrightarrow \varphi_0] + \mathcal{O}\left(\mathcal{A}^2\right)$$

Compare with (Coleman-Weinberg)

$$\mathcal{V}_{\mathrm{QFT}}[\varphi;\varphi_0] = V[\varphi] + \frac{\hbar}{2}\int \frac{\mathrm{d}^d k\,\mathrm{d}\omega}{(2\pi)^{d+1}} \ln\left\{1 + \frac{\frac{\delta^2 V}{\delta\varphi^2}}{\omega^2 + \boldsymbol{k}^2}\right\} - [\varphi \leftrightarrow \varphi_0] + \mathcal{O}(\hbar^2)$$

Interpretation

- Physical energy need not be conserved, but

$$\mathcal{S}_c \equiv \frac{1}{2} \int \int (\mathcal{L}\phi - F[\phi])\, g^{-1} (\mathcal{L}\phi - F[\phi]) \geq 0$$

with minima in $\mathcal{L}\phi - F[\phi] = 0$.

- The EA contains all the info about the ground state and its fluctuations:

stationary points $\leftrightarrow$ stochastic $\langle \bullet \rangle$ at $J = 0$

$$\frac{\delta \Gamma[\varphi; \varphi_0]}{\delta \varphi} = 0 \quad \leftrightarrow \quad \varphi = \langle \phi \rangle_{J=0}$$

Interpretation

- Since $P_\phi[\phi] = \mathcal{P}\left[\mathcal{L}\phi - F[\phi]\right]\sqrt{\mathcal{J}\mathcal{J}^\dagger}$, if we coarse-grain

$$\langle\phi\rangle_\Omega \equiv \frac{1}{\Omega}\int_\Omega \mathrm{d}x\,\phi(x)$$

then

$$\mathrm{Prob}(\langle\phi\rangle_\Omega = \bar{\phi}) = \int \mathcal{D}\phi\, P_\phi[\phi]\delta\left(\langle\phi\rangle_\Omega - \bar{\phi}\right) = \int \mathcal{D}\phi\, P_\phi[\phi]\int \mathrm{d}\lambda\, e^{i\lambda\Omega\left(\langle\phi\rangle_\Omega - \bar{\phi}\right)}$$

$$= \int \mathrm{d}\lambda\, \mathcal{Z}[i\lambda\mathcal{A}\,\theta(\Omega)]e^{-i\lambda\Omega\bar{\phi}} \asymp e^{-\frac{\Omega}{\mathcal{A}}\mathcal{V}[\bar{\phi};\phi_{\mathbf{0}}]+\mathcal{O}\left(\frac{1}{\Omega}\right)}$$

i.e. effective potential $\leftrightarrow$ PDF for the spacetime averaged field

Interpretation

- Transition probability

$$\text{Prob}\left[\phi_f(\boldsymbol{x}), t_f; \phi_i(\boldsymbol{x}), t_i\right] \propto \int_{\phi(\boldsymbol{x},t_i)=\phi_i(\boldsymbol{x})}^{\phi(\boldsymbol{x},t_f)=\phi_f(\boldsymbol{x})} \mathcal{D}\phi \, e^{-\frac{1}{\mathcal{A}}\mathcal{S}_c[\phi]} \sqrt{\mathcal{J}\mathcal{J}^\dagger}$$

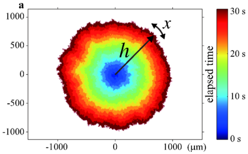
$$\asymp e^{-\frac{\Omega}{\mathcal{A}}\Gamma[\phi_{\text{int}}]}$$

where ϕ_{int} minimizes $\mathcal{S}_c[\phi]$ and interpolates from $\phi_i(\boldsymbol{x})$ to $\phi_f(\boldsymbol{x})$.

- In a nutshell, effective action/potential give info about the ground state in out-of-equilibrium fluctuating phenomena.

Example: massless KPZ

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\phi = F_0 + \frac{\lambda}{2}(\boldsymbol{\nabla}\phi)^2 + \eta$$



- Surface growth interpretation: $\phi(x,t)$ height of surface
- Fluid dynamical interpretation: $u = -\boldsymbol{\nabla}\phi$ fluid velocity,

$$u_t + \lambda u u_x = \nu u_{xx} - \eta_x$$

- Symmetries:

$$\begin{cases} x \to x' = x - \lambda\epsilon t \\ t' = t \\ \phi(x,t) \to \phi(x',t') = \phi(x,t) - \epsilon x \end{cases} \qquad \begin{cases} \phi \to \phi + c(t) \\ F_0 \to F_0 + \frac{\mathrm{d}c}{\mathrm{d}t} \end{cases}$$

i.e. Galilean transformation $u \to u' = u + \epsilon$ (**iff** time-white noise), and gauge transformation for u.

Effective potential - KPZ

- Homogeneous, static field configs $\rightarrow \mathcal{V}[\varphi;\varphi_0]=0$
- Most general static config for which $\mathcal{L}\phi=0$ is $\phi=-u\cdot x$
- @ zero loops,

$$\mathcal{V}[u;u_0]=\frac{1}{2}\left[\left(F_0+\frac{1}{2}\lambda u^2\right)^2-\left(F_0+\frac{1}{2}\lambda u_0^2\right)^2\right]+\mathcal{O}(\mathcal{A})$$

formally equivalent to $\lambda\phi^4$ QFT.

- Choose temporal white noise, so that we can set $u_0=0$
- Renormalized value of F_0 must be set to zero, as

$$\frac{1}{\Omega}\left\langle\int_\Omega \frac{\partial\phi}{\partial t}\,\mathrm{d}^d x\right\rangle=F_0+\mathcal{O}(\mathcal{A})$$

Effective potential - KPZ

- Let $g(k) = \theta(\Lambda - |k|)$, so that @ 1 loop

$$\mathcal{V}[u] = \mathcal{V}_0[u] + \frac{\mathcal{A}}{2}\nu \int_{|k|<\Lambda} \frac{\mathrm{d}^d k\,|k|}{(2\pi)^d} \left\{ \sqrt{k^2 + \frac{\lambda}{\nu^2}\left(F_0 + \frac{1}{2}\lambda u^2\right)} - \sqrt{k^2 + \frac{\lambda}{\nu^2}F_0} \right\}$$

- $u^{2n} \sim \Lambda^{d+2-2n}$, *i.e.* renormalizable for $d < 4$
- This depends on the choice of noise! Picking

$$g(k) = \left(\frac{k_0}{k}\right)^{\alpha} \theta(\Lambda - |k|)$$

then $u^{2n} \sim \Lambda^{d+2-2n-n\alpha}$.

d=1 case

$$\mathcal{V}_1[u] = \frac{\mathcal{A}}{4\pi}\nu \int_0^\Lambda \mathrm{d}k^2 \left\{ \sqrt{k^2 + \frac{\lambda}{\nu^2}\left(F_0 + \frac{1}{2}\lambda u^2\right)} - \sqrt{k^2 + \frac{\lambda}{u^2}F_0} \right\}$$

Consider

$$\mathcal{I}(a) \equiv \int_0^\infty \mathrm{d}k^2 \left\{ \sqrt{k^2 + a} - \sqrt{k^2} \right\}$$

$$\mathcal{I}''(a) = -\frac{1}{2a} \quad \rightarrow \quad \mathcal{I}(a) = ca - \frac{2}{3}a^{\frac{3}{2}}$$

$(\mathcal{I}(0) = 0)$ and reabsorb the constant c into

$$(F_0)_{\text{bare}} = (F_0)_{\text{renormalized}} + \mathcal{A} \cdot \delta F_0 + \mathcal{O}(\mathcal{A}^2) \quad \rightarrow \quad \delta F_0 = -\frac{c\lambda}{4\pi\nu}$$

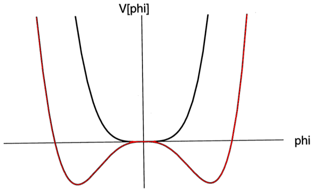
$$\mathcal{L} = \mathcal{L}_R + \delta\mathcal{L}\ .$$

Finally, set $(F_0)_{\text{renormalized}} \equiv 0$ into $\mathcal{V}[u]$.

Dynamical symmetry breaking

In $d = 1$,

$$\mathcal{V}[u] = \frac{\lambda^2}{8} u^4 - \frac{\mathcal{A}\lambda^3}{12\sqrt{2}\pi\nu^2} |u|^3 + \mathcal{O}(\mathcal{A}^2)$$

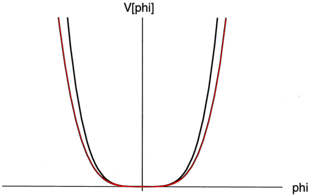


- SSB: a symmetry of the potential is not shared by 0-loop ground state
- DSB: symmetry preserved at classical level (noiseless), but broken due to fluctuations
- Non-zero minima, *i.e.* spontaneous onset of bulk fluid flows due to noise! (*e.g.* thin pipe)

Dynamical symmetry breaking

In $d = 3$,

$$\mathcal{V}[u] = \frac{\lambda^2}{8} u^4 + \frac{\mathcal{A}\lambda^5}{120\sqrt{2}\pi^2\nu^4} |u|^5 + \mathcal{O}(\mathcal{A}^2)$$



- SSB: a symmetry of the potential is not shared by 0-loop ground state
- DSB: symmetry preserved at classical level (noiseless), but broken due to fluctuations
- Non-zero minima, *i.e.* spontaneous onset of bulk fluid flows due to noise! (*e.g.* thin pipe)

Outline

Scale-dependence of parameters

- SDE, Fokker-Planck, Master equation $\rightarrow$ time evolution of statistical properties of the solution
- Interplay between fluctuations and interactions leads to scale dependence in the parameters and couplings in stochastic systems
- RG: how the dynamics of the system evolves as we change the temporal/spatial scale at which the phenomenon is observed

Perturbative RG and multiplicative noise

Cubic autocatalytic model (simplified Gray-Scott):

$$2\phi + U \xrightarrow{\lambda'} 3\phi$$
$$\phi \xrightarrow{r} \varnothing$$

If U is very abundant, calling $\lambda \equiv \lambda' U$

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -r\phi(t) + \lambda\phi^2(t) + \eta(t)$$

or

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -r\phi(t) + \lambda\phi^2(t) + \phi(t)\eta(t)$$

Perturbative RG and multiplicative noise

Additive noise:

$$\phi(\omega) = R_0(\omega)\left\{\eta(\omega) + \lambda \int \frac{\mathrm{d}\omega'}{2\pi}\phi(\omega-\omega')\phi(\omega')\right\}$$

Multiplicative noise:

$$\phi(\omega) = R_0(\omega)\left\{\int \frac{\mathrm{d}\omega'}{2\pi}\phi(\omega-\omega')\eta(\omega') + \lambda \int \frac{\mathrm{d}\omega'}{2\pi}\phi(\omega-\omega')\phi(\omega')\right\}$$

where we defined the free response

$$R_0(\omega) = \frac{1}{r + i\omega}$$

Perturbative RG and multiplicative noise

Look for the full response

$$\phi(\omega) = R(\omega)\eta(\omega)$$

Additive noise:

$$R(\omega)\eta(\omega) = R_0(\omega)\eta(\omega) + \lambda R_0(\omega)\int \frac{\mathrm{d}\omega'}{2\pi} R(\omega-\omega')\eta(\omega-\omega')R(\omega')\eta(\omega')$$
$$R = R_0 + \mathcal{O}(R^2)$$

Multiplicative noise:

$$\begin{aligned} R(\omega)\eta(\omega) = R_0(\omega)\int \frac{\mathrm{d}\omega'}{2\pi} R(\omega-\omega')\eta(\omega-\omega')\eta(\omega') \\ + \lambda R_0(\omega)\int \frac{\mathrm{d}\omega'}{2\pi} R(\omega-\omega')\eta(\omega-\omega')R(\omega')\eta(\omega') \end{aligned}$$

Perturbative RG and multiplicative noise

Look for the full response

$$\phi(\omega) = R(\omega)\eta(\omega)$$

Additive noise:

$$R(\omega)\eta(\omega) = R_0(\omega)\eta(\omega) + \lambda R_0(\omega) \int \frac{\mathrm{d}\omega'}{2\pi} R(\omega - \omega')\eta(\omega - \omega')R(\omega')\eta(\omega')$$

$$R = R_0 + \mathcal{O}(R^2)$$

Multiplicative noise:

$$R(\omega)\eta(\omega) = R_0(\omega) \int \frac{\mathrm{d}\omega'}{2\pi} R(\omega - \omega')\eta(\omega - \omega')\eta(\omega') + \lambda R_0(\omega) \int \frac{\mathrm{d}\omega'}{2\pi} R(\omega - \omega')\eta(\omega - \omega')R(\omega')\eta(\omega')$$

Effective potential for multiplicative noise

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = -F[\phi] + \mathcal{G}[\phi]\eta(t)$$

As before, write

$$1 = \int \mathcal{D}\phi\, \delta\left[\phi - \phi_{\mathsf{sol}}(x, t|\eta)\right]$$

where

$$\begin{aligned}\delta\left[\phi - \phi_{\mathsf{sol}}(x, t|\eta)\right] &= \delta\left[\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + F[\phi] - \mathcal{G}[\phi]\eta(t)\right]\sqrt{\mathcal{J}\mathcal{J}^\dagger} \\ &= \delta\left[\eta - \frac{1}{\mathcal{G}[\phi]}\left(\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + F[\phi]\right)\right] \det\left\{\frac{1}{\mathcal{G}[\phi]}\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{F[\phi]}{\mathcal{G}[\phi]}\right)' - \frac{\mathcal{G}'[\phi]}{\mathcal{G}^2[\phi]}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right\}\end{aligned}$$

Effective potential for multiplicative noise

Additive noise (space+time):

$$\mathcal{V}[\varphi;\varphi_0] \simeq \frac{1}{2}F^2[\varphi] + \frac{\mathcal{A}}{2}\int \frac{\mathrm{d}^d k\,\mathrm{d}\omega}{(2\pi)^{d+1}} \ln\left\{1 + \frac{g(\boldsymbol{k},\omega)F[\varphi]\frac{\delta^2 F}{\delta\varphi^2}}{\left[\mathcal{L}^\dagger(\boldsymbol{k},\omega) - \frac{\delta F}{\delta\varphi}^\dagger\right]\left[\mathcal{L}(\boldsymbol{k},\omega) - \frac{\delta F}{\delta\varphi}\right]}\right\}$$

Multiplicative noise:

$$\mathcal{V}[\varphi;\varphi_0] \simeq \frac{1}{2}\left(\frac{F[\varphi]}{\mathcal{G}[\varphi]}\right)^2 + \frac{\mathcal{A}}{2}\int \frac{\mathrm{d}\omega}{2\pi} \ln\left\{1 + \frac{g(\omega)\mathcal{G}[\varphi]F[\varphi]\left(\frac{F[\varphi]}{\mathcal{G}[\varphi]}\right)''}{\omega^2 + \mathcal{G}^2[\varphi]\left[\left(\frac{F[\varphi]}{\mathcal{G}[\varphi]}\right)'\right]^2}\right\}$$

Toy model

Rate law:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -r\phi + \lambda\phi^2 + \phi^{\frac{1}{2}}\eta$$

- $\phi^{\frac{1}{2}}$ represents intermediate chemical steps happening faster than $2\phi + U \to 3\phi$
- Dynamics is modified as we change the scale at which the phenomena is being observed. How do the parameters of the Langevin equation scale?
- If some steps are light-sensitive, shine noisy light and measure the dynamics $\to$ gain insight into intermediate steps

Toy model

- Choose $g(\omega) = 1 - \frac{|\omega|}{\omega_0}$ (*i.e.* $\sim \frac{1}{t^2}$ time correlations)
- Effective potential becomes

$$\mathcal{V}[\varphi;\varphi_0] \simeq \frac{1}{2}\left(r^2\varphi - 2r\lambda\varphi^2 + \lambda^2\varphi^3\right) + \frac{\mathcal{A}}{2}\int\frac{\mathrm{d}\omega}{2\pi}\ln\left\{1 + \frac{g(\omega)\left(-\frac{3}{4}\lambda\right)\left(r\varphi - \lambda\varphi^2\right)}{\omega^2 + \frac{1}{4}r^2 - \frac{3}{2}r\lambda\varphi + \frac{9}{4}\lambda^2\varphi^2}\right\}$$

- Key: keep only UV divergent terms

$$\begin{aligned}\mathcal{V}[\varphi;\varphi_0] &\simeq \left[\frac{1}{2}r^2 - \frac{3}{8}\lambda r\mathcal{A}\int\frac{\mathrm{d}\omega}{2\pi}\frac{g(\omega)}{\omega^2 + \frac{1}{4}r^2} + \text{finite}\right]\varphi \\ &\quad - \left[\frac{1}{2}(2r\lambda) - \frac{3}{8}\lambda^2\mathcal{A}\int\frac{\mathrm{d}\omega}{2\pi}\frac{g(\omega)}{\omega^2 + \frac{1}{4}r^2} + \text{finite}\right]\varphi^2 + \frac{1}{2}\lambda^2\varphi^3 \\ &\equiv \frac{1}{2}\left[(rZ_r)^2\varphi - 2(rZ_r)\lambda\varphi^2 + \lambda^2\varphi^3\right] + \text{finite}\end{aligned}$$

1-loop correction to r

We deduce ($\mathcal{L} = \mathcal{L}_R + \delta\mathcal{L}$)

$$r_0 \;=\; r + C \;=\; r\left(1 + \frac{C}{r}\right) \;\equiv\; rZ_r,$$

where C is the minimal counterterm necessary to cancel the divergence:

$$C = -\frac{3\lambda\mathcal{A}}{8\omega_0}\int\frac{\mathrm{d}\omega}{2\pi}\,\frac{|\omega|}{\omega^2+\frac{r^2}{4}}$$

1-loop correction to r

We deduce ($\mathcal{L} = \mathcal{L}_R + \delta\mathcal{L}$)

$$r_0 \;=\; r + C \;=\; r\left(1 + \frac{C}{r}\right) \;\equiv\; rZ_r,$$

where C is the minimal counterterm necessary to cancel the divergence:

$$\begin{aligned} C &\simeq -\frac{3\lambda\mathcal{A}^{(d)}}{8\omega_0}\int\frac{\mathrm{d}^d\omega}{(2\pi)^d}\;\frac{|\omega|}{\omega^2+\frac{r^2}{4}} \\ &= -\frac{3\lambda\mathcal{A}^{(d)}}{8\omega_0}\;\frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{r^2}{4}\right)^{\frac{d}{2}-\frac{1}{2}}\;\frac{\pi}{\Gamma\left(\frac{d}{2}\right)\sin\pi\left(\frac{d}{2}+\frac{1}{2}\right)} \\ &\xrightarrow{d=1-\epsilon} -\frac{3\lambda\mathcal{A}^{(d)}}{8\pi\omega_0}\;\frac{1}{\epsilon} + \text{finite} \end{aligned}$$

Running of r

Define the effective dimensionless rate

$$h^{(d)} \equiv \frac{\lambda \mathcal{A}^{(d)}}{r\omega_0} = \frac{\lambda \mathcal{A}}{r\omega_0} T^{d-1} = h\, T^{d-1} .$$

To find the running of h, start from the bare coupling

$$h_0^{(d)} = \frac{\lambda_0 \mathcal{A}_0^{(d)}}{r_0\omega_0} = \frac{\lambda \mathcal{A}}{r Z_r \omega_0} T^{-\epsilon} = \frac{1}{Z_r} h\, T^{-\epsilon}$$

and derive it by T using $\partial_T h_0 = 0$,

$$T\frac{dh}{dT} = \epsilon h - \frac{3}{8\pi} h^2 \quad \rightarrow \quad T\frac{dr}{dT} = -\epsilon r + \frac{3}{8\pi}\frac{\lambda \mathcal{A}}{\omega_0}$$

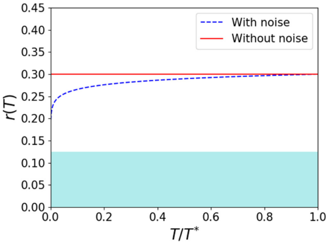
whose solution in the $\epsilon \to 0$ limit is

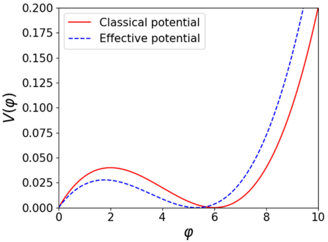
$$r(T) = r(T^*) + \frac{3}{8\pi}\frac{\lambda \mathcal{A}}{\omega_0} \ln\left(\frac{T}{T^*}\right)$$

Running of r

- Noise decreases the removal rate at smaller temporal scales, which constrains possible fast chemical mechanisms
- Concentration fixed points get modified by the noise
- Valid as long as

$$h = \frac{\lambda \mathcal{A}}{r\omega_0} \ll 1$$





Outline

To sum up

- 1-loop effective action for SPDEs subject to Gaussian colored noise, both additive and multiplicative
- Effective potential gives info about stable and metastable states in a nonequilibrium fluctuating system
- Dynamical symmetry breaking phenomena in KPZ equation with white Gaussian noise
- Running of the removal rate in a chemical reaction at different probing scales

Thanks for your attention!