Finite temperature models of Bose-Einstein condensation

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Outline

Basic system Hamiltonian

Start from a bosonic Hamiltonian in second quantization,

$$\hat{\mathcal{H}} = \int \mathrm{d}\mathbf{r}\, \hat{\Psi}^\dagger(\mathbf{r},t)\hat{h}_0(\mathbf{r})\hat{\Psi}(\mathbf{r},t) + \frac{1}{2}\int \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'\, \hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}^\dagger(\mathbf{r}',t)V(\mathbf{r}-\mathbf{r}')\hat{\Psi}(\mathbf{r}',t)\hat{\Psi}(\mathbf{r},t)$$

$$\hat{h}_0(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{r},t)\ .$$

Contact interaction approximation: for dilute gases at very low T,

$$V(\mathbf{r}-\mathbf{r}') \equiv g\delta(\mathbf{r}-\mathbf{r}')\ , \qquad g = \frac{4\pi\hbar^2 a_s}{m}\ .$$

The equations of motion in the Heisenberg picture read

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) = \left[\hat{\Psi}(\mathbf{r},t),\hat{\mathcal{H}}\right] = \left(\hat{h}_0(\mathbf{r}) + g\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\right)\hat{\Psi}(\mathbf{r},t)\ .$$

Bose-Einstein condensation

Condensation: macroscopic occupation of a single quantum state. Let's separate the contributions

$$\hat{\Psi}(\mathbf{r},t) = \hat{\phi}(\mathbf{r},t) + \hat{\delta}(\mathbf{r},t)\,, \qquad \begin{cases} \hat{\phi}(\mathbf{r},t) = \hat{a}_0(t)\varphi_0(\mathbf{r},t) \\ \hat{\delta}(\mathbf{r},t) = \sum_{i\neq 0} \hat{a}_i(t)\varphi_i(\mathbf{r},t)\,. \end{cases}$$

Bogoliubov approximation: since

$$\frac{1}{N_0}\left[\hat{a}_0(t), \hat{a}_0^\dagger(t)\right]|N_0\rangle = \frac{1}{N_0}|N_0\rangle \xrightarrow[N_0 \gg 1]{} 0$$

we may approximate the condensate field operator with a *condensate wavefunction*:

$$\hat{a}_0(t) \simeq \sqrt{N_0} \quad \Leftrightarrow \quad \hat{\phi}(\mathbf{r},t) \simeq \phi(\mathbf{r},t) = \sqrt{N_0}\varphi_0(\mathbf{r},t)\,.$$

Bogoliubov approximation

$\hat{\mathcal{H}}$ was invariant under global phase transformations of $\hat{\Psi}$, while $\phi(\mathbf{r}, t)$ breaks $U(1)$. As a result, N is not conserved: since $N_0 \pm 1 \simeq N_0$,

$$\left\langle \hat{\phi}(\mathbf{r}, t) \right\rangle = \phi(\mathbf{r}, t) \neq 0 \,, \qquad \left\langle \hat{\delta}(\mathbf{r}, t) \right\rangle = 0 \,.$$

We will define

$$n(\mathbf{r}, t) = \left\langle \hat{\Psi}^\dagger(\mathbf{r}, t)\hat{\Psi}(\mathbf{r}, t) \right\rangle = |\phi(\mathbf{r}, t)|^2 + \left\langle \hat{\delta}^\dagger(\mathbf{r}, t)\hat{\delta}(\mathbf{r}, t) \right\rangle \equiv n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t) \,.$$

Even at $T \simeq 0$, where the quantum depletion (in 3D) is

$$n - n_0 = \frac{8\sqrt{\pi}}{3}\sqrt{na^3} \,, \qquad na^3 \sim 10^{-3} \,,$$

thermal fluctuations dominate over quantum fluctuations. Then $\hat{\delta}$ will be used to describe the thermal cloud.

Resulting Hamiltonian

$$\hat{\mathcal{H}} = \mathcal{H}_0 + \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_3 + \hat{\mathcal{H}}_4$$

$$H_0 = \int \mathrm{d}r \left[\phi^* \hat{h}_0 \phi + \frac{g}{2}|\phi|^4\right]$$

$$\hat{\mathcal{H}}_1 = \int \mathrm{d}r \left[\hat{\delta}^\dagger \left(\hat{h}_0 + g|\phi|^2\right)\phi + \phi^* \left(\hat{h}_0 + g|\phi|^2\right)\hat{\delta}\right]$$

$$\hat{\mathcal{H}}_2 = \int \mathrm{d}r \left\{\hat{\delta}^\dagger \left(\hat{h}_0 + 2g|\phi|^2\right)\hat{\delta} + \frac{g}{2}\left[(\phi^*)^2\hat{\delta}\hat{\delta} + \phi^2\hat{\delta}^\dagger\hat{\delta}^\dagger\right]\right\}$$

$$\hat{\mathcal{H}}_3 = \int \mathrm{d}r\, g \left[\phi\hat{\delta}^\dagger\hat{\delta}^\dagger\hat{\delta} + \phi^*\hat{\delta}^\dagger\hat{\delta}\hat{\delta}\right]$$

$$\hat{\mathcal{H}}_4 = \int \mathrm{d}r\, \frac{g}{2}\hat{\delta}^\dagger\hat{\delta}^\dagger\hat{\delta}\hat{\delta}\,.$$

For a *mean field* treatment, we can either:

(i) Write the equation of motion and solve it in the time independent limit, or

(ii) Diagonalize $\hat{\mathcal{H}} - \mu\hat{N}$ in the Grand Canonical ensemble.

Outline

Gross-Pitaevskii equation

At $T = 0$ all the particles are in the condensate, so we can set $\hat{\delta} = \hat{\delta}^\dagger = 0$. The equation of motion reduces to the GPE

$$i\hbar \frac{\partial}{\partial t}\phi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}, t) + g|\phi(\mathbf{r}, t)|^2\right]\phi(\mathbf{r}, t) \,.$$

We can eliminate the time dependence by letting

$$\phi(\mathbf{r}, t) = \phi_0(\mathbf{r})e^{-\frac{i}{\hbar}\mu t}$$

and thus write the time-independent GPE

$$\mu\phi_0(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}, t) + g|\phi_0(\mathbf{r})|^2\right]\phi_0(\mathbf{r}) \,.$$

This can alternatively be found by minimizing at fixed N the energy functional

$$E[\phi] = \int \mathrm{d}r \left\{\frac{\hbar^2}{2m}|\nabla\phi_0(\mathbf{r})|^2 + V_{\text{ext}}(\mathbf{r}, t)|\phi_0(\mathbf{r})|^2 + \frac{g}{2}|\phi_0(\mathbf{r})|^4\right\} \,.$$

Hydrodynamics

Defining

$$\phi(\mathbf{r},t) = \sqrt{n_0(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)} \ , \qquad \mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t)$$

the GPE may be rewritten as

$$\begin{cases} \frac{\partial n_0}{\partial t} + \boldsymbol{\nabla}\cdot(n_0\mathbf{v}) = 0 \\ m\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = -\boldsymbol{\nabla}\mu_0 = -\boldsymbol{\nabla}\left(P_Q + V_{\text{ext}}\right) \end{cases}$$

where we defined the *quantum pressure*

$$P_Q = \frac{1}{2}gn_0^2 - \frac{1}{4}n_0\nabla^2\left(\log n_0\right) \ .$$

The first term is consistent with the usual $v^2 = \frac{\partial P}{\partial \rho}$, with $\rho = m|\phi|^2 = mn_0$, if $v = \sqrt{\frac{gn_0}{m}}$.

Bogoliubov equations

Inserting an ansatz

$$\phi(\mathbf{r},t) = e^{-\frac{i}{\hbar}\mu t}\left[\phi_0(\mathbf{r}) + \delta\phi(\mathbf{r},t)\right] \,, \qquad \delta\phi(\mathbf{r},t) = \sum_i \left[u_i(\mathbf{r})e^{-i\omega_i t} - v_i^*(\mathbf{r})e^{i\omega_i t}\right]$$

into the equation of motion gives Bogoliubov equations

$$\begin{pmatrix} \hat{L}(\mathbf{r}) & \hat{M}(\mathbf{r}) \\ -\hat{M}^*(\mathbf{r}) & -\hat{L}^*(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$\hat{L}(\mathbf{r}) = \hat{h}_0 + 2g|\phi_0(\mathbf{r})|^2 - \mu \,, \qquad \hat{M}(\mathbf{r}) = g\phi_0(\mathbf{r})^2 \,.$$

Uniform case: $V_{\text{ext}} = 0$, $n_c = |\phi_0|^2$ independent of $\mathbf{r}$.
Substituting the plane waves solutions $u_i(\mathbf{r}) = u_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}}$, $v_i(\mathbf{r}) = v_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}}$ gives the *quasiparticle spectrum*

$$\epsilon(\mathbf{p}) = \sqrt{\varepsilon_{\mathbf{p}}^2 + 2gn_0\varepsilon_{\mathbf{p}}} \to \begin{cases} gn_0 + \varepsilon_{\mathbf{p}} \,, & \text{large } \mathbf{p} \\ sp \,, & \text{small } \mathbf{p} \,. \end{cases}$$

Alternative derivation

Diagonalize $\hat{\mathcal{H}} \simeq \mathcal{H}_0 + \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2$ in the Grand Canonical:

$$\begin{aligned}
K_0 &= \int \mathrm{d}r \left[\phi_0^* (\hat{h}_0 - \mu) \phi_0 + \frac{g}{2} |\phi_0|^4 \right] \\
\hat{\mathcal{K}}_1 &= \int \mathrm{d}r \left[\hat{\delta}^\dagger \left(\hat{h}_0 + g|\phi_0|^2 - \mu \right) \phi_0 + \phi_0^* \left(\hat{h}_0 + g|\phi_0|^2 - \mu \right) \hat{\delta} \right] \\
\hat{\mathcal{K}}_2 &= \int \mathrm{d}r \left\{ \hat{\delta}^\dagger \left(\hat{h}_0 + 2g|\phi_0|^2 - \mu \right) \hat{\delta} + \frac{g}{2} \left[(\phi_0^*)^2 \hat{\delta}\hat{\delta} + \phi_0^2 \hat{\delta}^\dagger \hat{\delta}^\dagger \right] \right\} .
\end{aligned}$$

(i) Minimizing K_0 gives the time independent GPE

(ii) $\hat{K}_1$ is automatically null

(iii) $\hat{K}_2$ can be diagonalized by a Bogoliubov transformation

$$\hat{\delta}(\mathbf{r}, t) = \sum_i \left[u_i(\mathbf{r}) \hat{\beta}_i(t) - v_i^*(\mathbf{r}) \hat{\beta}_i^\dagger(t) \right] .$$

If $u_i(\mathbf{r})$, $v_i(\mathbf{r})$ satisfy Bogoliubov's equations, then

$$\hat{K}_2 = -\sum_i \epsilon_i \int \mathrm{d}r \, |v_i(\mathbf{r})|^2 + \sum_i \epsilon_i \hat{\beta}_i^\dagger \hat{\beta}_i .$$

Outline

Finite Temperature Mean Field Theory

We want to consider the full $\hat{\mathcal{H}}$. Let's approximate

$$\hat{\delta}^\dagger\hat{\delta}^\dagger\hat{\delta}\hat{\delta} \simeq 4\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle\hat{\delta}^\dagger\hat{\delta} + \left\langle\hat{\delta}^\dagger\hat{\delta}^\dagger\right\rangle\hat{\delta}\hat{\delta} + \left\langle\hat{\delta}\hat{\delta}\right\rangle\hat{\delta}^\dagger\hat{\delta}^\dagger - \left[2\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle^2 + \left\langle\hat{\delta}\hat{\delta}\right\rangle\left\langle\hat{\delta}^\dagger\hat{\delta}^\dagger\right\rangle\right]$$

$$\hat{\delta}^\dagger\hat{\delta}\hat{\delta} \simeq 2\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle\hat{\delta} + \hat{\delta}^\dagger\left\langle\hat{\delta}\hat{\delta}\right\rangle , \qquad \hat{\delta}^\dagger\hat{\delta}^\dagger\hat{\delta} \simeq 2\hat{\delta}^\dagger\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle + \left\langle\hat{\delta}^\dagger\hat{\delta}^\dagger\right\rangle\hat{\delta}$$

which give the (equilibrium) Wick theorem results

$$\left\langle\hat{\delta}^\dagger\hat{\delta}^\dagger\hat{\delta}\hat{\delta}\right\rangle = 2\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle\left\langle\hat{\delta}^\dagger\hat{\delta}\right\rangle + \left\langle\hat{\delta}\hat{\delta}\right\rangle\left\langle\hat{\delta}^\dagger\hat{\delta}^\dagger\right\rangle , \qquad \left\langle\hat{\delta}^\dagger\hat{\delta}\hat{\delta}\right\rangle = 0 .$$

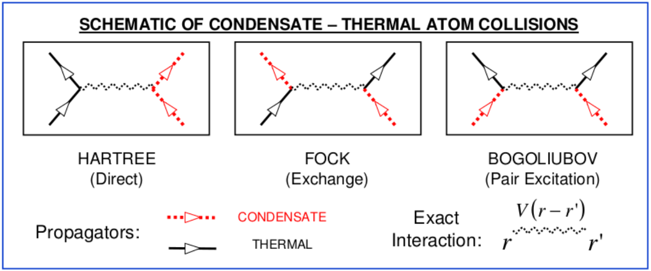
Defining the pair anomalous average $\tilde{m}(\mathbf{r},t) = \left\langle\hat{\delta}(\mathbf{r},t)\hat{\delta}(\mathbf{r},t)\right\rangle$,

$$\delta\mathcal{H}_0 = \delta\mathcal{H}_0^{HF} + \delta\mathcal{H}_0^{BOG} = -g\int \mathrm{d}r\,\tilde{n}^2 - \frac{g}{2}\int \mathrm{d}r\,\tilde{m}\tilde{m}^*$$

$$\delta\hat{\mathcal{H}}_1 = \delta\hat{\mathcal{H}}_1^{HF} + \delta\hat{\mathcal{H}}_1^{BOG} = g\int \mathrm{d}r\left(2\phi\tilde{n}\hat{\delta}^\dagger + \mathtt{h.c.}\right) + g\int \mathrm{d}r\left(\phi\tilde{m}^*\hat{\delta} + \mathtt{h.c.}\right)$$

$$\delta\hat{\mathcal{H}}_2 = \delta\hat{\mathcal{H}}_2^{HF} + \delta\hat{\mathcal{H}}_2^{BOG} = 2g\int \mathrm{d}r\,\tilde{n}\hat{\delta}^\dagger\hat{\delta} + \frac{g}{2}\int \mathrm{d}r\left(\tilde{m}^*\hat{\delta}\hat{\delta} + \tilde{m}\hat{\delta}^\dagger\hat{\delta}^\dagger\right) .$$

Finite Temperature Mean Field Theory



Hartree-Fock limit: discard all the terms containg two equal creation/annihilation operators, including

$$\hat{\mathcal{H}}_2^{\mathrm{BOG}} = \int \mathrm{d}r \left[(\phi^*)^2 \hat{\delta}\hat{\delta} + \phi^2 \hat{\delta}^\dagger \hat{\delta}^\dagger \right] .$$

Hartree-Fock limit

(i) The linear term $\hat{\mathcal{K}}_1 + \delta\hat{\mathcal{H}}_1^{HF}$ vanishes if we impose

$$\left[\hat{h}_0 + g|\phi_0|^2 + 2g\,\tilde{n}\right]\phi_0 = \mu\phi_0$$

which is a generalized time-independent GPE. This choice also leaves $\mathcal{K}_0 + \delta\mathcal{H}_0^{HF}$ minimized.

(ii) What remains is

$$\hat{\mathcal{K}}_2 - \hat{\mathcal{H}}_2^{\mathrm{BOG}} + \delta\hat{\mathcal{H}}_2^{HF} = \int \mathrm{d}r\,\hat{\delta}^\dagger\left[\hat{h}_0 + 2g(|\phi_0|^2 + \tilde{n}) - \mu\right]\hat{\delta}$$

which is already diagonal: single-particle energies get dressed as

$$\tilde{\varepsilon}_i(\mathbf{r}) = \varepsilon_i + 2g\left[|\phi_0|^2(\mathbf{r}) + \tilde{n}(\mathbf{r})\right] - \mu\ .$$

Thermal cloud

The equilibrium thermal cloud density is then

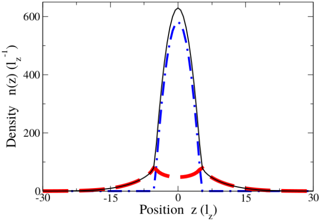
$$\tilde{n}(\mathbf{r}) = \sum_{i \neq 0} |\varphi_i(\mathbf{r})|^2 \left\langle \hat{a}_i^\dagger \hat{a}_i \right\rangle \ , \qquad \left\langle \hat{a}_i^\dagger \hat{a}_i \right\rangle = \frac{1}{e^{\beta \tilde{\varepsilon}_i(\mathbf{r})} - 1} \ .$$

If $V_{\text{ext}}(\mathbf{r})$ varies slowly (*local density approximation*), we may express semiclassically

$$\tilde{\varepsilon}(\mathbf{r}, \mathbf{p}) = \varepsilon(\mathbf{p}) + V_{\text{ext}}(\mathbf{r}) + 2g \left[|\phi_0|^2(\mathbf{r}) + \tilde{n}(\mathbf{r}) \right] - \mu$$

and thus compute

$$\tilde{n}(\mathbf{r}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi\hbar)^3} \frac{1}{e^{\beta \tilde{\varepsilon}_i(\mathbf{r}, \mathbf{p})} - 1} \ .$$



Hartree-Fock-Bogoliubov limit

Let's now include all quadratic non-condensate operators ($\tilde{m}$, $\tilde{m}^*$ as well). Proceeding as before,

(i) The linear term $\hat{\mathcal{K}}_1 + \delta\hat{\mathcal{H}}_1$ vanishes if we impose

$$\left[\hat{h}_0 + g|\phi_0|^2 + 2g\tilde{n}\right]\phi_0 + g\tilde{m}\phi_0^* = \mu\phi_0$$

which is yet another generalized time-independent GPE.

(ii) What remains is

$$\hat{\mathcal{K}}_2 + \delta\hat{\mathcal{H}}_2 = \int \mathrm{d}r \left\{\hat{\delta}^\dagger \left[\hat{h}_0 + 2g(|\phi_0|^2 + \tilde{n}) - \mu\right]\hat{\delta} + \frac{g}{2}\left[\left((\phi^*)^2 + \tilde{m}^*\right)\hat{\delta}\hat{\delta} + \left(\phi^2 + \tilde{m}\right)\hat{\delta}^\dagger\hat{\delta}^\dagger\right]\right\}$$

which can be diagonalized by a Bogoliubov transformation.

Hartree-Fock-Bogoliubov limit

(iii) This leads to the generalized Bogoliubov equations

$$\begin{pmatrix} \hat{L}(\mathbf{r}) & \hat{M}(\mathbf{r}) \\ -\hat{M}^*(\mathbf{r}) & -\hat{L}^*(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$\hat{L}(\mathbf{r}) = \hat{h}_0(\mathbf{r}) + 2g\left[|\phi_0(\mathbf{r})|^2 + \tilde{n}(\mathbf{r})\right] - \mu \,, \qquad \hat{M}(\mathbf{r}) = g\left[\phi_0(\mathbf{r})^2 + \tilde{m}(\mathbf{r})\right] \,.$$

The excitations are Bogoliubov quasiparticles.
For a static thermal cloud, $\left\langle \hat{\beta}_i^\dagger \hat{\beta}_j \right\rangle = \delta_{ij} f_i$.

(iv) Equilibrium averages are then

$$\tilde{n}(\mathbf{r}) = \left\langle \hat{\delta}^\dagger(\mathbf{r})\hat{\delta}(\mathbf{r}) \right\rangle = \sum_i \left(|u_i(\mathbf{r})|^2 + |v_i(\mathbf{r})|^2\right) f_i + |v_i(\mathbf{r})|^2$$

$$\tilde{m}(\mathbf{r}) = \left\langle \hat{\delta}(\mathbf{r})\hat{\delta}(\mathbf{r}) \right\rangle = \sum_i u_i(\mathbf{r}) v_i^*(\mathbf{r})(1 + 2f_i) \,.$$

HFB vs HF

PROs:
It lowers the free energy, so it gives in principle a better approximation to the many body wavefunction.

CONs:

(1) The expression for $\tilde{m}(\mathbf{r})$ diverges!
In the homogeneous case, for instance,

$$u_{\mathbf{p}} v^*_{\mathbf{p}} \propto \frac{1}{\varepsilon_{\mathbf{p}}} \ .$$

(2) The homogeneous Bogoliubov spectrum is gapped,

$$\epsilon^2_{\mathbf{p}} = \varepsilon^2_{\mathbf{p}} + 2g\varepsilon_{\mathbf{p}} \left(|\phi_0|^2 - \tilde{m}\right) + g^2 \left[\tilde{m}^2 - 2\tilde{m}|\phi_0|^2 - |\tilde{m}|^2 - 2\,\mathrm{Re}(\phi_0^2 \tilde{m}^*)\right] \ ,$$

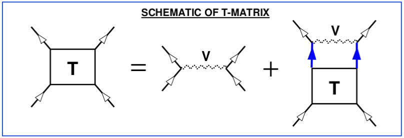
and this violates Goldstone theorem.

How to deal with the effective interaction?

- For low temperature dilute systems, atoms spend most of their time far away from each other: short-distance correlations are unimportant.
- Interactions are well described in terms of scattering processes on *asymptotic scattering states*, whose only effect is a change of phase of the wavefunction.
- At low energy, the only accessible scattering channel is s-wave. We may approximate the process by a *pseudopotential*

$$V(\mathbf{r}) \to g\delta(\mathbf{r}) \ .$$

How to implement the pseudopotential



- Construct an effective potential using Lippmann-Schwinger's equation: resumming the whole series, one gets the T-matrix. *Now* we can impose the pseudopotential approximation.
- g is fixed by assuming scattering to take place in the vacuum at low energies: then

$$g \equiv T^{(2)}(\mathbf{p} = \mathbf{p}' = 0; E = 0) = \frac{4\pi\hbar^2 a_s}{m} \ .$$

  Use it with an upper momentum cut-off: for $p \geq \frac{\hbar}{a_s}$, the real T-matrix would drop to zero.
- If you simply set $V(\mathbf{r}) \equiv g\delta(\mathbf{r})$ and ran perturbation theory, high-lying modes may be counted twice!

HFB troubleshooting

(1) The divergence of $\tilde{m}$ is due to double-counting of the high-lying modes, which we then subtract: define the renormalized

$$\tilde{m}^R(\mathbf{r}) = \tilde{m}(\mathbf{r}) - \lim_{i\to\infty} u_i(\mathbf{r})v_i^*(\mathbf{r})$$

to be replaced for $\tilde{m}(\mathbf{r})$ everywhere.

(2) A dirty way is to ignore all occurrences of $\tilde{m}$: this gives the HFB-Popov limit

$$\hat{\mathcal{H}}^{\mathrm{HFBP}} = \hat{\mathcal{H}}^{\mathrm{HF}} + \hat{\mathcal{H}}_2^{\mathrm{BOG}} = \sum_{i=0}^{2} \left(\hat{\mathcal{H}}_i + \delta\hat{\mathcal{H}}_i^{\mathrm{HF}}\right) .$$

The GPE is the same as in the HF limit, but it has a *gapless Bogoliubov spectrum*.

Static generalized many-body theories

What if scattering takes place in a medium?

(i) Intermediate states may be thermally populated ($\rightarrow$ bosonic enhancement of the transfer rate)

(ii) Intermediate states may be dressed quasiparticle states.

Upgrading of the 2-body to the *many-body* T-matrix is achieved by including $\tilde{m}(\mathbf{r}) = \left\langle \hat{\delta}\hat{\delta} \right\rangle$, which contains information about correlations between nearby atoms.

$\rightarrow$ Define a generalized effective interaction for collisions between two condensate atoms

$$g(\mathbf{r}) \equiv g \left(1 + \frac{\tilde{m}^R}{|\phi_0|^2} \right)$$

so that $g(\mathbf{r})|\phi_0|^2\phi_0 = g\left[|\phi_0|^2 + \tilde{m}^R\right]\phi_0$.

Static generalized many-body theories

Replacing for $g(\mathbf{r})$ where not already present, we get

$$\begin{pmatrix} \hat{L}(\mathbf{r}) & \hat{M}(\mathbf{r}) \\ -\hat{M}^*(\mathbf{r}) & -\hat{L}^*(\mathbf{r}) \end{pmatrix} \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} u_i(\mathbf{r}) \\ v_i(\mathbf{r}) \end{pmatrix}$$

$$\hat{L}(\mathbf{r}) = \hat{h}_0(\mathbf{r}) + 2g_c(\mathbf{r})|\phi_0(\mathbf{r})|^2 + 2g_t(\mathbf{r})\tilde{n}(\mathbf{r}) - \mu \,, \qquad \hat{M}(\mathbf{r}) = g_c(\mathbf{r}) \left[\phi_0(\mathbf{r})^2\right]$$

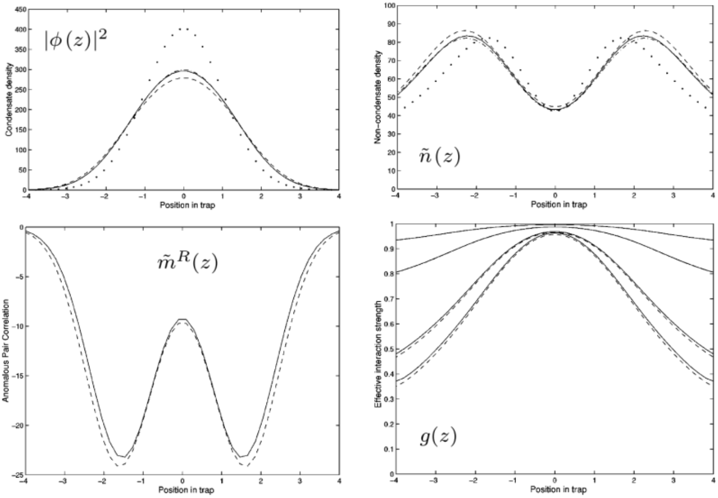
$$\left[\hat{h}_0 + g_c(\mathbf{r})|\phi_0|^2 + 2g_t(\mathbf{r})\tilde{n}(\mathbf{r})\right] \phi_0(\mathbf{r}) = \mu\phi_0(\mathbf{r}) \,.$$

What about collisions between a condensate and a thermal atom? We have two options:

$$\begin{cases} g_t(\mathbf{r}) = g_c(\mathbf{r}) = g(\mathbf{r}) \\ g_c(\mathbf{r}) = g(\mathbf{r}) \,, \qquad g_t(\mathbf{r}) = g \end{cases}$$

leading to two distinct *generalized HFB theories*.

Generalized HFB theories



What's next?

- The mean field approximations we took at the beginning have the effect of ignoring *particle exchanging collisions between condensed and thermal atoms* and *collisions between thermal atoms*, which lead to thermal population redistribution.
- For this reason, they can't be used to describe dynamical effects.
- This gave rise to four variants of mean field theories: HF, HFB, HFBP and generalized HFB.
- Going beyond mean field by taking into account correlations of three fluctuation operators, like $\left\langle \hat{\delta}^\dagger \hat{\delta} \hat{\delta} \right\rangle$, leads to a description of the dynamics of both the condensate and the thermal cloud and solves the problem of the gapped spectrum.


That's all Folks!