Generalized Hydrodynamics

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Maximal entropy states

Setting: homogeneous many-body 1d system of ∞ length, with short-range interactions. Let the system admit a set of conserved charges

$$Q_i = \int \mathrm{d}x\, \tilde{q}_i(x,t)\,, \qquad \frac{\partial Q_i}{\partial t} = 0$$

where $\tilde{q}_i(x,t)$ are a set of local densities s.t.

$$\partial_t \tilde{q}_i(x,t) + \partial_x \tilde{j}_i(x,t) = 0\,.$$

After a long enough time, we expect any finite region to relax to a state described by the density matrix ρ, the rest of the system playing the role of a bath. This ρ maximizes the entropy

$$S = -\operatorname{Tr}\{\rho \log \rho\} \qquad \rightarrow \qquad \rho \sim e^{-\sum_i \beta^i Q_i}\,.$$

Thermodynamics

Averages of observables, invariant under spacetime translations, are given by

$$\langle O \rangle = \mathrm{Tr}\{\rho O\} = \langle O \rangle_{\vec{\beta}}$$

$$-\frac{\partial}{\partial \beta^i} \langle O \rangle_{\vec{\beta}} = \int \mathrm{d}x \, \langle O \tilde{q}_i(x,0) \rangle^c_{\vec{\beta}}$$

where the set of Lagrange parameters $\vec{\beta}$ specifies the state. In particular,

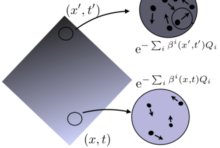
$$q_i \equiv \langle \tilde{q}_i(0,0) \rangle_{\vec{\beta}} \ , \qquad j_i \equiv \left\langle \tilde{j}_i(0,0) \right\rangle_{\vec{\beta}} \ .$$

Since there are as many β_i as there are conserved densities q_i, the set of averages $\vec{\mathbf{q}}$ can in fact be used to fully characterize the state.
This leads to the equations of state $j_i = j_i(\vec{\mathbf{q}})$.

Euler hydrodynamics

Real states are inhomogeneous/dynamical: $\langle O(x,t)\rangle$.

Assumption: local entropy maximisation



$$\langle O(x,t)\rangle \simeq \langle O(0,0)\rangle_{\vec{\beta}(x,t)}$$

where (x,t) is a fluid cell (*separation of scales*). Calling

$$q_i(x,t) \equiv \langle \tilde{q}_i(0,0)\rangle_{\vec{\beta}(x,t)} \ , \qquad j_i(x,t) \equiv \langle \tilde{j}_i(0,0)\rangle_{\vec{\beta}(x,t)}$$

leads to Euler hydrodynamic equations

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0 \ ,$$

which can be cast in their *quasilinear form* by defining the *flux Jacobian*

$$A_i^{\,j}(\vec{\mathbf{q}}(x,t)) \equiv \frac{\partial j_i(\vec{\mathbf{q}})}{\partial q_j} \qquad \rightarrow \qquad \partial_t q_i(x,t) + \sum_j A_i^{\,j}(x,t)\partial_x q_j(x,t) = 0 \ .$$

Normal modes, entropy

- Diagonalizing $A_i^{\ j}$ we get the *effective velocities*

$$RAR^{-1} = \mathrm{diag}(v_i^{\mathrm{eff}}) \ .$$

If we can find functions $n_i(\vec{\mathbf{q}})$ (normal modes) s.t.

$$R_i^{\ j} = \frac{\partial n_i}{\partial q_j} \qquad \rightarrow \qquad \partial_t n_i + v_i^{\mathrm{eff}} \partial_x n_i = 0$$

i.e. n_i is convectively transported at $v_i^{\mathrm{eff}}(\vec{\mathbf{n}})$.

- Define a *free energy* and *free energy flux* s.t.

$$q_i = \frac{\partial f}{\partial \beta^i} \ , \qquad j_i = \frac{\partial g}{\partial \beta^i} \ ,$$

hence the *entropy density* and its *current*

$$s = \sum_i \beta^i q_i - f \ , \qquad j_s = \sum_i \beta^i j_i - g \ .$$

Euler equations conserve entropy:

$$\partial_t s + \partial_x j_s = 0 \ .$$

Scattering map in integrable systems

Integrable system: ∞ conserved quantities with local densities. Maximal entropy states still have the form (*Generalized Gibbs Ensemble*)

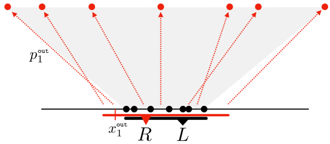
$$\rho \sim e^{-\sum_i \beta^i Q_i} \sim e^{-Q_w} \; .$$

Scattering map: a recipe

1. Choose a vacuum
2. Excite the system over some region $[0, L]$
3. Step to asymptotic coordinates, *i.e.* let the system evolve for $t \gg 1$ on the line, until the density is null and the excitations are far apart
4. Put a measure on such coordinates and send $L \to \infty$ to obtain the thermodynamics

Asymptotic coordinates

Consider a set of particles with repulsive interactions, described by the canonical coordinates



$$\{x_n, p_m\} = \delta_{n,m} \ .$$

Let evolve until $x_1 \ll x_2 \ll ... \ll x_n$, $p_n < p_{n-1}$. Here the potential won't be felt anymore and the evolution will be free:

$$\begin{cases} x_n(0) \\ p_n(0) \end{cases} \xrightarrow[t^* \gg 1]{\mathcal{H}} \begin{cases} x_n(t^*) \\ p_n(t^*) \equiv p_n^{\text{out}} \end{cases} \xrightarrow[-t^*]{\mathcal{H}_{\text{free}}} \begin{cases} x_n^{\text{out}} = x_n(t^*) - p_n^{\text{out}} t^* \\ p_n^{\text{out}} \end{cases}$$

This defines the canonical maps

$$\{x_n, p_n\} \xrightarrow{S_{\text{out}}} \{x_n^{\text{out}}, p_n^{\text{out}}\} \ , \qquad \{x_n, p_n\} \xrightarrow{S_{\text{in}}} \{x_n^{\text{in}}, p_n^{\text{in}}\} \ .$$

Some consequences

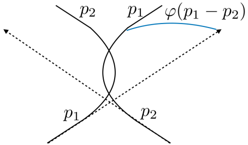
- Conserved charges, being local (or quasi-local), can only depend on the asymptotic momenta when the particles are far apart:

$$Q_w = \sum_i \beta^i Q_i \equiv \sum_n w(p_n^{\text{out}}) \,, \qquad w(p) = \sum_i \beta^i h_i(p) \,.$$

- Scattering can be viewed as $S_{\text{out}} \circ S_{\text{in}}^{-1}$.
- The charges Q_i commute with $\mathcal{H}$ and they can be thought as generators of nonlinear transformations acting on the asymptotic coordinates. This implies the factorization of scattering into 2-body processes.
- In 1+1 dimensions, this means elastic scattering:

$$\left\{p_n^{\text{in}}\right\} = \left\{p_n^{\text{out}}\right\} \,.$$

Quasiparticles



Quasiparticle: it's a *tracer* attached to real particles which jumps from particle to particle at collisions, following a given momentum. We label them by $p_n^{\mathtt{in}}$.

Each scattering event is described by a 2-body shift $\varphi(p_n - p_m)$. Summing over all of them,

$$x_n^{\mathtt{out}} = x_n^{\mathtt{in}} - \sum_{m>n} \varphi(p_n^{\mathtt{in}} - p_m^{\mathtt{in}}) + \sum_{m<n} \varphi(p_n^{\mathtt{in}} - p_m^{\mathtt{in}}) \, .$$

This can be used to approximate $S_{\mathtt{in}}$:

$$x_n^{\mathtt{in}} \simeq x_n + \left(\sum_{m>n, x_m<x_n} - \sum_{m<n, x_m>x_n} \right) \varphi(p_n^{\mathtt{in}} - p_m^{\mathtt{in}}) \, .$$

Quasiparticle description of thermodynamics

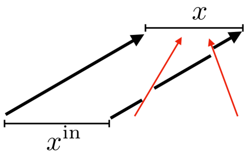
Let's compute the canonical partition function

$$\mathcal{Z} = \int_{x_n\in[0,L]} \prod_{n=1}^{N} \mathrm{d}x_n\,\mathrm{d}p_n\, e^{-\sum_i \beta^i Q_i}$$

$$\xrightarrow[S_{\mathrm{in}}]{} \int \prod_n \mathrm{d}p_n^{\mathrm{in}}\, e^{-\sum_n w(p_n^{\mathrm{in}})} \underbrace{\int_{x_n\in[0,L]} \prod_n \mathrm{d}x_n^{\mathrm{in}}}_{R^N}$$

where the volume contribution from quasiparticle n is

$$R = L + \sum_{m\neq n} \varphi(p_n^{\mathrm{in}} - p_m^{\mathrm{in}})\ .$$



Quasiparticle description of thermodynamics

In the limit $N, L \to \infty$ at fixed density, define the density of quasiparticles

$$\rho_p(p_n) = \lim_{N,L\to\infty} \frac{1}{L(p_{n+1} - p_n)}$$

so that

$$\frac{1}{L}\sum_n f(p_n^{\mathrm{in}}) \quad \to \quad \int \mathrm{d}p\, \rho_p(p) f(p)\,.$$

Rewrite the volume contribution in term of the density of space $\rho_s(p)$,

$$\frac{R}{L} = 1 + \frac{1}{L}\sum_{m\neq n} \varphi(p_n^{\mathrm{in}} - p_m^{\mathrm{in}}) \quad \to \quad 2\pi\rho_s(p_n^{\mathrm{in}}) = 1 + \int \mathrm{d}p\, \rho_p(p)\varphi(p_n^{\mathrm{in}} - p)\,.$$

Finally, construct the grancanonical partition function and perform a large deviation analysis.

Thermodynamic Bethe Ansatz: ingredients

1. Spectral space
2. Scattering shift $\varphi(p - p')$ (from 2-body scattering)
3. Free energy function $F(\epsilon)$, or occupation function $n(\epsilon) = \frac{\mathrm{d}F(\epsilon)}{\mathrm{d}\epsilon}$, *e.g.* for fermions/bosons

$$n(\epsilon) = (e^{\epsilon} \pm 1)^{-1} \, .$$

4. The functions $h_i(p)$ which specify the conserved quantities, $Q_i = \sum_n h_i(p_n^{\mathrm{in}})$. For example, in QM

$$Q_i \, |p_1, p_2, ..., p_N\rangle = \sum_n h_i(p_n) \, |p_1, p_2, ..., p_N\rangle \, .$$

5. The *source term* $w(p) = \sum_i \beta^i h_i(p)$, which specifies the state. For example, in QM, $w(p)$ is the one-particle eigenvalue of $\sum_i \beta^i Q_i$.

Thermodynamic Bethe Ansatz: results

- Energy contribution of the quasiparticle p

$$\epsilon(p) = w(p) + \int \frac{\mathrm{d}p'}{2\pi} \varphi(p - p') F(\epsilon(p'))$$

- Dressing operation

$$f^{\mathrm{dr}}(p) = f(p) + \int \frac{\mathrm{d}p'}{2\pi} \varphi(p - p') n(p') f^{\mathrm{dr}}(p')$$

where $n(p)$ is the occupation function

$$n(p) = \left. \frac{\mathrm{d}F(\epsilon)}{\mathrm{d}\epsilon} \right|_{\epsilon = \epsilon(p)} .$$

We can then rewrite

$$\rho_p(p) = \frac{1}{2\pi} 1^{\mathrm{dr}}(p) n(p) , \quad \rho_s(p) = \frac{1}{2\pi} 1^{\mathrm{dr}}(p) \quad \to \quad n(p) = \frac{\rho_p(p)}{\rho_s(p)} .$$

Charges and currents

Charge/entropy densities are constructed as

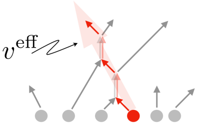
$$q_i = \int \mathrm{d}p\, \rho_p(p) h_i(p)\,, \qquad s = \int \mathrm{d}p\, \rho_s(p) \left[\epsilon(p) n(p) - F(\epsilon(p))\right]\,.$$

Currents require a definition of *time*. We need as an extra ingredient the energy function

$$E(p) = h_k(p) \quad \text{s.t.} \quad \mathcal{H} \equiv Q_k = \sum_n h_k(p_n^{\mathrm{in}})\,.$$

This allows to define an effective velocity

$$v^{eff}(p) = \frac{(E')^{\mathrm{dr}}(p)}{1^{\mathrm{dr}}(p)} = \frac{(E')^{\mathrm{dr}}(p)}{2\pi\rho_s(p)}$$



and finally the fluxes

$$j_i = \int \mathrm{d}p\, \rho_p(p) v^{eff}(p) h_i(p)\,, \quad j_s = \int \mathrm{d}p\, \rho_s(p) v^{eff}(p) \left[\epsilon(p) n(p) - F(\epsilon(p))\right]\,.$$

Fundamental equations

In scattering theory (with TBA),

$$\{\beta^i\} \leftrightarrow \{q_i\} \leftrightarrow \rho_p(p) \leftrightarrow n(p) \leftrightarrow w(p) \leftrightarrow \epsilon(p)$$

are all equivalent ways of describing the state of the system. In hydrodynamics, each *fluid cell* (x, t) has its maximal entropy state:

$$\rho_p(p) \rightarrow \rho_p(p, x, t) \,.$$

The conservation equations become

$$\int \mathrm{d}p \, h_i(p) \left\{ \partial_t \rho_p + \partial_x \left(v^{\mathrm{eff}} \rho_p \right) \right\} = 0 \,.$$

If $\{h_i\}$ form a complete set, we get the GHD equations

$$\partial_t \rho_p(p, x, t) + \partial_x \left[v^{\mathrm{eff}}(p, x, t) \rho_p(p, x, t) \right] = 0$$

$$v^{\mathrm{eff}}(p, x, t) = E'(p) + \int \mathrm{d}p' \, \varphi(p - p') \rho_p(p', x, t) \left[v^{\mathrm{eff}}(p', x, t) - v^{\mathrm{eff}}(p, x, t) \right] \,.$$

Conservation laws and normal modes

- Again, entropy is conserved:

$$\partial_t s + \partial_x j_s = 0 \; .$$

- The occupation function $n(p)$ gives the GHD normal modes

$$\partial_t n(p,x,t) + v^{\mathtt{eff}}(p,x,t)\partial_x n(p,x,t) = 0$$

  i.e. the density of quasiparticle per unit available space in asymptotic coordinates is convectively transported by the GHD flow, along the characteristic curve with $v^{\mathtt{eff}}(p,x,t)$.

- Any $r(n)$ gives rise to a conservation law:

$$\partial_t \left[\rho_p \cdot r(n)\right] + \partial_x \left[v^{\mathtt{eff}} \cdot \rho_p \cdot r(n)\right] = 0 \; .$$

  Choosing in particular $r = \frac{1}{n}$,

$$\partial_t \rho_s + \partial_x \left[v^{\mathtt{eff}} \rho_s\right] = 0 \; .$$

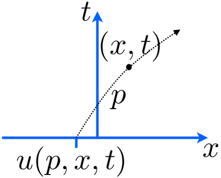
Solution by characteristics

Define the characteristic curve starting at position u

$$t \mapsto x(p,u,t) \ : \ \begin{cases} x(p,u,0) = u \\ \partial_t x(p,u,t) = v^{\mathrm{eff}}(p, x(p,u,t), t) \ . \end{cases}$$

By GHD,

$$\frac{\mathrm{d}}{\mathrm{d}t} n(p, x(p,u,t), t) = 0 \ .$$



Let's invert $x(p,u,t) \to u(p,x,t)$.
Climbing backwards from (x,t), since n is constant,

$$n(p,x,t) = n(p, u(p,x,t), 0)$$

so if we know $u = u(p,x,t)$ we have solved the IVP.

Equation for $u(p, x, t)$

The function $u(p, x, t)$ satisfies, just like n,

$$\partial_t u(p, x, t) + v^{\text{eff}}(p, x, t)\partial_x u(p, x, t) = 0$$

with the initial condition $u(p, x, 0) = x$.

Example: free model
$v^{\text{eff}}(p) \equiv v(p) = E'(p)$ for any state. Then $x = u + v(p)t$.

Suppose the system is asymptotically homogeneous on the far left ($x \leq x_0 \ll 0$) and does not evolve. Let

$$\hat{v}(p) \equiv 2\pi\rho_s(p, -\infty, 0)v^{\text{eff}}(p, -\infty, 0) .$$

Then u is determined by

$$2\pi \left\{ \int_{x_0}^{u} \mathrm{d}y\, \rho_s(p, y, 0) - \int_{x_0}^{x} \mathrm{d}y\, \rho_s(p, y, t) \right\} + \hat{v}(p) \cdot t = 0 .$$

Geometric interpretation

Define a new space coordinate

$$\mathrm{d}\hat{x} = 2\pi \rho_s(p,x,t)\,\mathrm{d}x \ , \qquad \hat{x}(p,t) = \int_{x_0}^{x} 2\pi\rho_s(p,y,t)\,\mathrm{d}y \ .$$

In these terms, the equation for u becomes simply

$$\hat{x}(p,t) = \hat{u}(p,0) + \hat{v}(p)\cdot t \ .$$

This change of metric trivializes the fluid equation,

$$\partial_t \hat{n}(p,\hat{x},t) + \hat{v}(p)\partial_{\hat{x}}\hat{n}(p,\hat{x},t) = 0 \ ,$$

because $\hat{x}$ is the space perceived by asymptotic particles, whose evolution is trivial:

$$x_n^{\mathrm{in}}(t) = x_n^{\mathrm{in}} + p_n^{\mathrm{in}}\cdot t \ , \qquad p_n^{\mathrm{in}}(t) = p_n^{\mathrm{in}} \ .$$

Being $n = \frac{\rho_p}{\rho_s}$ the density in asymptotic phase space,

$$\text{GHD} = S_{\mathrm{in}}^{-1}(\text{ Liouville equations }) \ .$$

External force fields

Let's add a potential $\mathcal{H} \mapsto \mathcal{H} + V(x)$ which breaks momentum conservation. Can we still use GHD?

→ Yes, if $V(x)$ varies slowly: then each fluid cell will still locally maximize entropy.

Example: Lieb-Liniger model with external $V(x)$

$$\mathcal{H} = \int \mathrm{d}x \left\{ \frac{1}{2}\partial_x \Psi^\dagger \partial_x \Psi + g\Psi^\dagger\Psi^\dagger\Psi\Psi + V(x)\Psi^\dagger(x)\Psi(x) \right\} .$$

Rewrite $\mathcal{H}(x) = \mathcal{H}_{LL} + V(x)Q_0$, where

$$\mathcal{H}_{LL} = \int_x \mathrm{d}y \left\{ \frac{1}{2}\partial_y \Psi^\dagger \partial_y \Psi + g\Psi^\dagger\Psi^\dagger\Psi\Psi \right\} , \qquad Q_0 = \int_x \mathrm{d}y \, \Psi^\dagger\Psi .$$

This gives the *local energy function*

$$E(p,x) = \frac{p^2}{2} + V(x) .$$

GHD equations with external $V(x)$

In general, one could couple a field to any charge:

$$\mathcal{H}(x) = \mathcal{H}_{LL} + \sum_i V^i(x) Q_i \quad \rightarrow \quad E(p,x) = \frac{p^2}{2} + \sum_i V^i(x) h_i(x) \, .$$

- The *effective velocity* gets modified as

$$v^{eff}(p,x,t) = \frac{\left[E'(\cdot,x)\right]^{\mathrm{dr}}(p,x,t)}{1^{\mathrm{dr}}(p,x,t)}$$

- GHD equations acquire a new term

$$\partial_t \rho_p + \partial_x \left(v^{\mathrm{eff}} \rho_p \right) + \partial_p \left(a^{\mathrm{eff}} \rho_p \right) = 0$$

  where we introduced the effective acceleration

$$a^{eff}(p,x,t) = \frac{\left[-\partial_x E(\cdot,x)\right]^{\mathrm{dr}}(p,x,t)}{1^{\mathrm{dr}}(p,x,t)} \, .$$

The flea gas algorithm

→ A molecular dynamics algorithm which solves GHD. We want to reproduce

$$\Delta x = E'(p)\Delta t + \sum_n \varphi(p - p_n) \cdot (\pm 1)$$

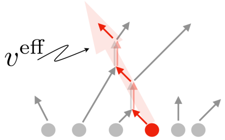
as it can be shown that

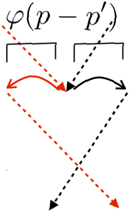
$$\langle \Delta x \rangle = v^{\mathrm{eff}}(p)\Delta t \, .$$

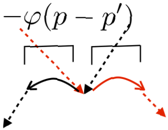
(i) Particles travel freely at $v(p)$ and, when they collide, they jump instantly.

(ii) They keep track of their encounters.

(iii) We account for *inside-jumps* via a recursive procedure.







Take home message

Hydrodynamics

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TBA on integrable systems

$\Downarrow$

GHD = S_{in}^{-1}(Liouville equations)

Thanks for your attention!


That's all Folks!