Growth processes and Random Matrices

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3 July 2020

Outline

Interacting particle systems
- Exclusion processes
- Growth phenomena
- Hydrodynamic limit
- KPZ equation

Random Matrix Theory
- Gaussian ensembles
- Largest eigenvalue statistics
- Ulam's problem
- Mapping growth problems on RMT

Explicit examples
- dTASEP with step initial conditions
- KPZ: map on directed polymer

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  - Hydrodynamic limit
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- Random Matrix Theory
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Interacting particle systems

- Non equilibrium stationary states (NESS): systems which are kept out of equilibrium by external influences. Macroscopic state variables are *time-independent*, but there are non-vanishing *microscopic currents*.
- Examples: heat conduction (Fourier's law), diffusion (Fick's law), electric conduction (Ohm's law).
- An alternative to the Hamiltonian description is to start from a stochastic microscopic dynamics in terms of *interacting particle systems*: they are lattice models with a discrete set of states associated to each site, and local interactions.

Exclusion processes

Setting: identical particles on a lattice $\Omega \subset \mathbb{Z}^d$.
A configuration is specified by $\mathbf{n} = \{n_x\}_{x \in \Omega} \in \{0,1\}^\Omega$.

Dynamics:

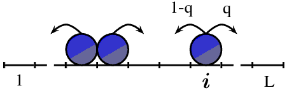
(i) Each particle carries a clock which rings according to a Poisson process

(ii) When the clock rings, particle at x attempts to jump in y with probability $q_{xy}(\mathbf{n})$

(iii) If y is empty, the jump is performed.

Assume: $d = 1$, homogeneous, no interactions, nearest neighbour hoppings. Then

$$q_{xy} = q\delta_{y,x+1} + (1-q)\delta_{y,x-1}$$

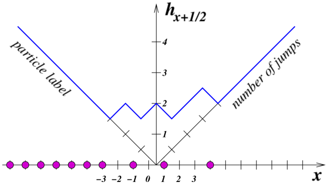


$\rightarrow$ SSEP, ASEP, TASEP...

Mapping to a growth model (Rost '81)

Let's map on the height configuration $\{n_x\} \longleftrightarrow \left\{h_{x+\frac{1}{2}}\right\}$ by fixing $h_{\frac{1}{2}} = 0$ and

$$h_{x+\frac{1}{2}} - h_{x-\frac{1}{2}} = \frac{1}{2} - n_x = \pm\frac{1}{2} \,.$$



This is known as single step model. Notice

$h_{x+\frac{1}{2}}(t) - h_{x+\frac{1}{2}}(0) =$ net # of particles which crossed the bond $(x, x+1)$ from left to right up to time t.

Choosing the initial condition

$$\begin{cases} n_x = 1 - \theta(x) \\ x_j(t=0) = -j \end{cases} \qquad \longleftrightarrow \qquad h_{x+\frac{1}{2}}(0) = \frac{1}{2}|x|$$

we get a *corner growth model*. Rotating by 45° gives the net # of jumps a given particle has performed as a function of its label j.

Growth phenomena

- Experimentally we observe, at late times,

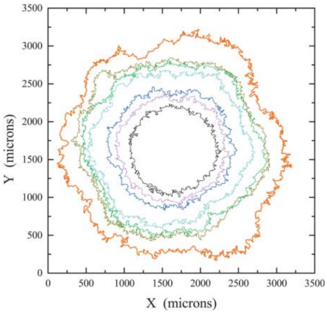
$$\langle h(t) \rangle \sim t$$

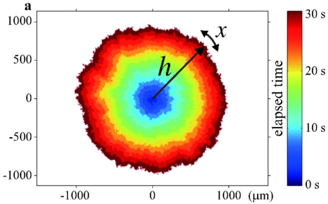
$$w = \sqrt{\langle (h - \langle h \rangle)^2 \rangle} \sim t^{\frac{1}{3}} .$$

Smells like universal!

- Many simple growth models (Eden, PNG, DP in a random medium, AEP...) indeed predict

$$h(t) = vt + \chi t^{\frac{1}{3}} .$$





Hydrodynamics

How does deterministic evolution emerge from stochastic microscopic dynamics on large scales?

Particle density: $\rho = \langle n_x \rangle$
Stationary particle current: net # of particles jumping $x \to x+1$ per unit time. We look for $J(\rho)$.

- Example: ASEP on a ring

$$\langle n_x(1 - n_{x+1}) \rangle = \langle n_{x+1}(1 - n_x) \rangle = \frac{N}{L} \cdot \frac{L-N}{L-1}$$

so that at fixed $\rho = N/L$

$$J = q \langle n_x(1 - n_{x+1}) \rangle - (1-q) \langle n_{x+1}(1 - n_x) \rangle = \frac{(2q-1)\rho(1-\rho)}{1 - \frac{1}{L}}$$

$$\xrightarrow[L\to\infty]{} (2q-1)\rho(1-\rho) \, .$$

Hydrodynamics

- Start from a slowly varying $\rho(x,0)$ (say, on a scale l). Since particle density is locally conserved,

$$\partial_t \rho(x,t) + \partial_x j(x,t) = 0 \ .$$

For $l \to \infty$, we expect $j(x,t) \to J(\rho(x,t))$, so that

$$\boxed{\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}J(\rho) = 0}$$

- What about fluctuations? Expanding around $\rho(x,t) = \bar{\rho} + u(x,t)$ up to 2^{nd} order we get

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \lambda u \frac{\partial u}{\partial x} = 0$$

where

$$c \equiv \left.\frac{\mathrm{d}J}{\mathrm{d}\rho}\right|_{\bar{\rho}} , \qquad \lambda \equiv \left.\frac{\mathrm{d}^2 J}{\mathrm{d}\rho^2}\right|_{\bar{\rho}} \ .$$

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From Euler to KPZ

- Stepping to a comoving frame we get the *inviscid Burgers equation*

$$u_t + \lambda u u_x = 0 \,.$$

- Let's introduce fluctuation/dissipation: we get the stochastic Burgers equation

$$u_t + \lambda u u_x = \nu u_{xx} - \zeta_x$$

$$\langle \zeta \rangle = 0 \,, \qquad \left\langle \zeta(x,t)\zeta(x',t') \right\rangle = D\delta(x-x')\delta(t-t')$$

which has a conservation form, with a current

$$\partial_x j(x,t) = \lambda u u_x - \nu u_{xx} + \zeta_x \,.$$

- Let $u(x,0) = 0$ and define the height function

$$h(x,t) = \int_0^t j(x,s)\,\mathrm{d}s$$

so that $\partial_x h = -u$. This gives the KPZ equation

$$\boxed{\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}\left(\boldsymbol{\nabla} h\right)^2 + \zeta}$$

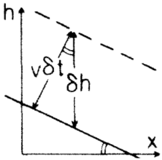
KPZ (1986): an effective model for growth

$$\boxed{\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\boldsymbol{\nabla} h)^2 + \zeta}$$

- $\nabla^2 h$ tries to smoothen the surface (opposing the noise), while $(\boldsymbol{\nabla} h)^2$ drives the system out of equilibrium.
- This is the simplest nonlinear, local differential equation governing the growth of a profile (higher order terms would be irrelevant).
- Growth occurs mainly at an "active" zone on the surface of the cluster, in a direction normal to the interface:

$$\delta h = \sqrt{(v\delta t)^2 + (v\delta t \boldsymbol{\nabla} h)^2}$$

$$\dot{h} = v\sqrt{1 + (\boldsymbol{\nabla} h)^2} \simeq v + \frac{v}{2}(\boldsymbol{\nabla} h)^2 + \dots$$



KPZ (1986): scaling

- Construct the adimensional quantity

$$\bar{h} = \frac{h}{\left(\frac{D^2}{\nu^2}|\lambda|t\right)^{\frac{1}{3}}} \, .$$

  We expect its fluctuations to exhibit a universal distribution.

- KPZ proved by Dynamical Renormalization Group that the interface width grows as

$$w(R,t) \sim R^{\chi} w_0(R,t) \sim t^{\frac{\chi}{z}} w_0(\mathcal{O}(1)) \sim t^{\frac{1}{3}} \, ,$$

  being $z = \frac{3}{2}$, $\chi = \frac{1}{2}$ in $d = 1$.

- Does the universality hold beyond the 2^{nd} moment?

Outline

Reminder: Gaussian random matrices

- Gaussian random matrix: the *jpdf* of its entries is given by

$$\mathcal{P}[M] \propto e^{-\beta \frac{N}{2} \operatorname{Tr} M^2}$$

and that of its eigenvalues by (Wigner '51)

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) \propto e^{-\frac{\beta}{2} N \sum_{i=1}^N \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta .$$

- In general, rotationally invariant matrices have

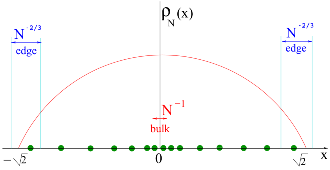
$$\begin{aligned}\mathcal{P}(\lambda_1, \ldots, \lambda_N) &= C_N |\Delta(\vec{\lambda})|^\beta \varphi(\operatorname{Tr} M, \operatorname{Tr} M^2, \ldots, \operatorname{Tr} M^N) \\ &\equiv \frac{1}{\mathcal{Z}_N} |\Delta(\vec{\lambda})|^\beta e^{-\sum_{i=1}^N V(\lambda_i)} .\end{aligned}$$

- The N real eigenvalues are strongly correlated variables. They admit the Coulomb gas interpretation (Dyson '62)

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) = \frac{1}{\mathcal{Z}_N} e^{-\frac{\beta}{2} \left\{ N \sum_{i=1}^N \lambda_i^2 - \sum_{j \neq k} \log |\lambda_j - \lambda_k| \right\}} .$$

Spectral density

$$\rho(\lambda, N) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle$$
$$\xrightarrow[N\to\infty]{} \rho(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2} \, .$$



There are 2 length scales in this problem:

1. Bulk interparticle distance

$$\int_0^{l_{\text{bulk}}} \rho(\lambda, N)\, \mathrm{d}\lambda \cong \frac{1}{N} \quad \to \quad l_{\text{bulk}} \sim N^{-1} \, .$$

2. Edges

$$\int_{\sqrt{2}-l_{\text{edge}}}^{\sqrt{2}} \rho(\lambda, N)\, \mathrm{d}\lambda \cong \frac{1}{N} \quad \to \quad l_{\text{edge}} \sim N^{-\frac{2}{3}} \, .$$

$$\longrightarrow \quad l_{\text{edge}} \gg l_{\text{bulk}} \quad .$$

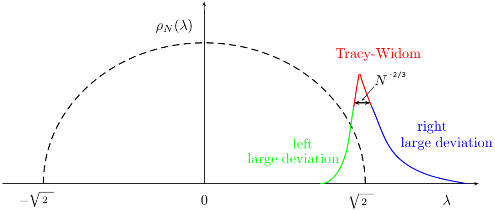
Largest eigenvalue statistics

$$\langle \lambda_{\max} \rangle = \sqrt{2}\,, \qquad |\lambda_{\max} - \sqrt{2}| \sim l_{\mathrm{edge}} \sim N^{-\frac{2}{3}}\,.$$

Tracy&Widom ('94) show that, for large N, typical fluctuations behave as

$$\boxed{\lambda_{\max} = \sqrt{2} + \frac{1}{\sqrt{2}} N^{-\frac{2}{3}} \chi_\beta}$$

$$f_\beta(x) \sim \begin{cases} e^{-\frac{2}{3}\beta x^{\frac{3}{2}}}, & x \to \infty \\ e^{-\frac{\beta}{24}|x|^3}, & x \to -\infty\,. \end{cases}$$



For $\beta = 2$ (GUE),

$$\boxed{F_2(s) = \det(\mathbb{I} - P_s A P_s)}$$

where P_s projects on $[s, +\infty)$ and A is the Airy kernel

$$A(a,b) = \int_0^\infty Ai(a+t)Ai(b+t) = \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}\,.$$

Ulam's problem (1961)

(i) Take a sequence of n integers and consider any one of the $n!$ permutations.

(ii) For each of them, construct all possible increasing subsequences and take the longest; let L_n be its length.

L_n is a random variable: it fluctuates from one permutation to another, and each one occurs with equal probability $\frac{1}{n!}$.

Ulam's problem (LIS): what is the statistics of L_n?

Baik, Deift, Johansson (1999):

$$\boxed{L_n = 2\sqrt{n} + n^{\frac{1}{6}}\chi_2}$$

with the same χ_2 as in GUE (Tracy-Widom).

Mapping growth problems on RMT

Discrete KPZ growth models $\longrightarrow$ **Ulam problem** $\longrightarrow$ **Tracy-Widom RMT**

In growth models,

$$h(t) = vt + t^{\frac{1}{3}}\chi_2 \, .$$

In Ulam's problem,

$$L_n = 2\sqrt{n} + n^{\frac{1}{6}}\chi_2 \, .$$

Under the exact mapping $h(t) \longleftrightarrow L_{n=t^2}$ one finds the distribution $\mathcal{P}(H,t)$ of the centered height function

$$H = h(x,t) - \langle h(x,t) \rangle$$

to be

$$\mathcal{P}(H,t) \xrightarrow[t \gg 1]{} \frac{1}{c t^{\frac{1}{3}}} f_2\left(\frac{H}{c t^{\frac{1}{3}}}\right)$$

where $f_2(x)$ is universal.

Outline

dTASEP ($d = 1$)

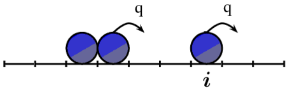
Particles on a line, $x_j(t) \in \mathbb{Z}$ at times $t = 0, 1, 2 \dots$
Step initial conditions: $x_j(t=0) = -j$, $j = 0, 1, 2 \dots$
At each time step, a jump to the right is attempted independently by all particles with probability q, but it's discarded if the receiving site is occupied.

Define the flux

$h_r(t) = \#$ of particles which crossed the bond $(r, r+1)$ up to time t.

The *stationary current* can be computed exactly from hydrodynamics ($\rho = \frac{1}{2}$),

$$J_q(\rho) = \frac{1}{2}\left(1 - \sqrt{1 - 4q\rho(1-\rho)}\right)$$



so we expect $\boxed{h_0(t) \sim J_q t}$

dTASEP

Defining

$$Z_q(t) \equiv \frac{h_0(t) - J_q t}{t^{\frac{1}{3}}}$$

we expect $\sigma^2_{Z_q} \xrightarrow{t \gg 1} \mathit{finite}$.
This reminds of CLT, but the limiting distribution is not Gaussian. T&W found for GUE

$$Prob\left(\frac{\lambda_{\max}(M) - \sqrt{2N}}{(8N)^{-\frac{1}{6}}} \leq s\right) \xrightarrow[N\to\infty]{} F_2(s)\,.$$

Theorem (Johansson 2000)

$$\boxed{Prob\left(\frac{h_0(t) - J_q t}{V t^{\frac{1}{3}}} \leq s\right) \xrightarrow[t\to\infty]{} 1 - F_2(-s)}$$

with $V = \left(2^{-4} q \sqrt{1-q}\right)^{\frac{1}{3}}$.

PROOF part 1: dTASEP $\to$ waiting times

Define the jumping times

$$T(j,k) \equiv \min\{t \in \mathbb{N} : x_j(t) = x_j(0) + (k+1) = -j + (k+1)\}$$

= time at which particle j has completed its $(k+1)^{\text{th}}$ jump.

By construction, since $x_k(0) = -k$, at time $T_k = T(k,k)$

$$x_0(T_k) > x_1(T_k) > \cdots > x_k(T_k) = 1 > 0 > x_{k+1}(T_k) > \ldots$$

so exactly $(k+1)$ particles crossed the bond (0,1):

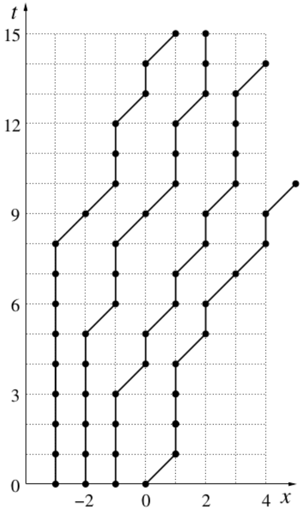
$$h_0(T_k) = k+1\,, \qquad h_0(t < T_k) \leq k$$

$$\boxed{Prob(h_0(t) \leq k) = Prob(T_k > t) = 1 - Prob(T(k,k) \leq t)}$$

Define the waiting times

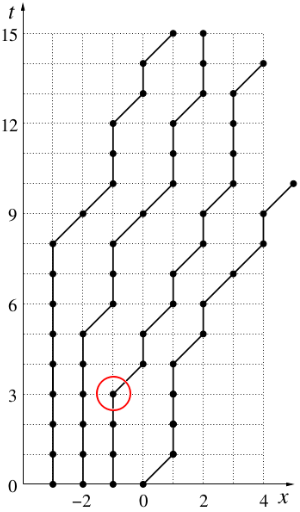
$w_{jk} = \#$ of times particle j stays on site $x_j(0) + k$ after it becomes possible to jump to the next, $x_j(0) + (k+1) = k+1-j$.

PROOF part 1: dTASEP $\to$ waiting times



$$\longrightarrow \begin{pmatrix} 0 & 3 & 1 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & 3 & \dots \\ 1 & 2 & 0 & 2 & ? & \dots \\ 2 & 0 & 2 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

PROOF part 1: dTASEP $\to$ waiting times



$$\longrightarrow \begin{pmatrix} 0 & 3 & 1 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & 3 & \dots \\ 1 & 2 & 0 & 2 & ? & \dots \\ 2 & 0 & 2 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

PROOF part 1: dTASEP $\to$ waiting times

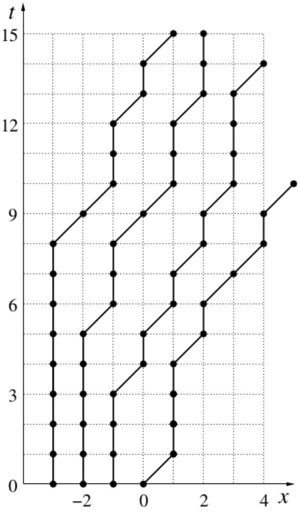
- Now notice $(j, k > 0)$

$$
\begin{aligned}
T(j,k) &= 1 + w_{jk} + (\texttt{time at which this jump becomes possible}) \\
&= 1 + w_{jk} + \max\{T(j,k-1), T(j-1,k)\} \\
&\xrightarrow[\texttt{by induction}]{} 1 + j + k + \max_{\phi:(0,0)\to(j,k)} \left(\sum_{(i,j)\in\phi} w_{ij} \right)
\end{aligned}
$$

  i.e. maximize the total waiting time with the constraint that only right-downward steps are allowed.

- In order to compute $Prob\,(T(k,k) \leq t)$, we only need the topleft $(k+1)x(k+1)$ corner of the matrix of waiting times w_{ij}.

PROOF part 1: dTASEP $\to$ waiting times



$$\longrightarrow \begin{pmatrix} 0 & 3 & 1 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & 3 & \dots \\ 1 & 2 & 0 & 2 & ? & \dots \\ 2 & 0 & 2 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

PROOF part 1: dTASEP $\rightarrow$ waiting times

Let's count the number of $(k+1)x(k+1)$ matrices $w_{ij} \in \mathbb{N}$ which satisfy

$$T(k,k) = 2k+1+\max_{\phi:(0,0)\rightarrow(k,k)}\left(\sum_{(i,j)\in\phi} w_{ij}\right) \leq t\,. \qquad (*)$$

To each of these matrices W we give a (normalized) weight

$$f_W \equiv \underbrace{q^{(k+1)^2}}_{\text{\# of decisions to jump}} \cdot \underbrace{(1-q)^{\sum_{ij} w_{ij}}}_{\text{\# of decisions to stay}}$$

so that

$$\boxed{Prob\left(T(k,k)\leq t\right) = \sum_{W\in(*)} q^{(k+1)^2}\cdot(1-q)^{|W|_1}}$$

PROOF part 2: waiting times $\to$ random words

To each $W = w_{ij}$ we associate a list of pairs (i, j) listed in lexicographical order, with the rule

$w_{ij} \equiv$ how often the pair (i, j) appears in the list.

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & 3 & \dots \\ 1 & 2 & 0 & 2 & ? & \dots \\ 2 & 0 & 2 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \longrightarrow \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & \dots \\ 1 & 1 & 1 & 2 & 0 & 0 & 3 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 2 & 2 & \dots \end{matrix}$$

This may be regarded as a random list of 2-letters words drawn from the alphabet $\{0, 1, \dots, k\}$.

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This may be regarded as a random list of 2-letters words drawn from the alphabet $\{0, 1, \dots, k\}$.

PROOF part 2: waiting times $\to$ random words

But any right-downward path $\phi \in W$ corresponds to a subsequence in this list which is weakly increasing in both rows, whence

$$\boxed{Prob\,(T(k,k) \leq t) = \sum_{\phi \in D(k,t)} q^{(k+1)^2} \cdot (1-q)^{|\phi|}}$$

where

$D(k,t) =$ set of finite sequences ϕ of lexicographically ordered 2-letter words from the alphabet $\{0, 1, \ldots, k\}$ and for which the length of the longest subsequence of ϕ (weakly increasing in both letters) is at most $t - 2k - 1$.

PROOF part 3: random words → RMT

Random words → Semi-Standard Young Tableaux → Schur polynomials →

$$\boxed{Prob\,(T(k,k) \leq t) = \frac{C_{q,k}}{(k+1)!} \sum_{\substack{\vec{\mathbf{y}} \in \mathbb{Z}^{k+1} \\ 0 \leq y_i \leq t-k-1}} \Delta(\vec{\mathbf{y}})^2 \prod_{i=0}^{k} (1-q)^{y_i}}$$

to be compared with the expression for GUE

$$Prob\,(\lambda_{\max} \leq \Lambda) = \frac{1}{\mathcal{Z}_N} \int_{-\infty}^{\Lambda} \cdots \int_{-\infty}^{\Lambda} \Delta(\vec{\lambda})^2 \prod_{j=1}^{N} e^{-\lambda_j^2}\, \mathrm{d}\lambda_j \;.$$

In both cases we adopt the technique of orthogonal polynomials: both Meixner and Hermite polynomials have their asymptotic behavior (beyond their largest zero) described in terms of Airy functions.

PROOF part 4: asymptotics

Define the weight functions ($x \in \mathbb{Z}$): $w_q(x) = \theta(x)(1-q)^x$.
Take $(p_j)_{j\geq 0}$ polynomials of degree j with leading coefficient γ_j and let

$$\varphi_j(x) \equiv p_j(x)\sqrt{w_q(x)}\,.$$

By a property of Vandermonde determinants,

$$\left(\det \varphi_j(y_i)\Big|_{0\leq i,j\leq k}\right)^2 = (\gamma_0 \dots \gamma_k)^2\,\Delta(\vec{y})^2 \prod_{i=0}^{k}(1-q)^{y_i}$$

and since moreover $(\det A)^2 = \det A^T A = \det\left(\sum_j A_{ji}A_{jk}\right)$ we can rewrite

$$\boxed{Prob\,(T(k,k)\leq t) = \frac{C_{q,k}}{(\gamma_0 \dots \gamma_k)^2}\det S}$$

$$(S)_{ij} = \sum_{x\in\mathbb{Z}} \varphi_i(x)\varphi_j(x)\left(1-\chi_{[t-k,\infty)}(x)\right)$$

PROOF part 4: asymptotics

Choose $(p_j)_{j\geq 0}$ to be Meixner polynomials:

$$\sum_{x\in\mathbb{Z}} \varphi_i(x)\varphi_j(x) = \sum_{x\in\mathbb{Z}} p_i(x)p_j(x)w_q(x) = \delta_{ij} \; .$$

Letting $S = \mathbb{I} - R(t-k)$, with

$$(R(s))_{ij} = \sum_{x\in\mathbb{Z}} \varphi_i(x)\varphi_j(x)\chi_{[s,\infty)}(x) = \sum_{x\geq s} \varphi_i(x)\varphi_j(x)$$

we can rewrite

$$\boxed{Prob\,(T(k,k)\leq t) = \frac{C_{q,k}}{\left(\gamma_0\ldots\gamma_k\right)^2}\det[\mathbb{I} - R(t-k)]}$$

PROOF part 4: asymptotics

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Letting $S = \mathbb{I} - R(t-k)$, with

$$(R(s))_{ij} = \sum_{x\in\mathbb{Z}} \varphi_i(x)\varphi_j(x)\chi_{[s,\infty)}(x) = \sum_{x\geq s} \varphi_i(x)\varphi_j(x)$$

we can rewrite

$$\boxed{Prob\,(T(k,k)\leq t) = \cancelto{1}{\frac{C_{q,k}}{(\gamma_0\ldots\gamma_k)^2}}\ \det[\mathbb{I} - R(t-k)]}$$

PROOF part 4: asymptotics

Finally, we may interpret $R(s) = A(s) \circ B(s)$, with

$$B(s): \; \vec{\mathbf{u}} \in \mathbb{R}^{k+1} \quad \mapsto \quad f(x) = \sum_j u_j \, \varphi_j(x) \Big|_{[s,\infty)}$$

$$A(s): \; f(x) \quad \mapsto \quad \sum_{x \geq s} f(x) \varphi_j(x) \, , \quad j = 0, 1, \ldots, k$$

and using $\det(\mathbb{I} - AB) = \det(\mathbb{I} - BA)$ we find

$$\boxed{Prob\,(T(k,k) \leq t) = \det[\mathbb{I} - \Gamma_k(t-k)]}$$

$$\Gamma_k(s): \; f(x) \quad \mapsto \quad \left(\sum_{y \geq s} \sigma_k(x,y) f(y) \right)_{x \geq s} \, , \quad \sigma_k(x,y) = \sum_{j=0}^{k} \varphi_j(x) \varphi_j(y)$$

$$\boxed{Prob(h_0(t) \leq k) = 1 - Prob(T(k,k) \leq t) = 1 - \det[\mathbb{I} - \Gamma_k(t-k)]}$$

Q.E.D.

What about KPZ itself?

$$\boxed{\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} \left(\boldsymbol{\nabla} h \right)^2 + \zeta}$$

It remained rather poorly understood until 2010!
Non-rigorous approach: apply to KPZ equation the map

$$\begin{cases} Z(x,t) = e^{\frac{\lambda}{2\nu} h(x,t)} \\ T = 2\nu \ , \ V(x,t) = -\lambda \zeta(x,t) \end{cases} \longrightarrow \boxed{\frac{\partial Z}{\partial t} = \frac{T}{2} \nabla^2 Z - \frac{1}{T} V(x,t) Z}$$

But this may be regarded as a Bloch equation for the propagator of a directed polymer in a random potential

$$Z(x,t|0,0) = \int_{\phi(0)=0}^{\phi(t)=x} \mathcal{D}\phi(\tau) e^{-\frac{1}{T} \int_0^t \mathrm{d}\tau \left\{ \frac{1}{2} \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \right)^2 + V(x(\tau),\tau) \right\}}$$

$$Z(x,0|0,0) = \delta(x) \ , \qquad \langle V(x,t) V(x',t') \rangle = D\lambda^2 \delta(x-x') \delta(t-t') \ .$$

What about KPZ itself?

We are after the *pdf* of the height field

$$h(x,t) \equiv \frac{2\nu}{\lambda} \log Z \propto \underbrace{F}_{\texttt{free energy}} .$$

This can be found using replicas: the moments

$$Z_n \equiv \langle Z(x_1, t|y_1, 0) \ldots Z(x_n, t|y_n, 0) \rangle$$

satisfy, by Feynman-Kac formula,

$$\partial_t Z_n = -\mathcal{H}_n Z_n \, , \qquad \mathcal{H}_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - 2D\lambda^2 \sum_{1 \le i < j \le n} \delta(x_i - x_j)$$

i.e. the QM problem in imaginary time of n attractive particles subject to the Lieb-Liniger Hamiltonian.
The exact solution is known by Bethe Ansatz!

Take home messages

- Exclusion processes $\leftrightarrow$ growth processes are examples of nonequilibrium phenomena in Physics
- KPZ universality class: $h(t) \sim t$, $w(t) \sim t^{\frac{1}{3}}$
- Growth models $\longrightarrow$ LIS $\longrightarrow$ RMT :
  fluctuations are described by Tracy-Widom distribution.

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Thanks for your attention!