Quantum Chaos and ETH

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Outline

- Microscopic vs macroscopic laws
  - Equilibrium and thermalization
  - Classical chaos
- Chaos and Random Matrix Theory
  - Quantum chaos
  - Structure of RM matrix elements
- Measures of chaos
  - Delocalization in phasespace
  - Delocalization in energy space
- Eigenstate thermalization
  - Eigenstate Thermalization Hypothesis
  - ETH predictions

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From micro to macro

Aim: find the connection between the macroscopic behavior of physical systems and their underlying microscopic dynamics.
Problem: microscopic laws are time-reversible while Thermodynamics is not.

To study the equilibrium properties of a *classical* system subject to some macroscopic constraints, one introduces a fictitious ensemble where all the configurations compatible with the constraints are weighted with a certain probability.

But in actual experiments there is no ensemble...

1^{st} line of thought: ergodicity

Ergodic hypothesis: the hypersurface in phasespace compatible with the macroscopic constraints is completely accessible and gets visited by the system as it time evolves. In the long time limit, the time spent in each region is proportional to its volume.

As a consequence,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, O(\mathrm{x}(t)) = \int \mathrm{d}\mathrm{x}\, \rho(\mathrm{x}) O(\mathrm{x}) \,.$$

Isolated system: $\rho(\mathrm{x}) = \rho_{mc}(\mathrm{x})$.

1^{st} line of thought: ergodicity

Problems:

(i) The time scales for thermalization grow exponentially in the number of degrees of freedom. Notably, relaxation is slow in near-integrable, turbulent and glassy systems.

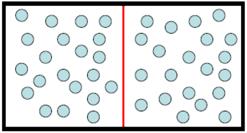
(ii) Ergodicity only implies thermalization in a *weak sense*. One would like to predict

$$\lim_{t\to\infty} O(t) \,,$$

namely, $O(t)$ approaches an equilibrium value and remains close to it at *most* later times. *(Caveat: Poincaré recurrence)*

2^{nd} line of thought: tipicality

Most configurations compatible with the macroscopic constraints give the same macroscopic values for the observables.



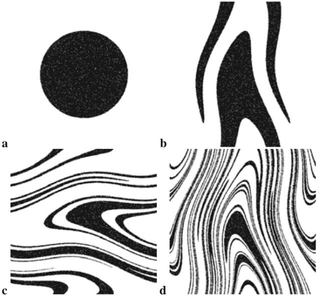
Typical configurations vastly outnumber atypical ones. Since under chaotic dynamics each configuration is reached with equal probability, atypical configurations quickly evolve into typical ones (while the opposite is rare).

But typical states are difficult to achieve in real (*non-adiabatic*) situations and in the presence of external perturbations. How are they reached? Examples: moving piston, light pulses...

Chaotic systems

- Non periodic asymptotic behavior.
- Sensitive dependence on initial conditions,

$$s(t) \simeq s_0 e^{\lambda t} \; .$$



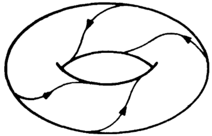
Integrable systems

$\mathcal{H}(\mathrm{q},\mathrm{p})$, $\mathrm{q},\mathrm{p} \in \mathbb{R}^N$ is integrable if it allows N independent conserved quantities in involution, *i.e.*

$$\{\mathcal{H}, I_j\} = \{I_i, I_j\} = 0 \ .$$

Then $\mathcal{H}$ can be expressed in action/angle variables as

$$\mathcal{H}(\mathrm{I},\phi) = \mathcal{H}(\mathrm{I})$$
$$\rightarrow \phi_i(t) = \phi_i(0) + \omega_i(\mathrm{I})t \ .$$



$\rightarrow$ A *MBS* is chaotic if it doesn't have an extensive number of conserved quantities.

Integrable systems

Theorem (KAM)

$$\mathcal{H}(\mathsf{I},\phi) = \mathcal{H}_0(\mathsf{I}) + \epsilon\mathcal{H}_1(\mathsf{I},\phi)$$

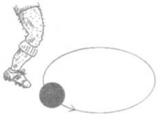
($\det\left(\frac{\partial^2\mathcal{H}_0}{\partial I_i \partial I_j}\right) \neq 0$). Then most of the invariant toruses which foliate phasespace in the integrable limit persist. Their measure tends to 1 as $\epsilon \to 0$.

$\rightarrow$ This allows for a crossover between regular and chaotic dynamics.

Example: Kicked rotator

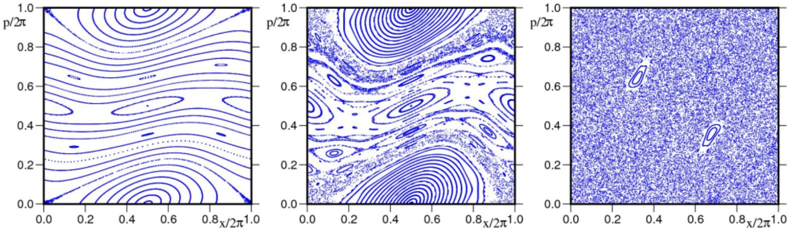
Time-dependent protocol (no energy conservation):

$$\mathcal{H} = \frac{p^2}{2} + k\cos(x) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \,.$$



Poincaré map:

$$\begin{cases} p_{n+1} = p_n + k\sin(x_n) \\ x_{n+1} = x_n + Tp_n \,. \end{cases}$$



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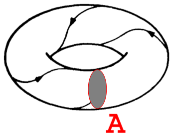
What about quantum systems?

- Schrödinger equation is linear $\rightarrow$ no exponential departing of trajectories (no chaos).
- No phase-space trajectories at all, due to uncertainty principle.

Single-particle classical limit (WKB):

$$\oint p \, \mathrm{d}q \simeq 2\pi\hbar n$$



i.e. the only allowed trajectories are those in which $A = nh$, where n is the quantum eigenvalue index. But what about open orbits?

Quantum chaos?

How does a quantum system reflect the regular/chaotic behavior of its classical counterpart?
$\rightarrow$ Look at its energy spectrum E_n.

For high enough energy, the levels get close together and it makes sense to speak of level statistics,

$$\rho(E) = \sum_n \delta(E - E_n) \; .$$

One interesting quantity to look at is the average spacing between energy levels,

$$\mathcal{P}(\omega) = P(E_{n+1} - E_n = \omega) \; .$$

Integrable systems

Example: array of independent quantum harmonic oscillators with incommensurate frequencies.
The levels are

$$E = \sum_j n_j \Omega_j \ .$$

For big enough E, nearby energy levels can come from very different sets $\{n_j\}$, which are then to be treated as uncorrelated random numbers:

$$\text{Prob}(n \text{ levels in } [E, E + \mathrm{d}E]) = \frac{\lambda^n}{n!} e^{-\lambda} \, \mathrm{d}E$$

($\lambda(E)$ average number of levels in the interval). Then

$$\mathcal{P}(\omega) = e^{-\omega} \ .$$

Integrable systems

Berry-Tabor conjecture ('77): *for quantum systems whose classical counterpart is integrable, the energy eigenvalues exhibit Poisson statistics.*

Integrable systems

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In many cases (*e.g.* atomic spectra) we don't even know the underlying Hamiltonian.

Idea (Wigner): draw $\mathcal{H}$ at random from a large collection of Hamiltonians subject to the desired symmetries.

Random matrices

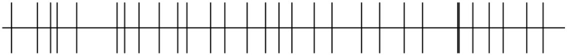
- $\mathcal{H}$ is Hermitian (real if there is *time reversal*).
- Choose a weight that is invariant under orthogonal/unitary rotations: $\sim \operatorname{Tr}\mathcal{H}^2$
- Gaussian because of CLT.

This gives the Gaussian Orthogonal and Unitary Ensembles (GOE,GUE)

$$\mathcal{P}[\mathcal{H}] \propto e^{-\beta \frac{N}{2} \operatorname{Tr}\mathcal{H}^2}$$

($\beta = 1, 2$ is the Dyson index). The eigenvalues are distributed as (Wigner '51)

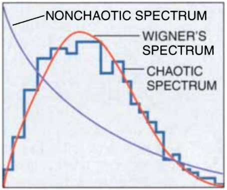
$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) \propto e^{-\frac{\beta}{2} N \sum_{i=1}^{N} \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta .$$



Random matrices

The resulting level spacing statitics is expressed by the Wigner surmise

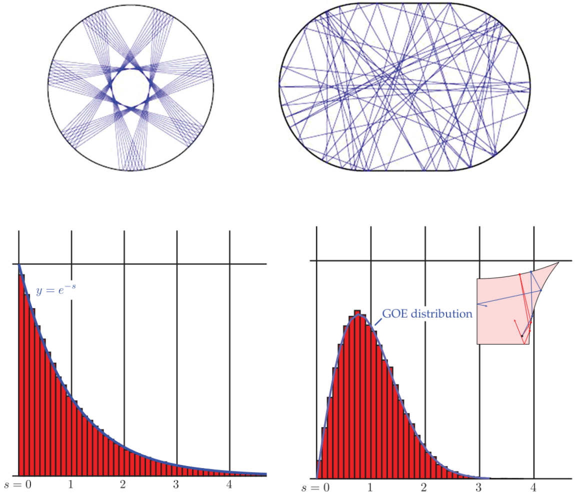
$$\mathcal{P}(\omega) = A_\beta \omega^\beta e^{-B_\beta \omega^2} .$$



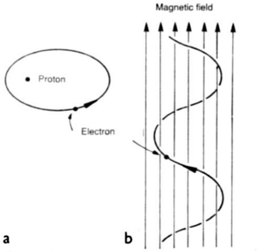
BGS conjecture (Bohigas, Giannoni, Schmit '84): *level statistics of quantum systems which have a classically chaotic counterpart are described by RMT.*

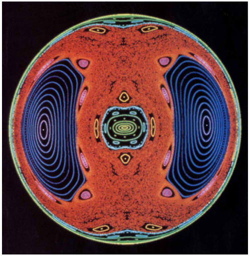
Even in the absence of a classical counterpart, the emergence of Wigner-Dyson statistics is used as a criterium.

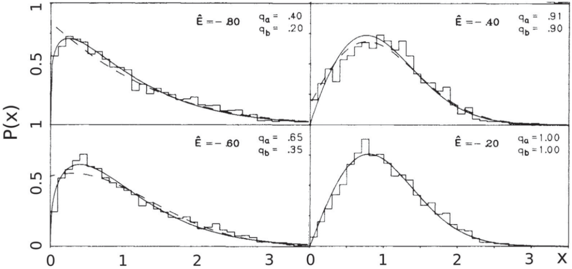
Example: particle in a cavity



Example: hydrogen atom in a magnetic field

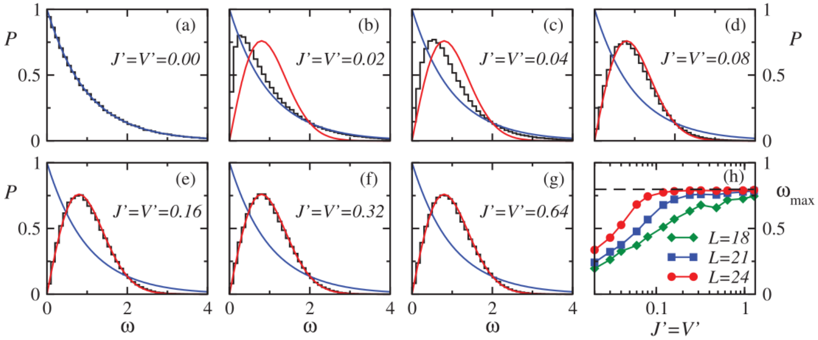






Example: chain of spinless fermions

$$\mathcal{H} = \sum_{j=1}^{L} \left[-J \left(\hat{f}_j^\dagger \hat{f}_{j+1} + \texttt{h.c.} \right) + V \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+1} + \frac{1}{2} \right) \right.$$
$$\left. - J' \left(\hat{f}_j^\dagger \hat{f}_{j+2} + \texttt{h.c.} \right) + V' \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+2} + \frac{1}{2} \right) \right]$$



Eigenstates in RMT

The eigenvectors of a random matrix are basically random unit vectors:

$$\mathcal{P}(\psi_1 \dots \psi_N) \propto \delta\left(\sum_j |\psi_j|^2 - 1\right) .$$

Indeed,

- By rotational invariance, their distribution can only depend on the norm;
- Two uncorrelated random vectors in a large dimensional space are nearly orthogonal (correlations due to the orthogonality condition can be ignored).

Eigenstates distribution

Let x be any of the components ψ_i. Marginalizing the distribution

$$\mathcal{P}(\psi_1 \dots \psi_d) = \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}}}\, \delta\left(\sum_{j=1}^{d} \psi_j^2 - 1\right)$$

over all the other components, one finds

$$p(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})}(1 - x^2)^{\frac{d-3}{2}}$$
$$\xrightarrow[d \gg 1]{} \sqrt{\frac{d}{2\pi}} e^{-\frac{d}{2}x^2}$$

i.e. a Gaussian variable with variance $\sigma^2 = \frac{1}{d}$.

Structure of the matrix elements

This has consequences on the structure of the matrix elements

$$O_{mn} = \langle m|\hat{O}|n\rangle = \sum_i O_i \langle m|i\rangle \langle i|n\rangle = \sum_i O_i \left(\psi_i^m\right)^* \psi_i^n \, .$$

Averaging over random eigenkets,

$$\left\langle \left(\psi_i^m\right)^* \psi_j^n \right\rangle = \frac{1}{d}\delta_{mn}\delta_{ij}$$

$$\boxed{\langle O_{mn}\rangle = \delta_{mn}\frac{1}{d}\sum_i O_i = \delta_{mn}\bar{O}}$$

Structure of the RM matrix elements

Fluctuations are also suppressed as $\frac{1}{d}$:

$$\left\langle O_{mm}^2 \right\rangle - \left\langle O_{mm} \right\rangle^2 = \frac{3-\beta}{d} \overline{O^2} \quad \text{(diagonal)}$$

$$\left\langle |O_{mn}|^2 \right\rangle - \left\langle |O_{mn}| \right\rangle^2 = \frac{1}{d} \overline{O^2} \quad \text{(off-diagonal)} \, .$$

To leading order, we found

$$\boxed{O_{mn} \simeq \bar{O} \delta_{mn} + \sqrt{\frac{\overline{O^2}}{d}} R_{mn}}$$

$$\left\langle R_{mn} \right\rangle = 0 \, , \quad \left\langle R_{mn}^2 \right\rangle = 1 \, .$$

Information entropy

A measure of delocalization of the eigenstates

$$|n\rangle = \sum_i c_n^i \, |i\rangle$$

over a fixed basis $|i\rangle$ is the information entropy

$$S_n \equiv -\sum_i |c_n^i|^2 \log |c_n^i|^2 \,.$$

The eigenstates are evenly delocalized in the GOE, where

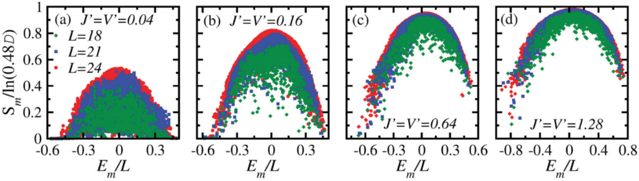
$$S_{\mathtt{GOE}} = -d \int_{-\infty}^{\infty} x^2 \ln x^2 \sqrt{\frac{d}{2\pi}} e^{-\frac{d}{2}x^2} = \ln(0.48d) + \mathcal{O}\left(\frac{1}{d}\right) \,.$$

The structure of many-body eigenstates

However, numerical simulations show that real many-body eigenstates have in general more structure, *i.e.* $S \leq S_{\text{GOE}}$.

Example: chain of spinless fermions

$$\mathcal{H} = \sum_{j=1}^{L} \Big[-J \left(\hat{f}_j^\dagger \hat{f}_{j+1} + \text{h.c.} \right) + V \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+1} + \frac{1}{2} \right) \\ -J' \left(\hat{f}_j^\dagger \hat{f}_{j+2} + \text{h.c.} \right) + V' \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+2} + \frac{1}{2} \right) \Big]$$



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Measures of chaos

- We have seen that the level spacing statistics is a manifestation of quantum chaos in many-body systems.
- But what is a sensible definition of quantum chaos, in the absence of a well-defined notion of phasespace trajectories?
- It turns out one can treat classical and quantum chaos on the same footing by analyzing delocalization of the system either in phasespace or in energy space.
- We will need appropriate entropy measures to characterize such delocalization.

Delocalization in phasespace

Paradigm: *quench*
Let an eigenstate of $\mathcal{H}_0$ evolve in time with $\mathcal{H}$.

In classical systems, a physical manifestation of chaos is delocalization in phase space after the quench, as measured by the entropy

$$S = -\int \frac{\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^d}\rho(\boldsymbol{x},\boldsymbol{p})\log\rho(\boldsymbol{x},\boldsymbol{p})\;.$$

This is conserved, in an isolated system, because of Liouville theorem. One has to focus instead on the reduced probability distribution of $N_A < N$ particles,

$$\rho(\boldsymbol{x}_1\ldots\boldsymbol{x}_{N_A},\boldsymbol{p}_1\ldots\boldsymbol{p}_{N_A},t) = \int \mathrm{d}\boldsymbol{x}_{N_A+1}\,\mathrm{d}\boldsymbol{p}_{N_A+1}\ldots\mathrm{d}\boldsymbol{x}_N\,\mathrm{d}\boldsymbol{p}_N\,\rho(\boldsymbol{x},\boldsymbol{p},t)\;.$$

Delocalization in phasespace

Similarly, in quantum systems one has that the Von Neumann entropy

$$S_{\mathrm{vn}} = -\operatorname{Tr}[\hat{\rho} \log \hat{\rho}]$$

is conserved under unitary evolution. Then one has to consider instead the reduced density matrix of a subsystem A wrt a subsystem B,

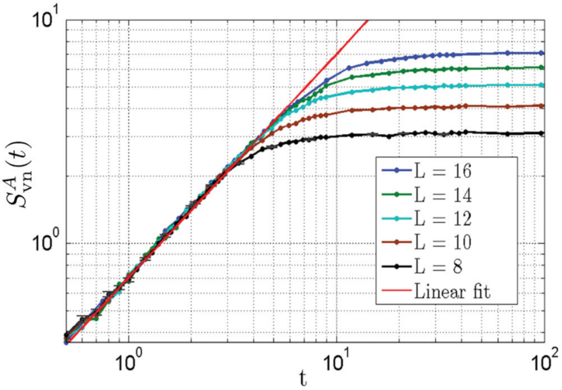
$$\hat{\rho}_A = \operatorname{Tr}_B \hat{\rho} \quad \rightarrow \quad S^A_{\mathrm{vn}} = -\operatorname{Tr}_A[\hat{\rho}_A \log \hat{\rho}_A] \, .$$

In the case where $\hat{\rho} = |\psi\rangle\langle\psi|$ is a pure state, this coincides with its entanglement entropy.

The structure of many-body eigenstates

Example: spin $\frac{1}{2}$ transverse field Ising chain

$$\hat{\mathcal{H}} = J\sum_{j=1}^{L-1}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z + g\sum_{j=1}^{L}\hat{\sigma}_j^x + h\sum_{j=2}^{L-1}\hat{\sigma}_j^z + (h-J)(\hat{\sigma}_1^z + \hat{\sigma}_L^z)$$



Delocalization in energy space

As an alternative, one can study the time-averaged probability distribution

$$\rho_T(\boldsymbol{x}, \boldsymbol{p}) \equiv \frac{1}{T} \int_0^T \mathrm{d}t \, \rho(\boldsymbol{x}, \boldsymbol{p}, t)$$

and compute its entropy, which is a non-decreasing function of T (Jensen's inequality).

For ergodic systems, this is expected to grow to its maximum allowed value,

$$S[\rho_T(\boldsymbol{x}, \boldsymbol{p})] \xrightarrow[T \to \infty]{} S[\rho_{\mathrm{MC}}(\boldsymbol{x}, \boldsymbol{p})] \, .$$

Delocalization in energy space

This suggests we may compute, for a quantum system,

$$\hat{\rho}_T \equiv \frac{1}{T}\int_0^T \mathrm{d}t\,\hat{\rho}(t) = \frac{1}{T}\int_0^T \mathrm{d}t \sum_{mn} \rho_{mn}(t)\,|m\rangle\langle n| \ .$$

Since

$$\rho_{mn}(t) = \rho_{mn}(t_0) e^{-i(E_m - E_n)(t-t_0)} \ ,$$

in the long time limit (assuming there are no degeneracies) the initial density matrix is projected onto the diagonal subspace of $\hat{\mathcal{H}}$ (*i.e.* energy space).

This gives the diagonal ensemble

$$\hat{\rho}_{\mathrm{DE}} \equiv \lim_{T\to\infty} \hat{\rho}_T = \sum_n \rho_{nn}\,|n\rangle\langle n| \ .$$

Diagonal entropy

It follows that a measure of delocalization in energy space is the diagonal entropy

$$S_d = -\sum_n p_n \log p_n \ .$$

Notice that, since

$$|\psi_I\rangle = \sum_n c_n |n\rangle \ , \qquad p_n \equiv \rho_{nn} = |\langle n|\psi_I\rangle|^2 = |c_n|^2 \ ,$$

this is nothing but the Von Neumann entropy of the diagonal ensemble, or the information entropy of the initial state $\hat{\rho} = |\psi_I\rangle\langle\psi_I|$ in the basis $|n\rangle$ of the post-quench Hamiltonian.

Tipicality strikes back

Problem: $\hat{\rho}_{\mathtt{DE}} \neq \hat{\rho}_{\mathtt{MC}}$!

Key: it's enough that

$$\bar{O} \xrightarrow[t\to\infty]{} \left\langle \hat{O} \right\rangle_{\mathtt{MC}}$$

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Eigenstate thermalization: setup

$$|\psi_I\rangle \to |\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |n\rangle \ , \qquad \begin{cases} \hat{\mathcal{H}} |n\rangle = E_n |n\rangle \\ c_n = \langle n|\psi_I\rangle \ . \end{cases}$$

The density matrix $\hat{\rho}(t)$ remains that of a pure state at all times. Observables evolve as

$$O(t) = \langle\psi(t)|\hat{O}|\psi(t)\rangle = \sum_m |c_m|^2 O_{mm} + \sum_{m,n\neq m} c_m^* c_n e^{i(E_m - E_n)t} O_{mn} \ .$$

Eigenstate thermalization

Object of desire: *strong sense* thermalization, *i.e.*

(i) After some relaxation time, $O(t)$ agrees with the microcanonical expectation value;

(ii) Temporal fluctuations remain small at most later times.

Problem: since $|c_n|^2$ is conserved in time, how is

$$\sum_m |c_m|^2 O_{mm}$$

to agree with the microcanonical ensemble prediction?

Eigenstate thermalization: RMT

Observation: in Random Matrix Theory, where

$$O_{mn} \simeq \bar{O}\delta_{mn} + \sqrt{\frac{\overline{O^2}}{d}} R_{mn} \,,$$

we would get

$$\sum_m |c_m|^2 O_{mm} \simeq \bar{O} \sum_m |c_m|^2 = \bar{O} \,.$$

But in contrast to random matrices, in real systems

(i) $\langle O \rangle$ is energy dependent;

(ii) Relaxation times are observable dependent.

$\rightarrow$ O_{mm}, O_{mn} contain more information than in RMT.

Eigenstate Thermalization Hypothesis

ETH:

$$O_{mn} \simeq O(\bar{E})\delta_{mn} + e^{-\frac{S(\bar{E})}{2}} f_0(\bar{E},\omega) R_{mn}$$

where $S(E)$ is the thermodyamic entropy,

$$\bar{E} \equiv \frac{E_m + E_n}{2}\ , \qquad \omega = E_n - E_m$$

and, in contrast to RMT,

- $O(\bar{E}) = \left\langle \hat{O} \right\rangle_{\mathrm{MC}} (\bar{E})$ is a smooth function of $\bar{E}$ and depends on the energy of the eigenstates;
- On top of the Gaussian fluctuations $\langle R_{mn} \rangle = 0$, $\left\langle R_{mn}^2 \right\rangle = 1$ there is a smooth envelope $f_0(\bar{E},\omega)$.

ETH predictions: long time average

We have seen that

$$\bar{O} \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, O(t) = \sum_m |c_m|^2 O_{mm} \equiv \mathrm{Tr}\left[\hat{\rho}_{\mathrm{DE}} \hat{O}\right] .$$

How can this agree with the microcanonical value

$$O_{\mathrm{MC}} = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{MC}} \hat{O}\right] \ ?$$

Prepare an initial state $|\psi_I\rangle = \sum_n c_n |n\rangle$ choosing the coefficients c_n so that:

$$\langle E\rangle = \langle\psi_I|\hat{\mathcal{H}}|\psi_I\rangle \equiv \mathrm{Tr}\left[\hat{\rho}_{\mathrm{MC}} \hat{\mathcal{H}}\right] \tag{1}$$

$$\delta E^2 = \langle\psi_I|\hat{\mathcal{H}}^2|\psi_I\rangle - \langle\psi_I|\hat{\mathcal{H}}|\psi_I\rangle^2 \ll 1 \tag{2}$$

ETH predictions: long time average

Taylor expanding the ETH ansatz around the microcanonical energy,

$$O_{mm} \simeq O(E_m) \simeq O(\langle E \rangle) + O'(\langle E \rangle)\,(E_m - \langle E \rangle) + \frac{O''(\langle E \rangle)}{2}\,(E_m - \langle E \rangle)^2 + \dots$$

we get

$$\begin{aligned}
\bar{O} &\equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, O(t) = \sum_m |c_m|^2 O_{mm} \\
&\simeq O(\langle E \rangle) + O'(\langle E \rangle) \sum_m |c_m|^2 (E_m - \langle E \rangle) + \frac{O''(\langle E \rangle)}{2} \sum_m |c_m|^2 (E_m - \langle E \rangle)^2 \\
&= O(\langle E \rangle) + \frac{O''(\langle E \rangle)}{2} (\delta E)^2 \\
&\simeq O_{\mathrm{MC}} + \frac{1}{2}\left[(\delta E)^2 - (\delta E_{\mathrm{MC}})^2\right] O''(\langle E \rangle)\,.
\end{aligned}$$

Notice δE_{MC} is subextensive and so is generally δE if $\mathcal{H}$ is local; so we proved that

$$\bar{O} \simeq O_{\mathrm{MC}}\,.$$

ETH predictions: fluctuations

Finally, the long time average of the temporal fluctuations is given by

$$
\begin{aligned}
\sigma_O^2 \equiv \overline{\left(O(t)-\bar{O}\right)^2} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\,[O(t)]^2 - \bar{O}^2 \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t \sum_{i,j,m,n} O_{ij}O_{mn}c_i^* c_j c_m^* c_n e^{i(E_i-E_j+E_m-E_n)t} - \bar{O}^2 \\
&= \sum_{i,j,m,n} O_{ij}O_{mn}c_i^* c_j c_m^* c_n \left[\delta(i-j+m-n) - \delta_{ij}\delta_{mn}\right] \\
&= \sum_{m,n\neq m} |c_m|^2|c_n|^2|O_{mn}|^2 \leq \max|O_{mn}|^2 \sum_{m,n}|c_m|^2|c_n|^2 \\
&= \max|O_{mn}|^2 \propto e^{-S(\bar{E})}\,.
\end{aligned}
$$

At long times, fluctuations of the expectation value of $\hat{O}$ from $\left\langle \hat{O} \right\rangle_{\mathrm{MC}}$ are exponentially suppressed in the system size: this is ergodicity in strong sense.

ETH predictions: fluctuations

Notice in experiments one observes instead

$$\begin{aligned}
\overline{\delta O^2} &\equiv \overline{\langle\psi(t)|\left(\hat{O}-\bar{O}\right)^2|\psi(t)\rangle} \\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\ \langle\psi(t)|\left(\hat{O}-\bar{O}\right)^2|\psi(t)\rangle \\
&= \sum_n |c_n|^2 (O^2)_{nn} - \bar{O}^2 \\
&\simeq \frac{1}{2}\left\{\left[O^2\right]''(\langle E\rangle) - 2O(\langle E\rangle)O''(\langle E\rangle)\right\}\left[(\delta E)^2-(\delta E_{\mathrm{MC}})^2\right]
\end{aligned}$$

which is nonzero even if we take $|\psi_I\rangle = |n\rangle$ (in which case $\sigma_O^2 = 0$).

Which observables?

→ Few body operators ($n \ll N$).
For instance, it does not work with $\hat{P}_n = |n\rangle\langle n|$.

Example: chain of hard-core bosons

$$\mathcal{H} = \sum_{j=1}^{L} \Big[-J \left(\hat{b}_j^\dagger \hat{b}_{j+1} + \mathrm{h.c.} \right) + V \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+1} + \frac{1}{2} \right) \\ - J' \left(\hat{b}_j^\dagger \hat{b}_{j+2} + \mathrm{h.c.} \right) + V' \left(\hat{n}_j - \frac{1}{2} \right) \left(\hat{n}_{j+2} + \frac{1}{2} \right) \Big] .$$

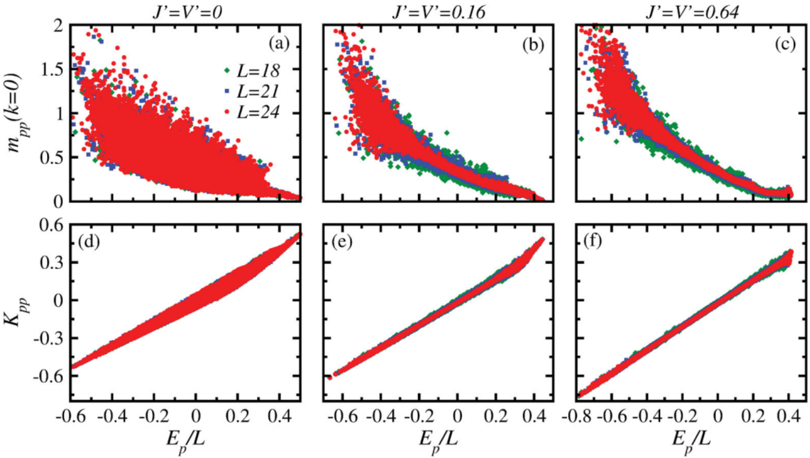
Mode occupation:

$$\hat{m}(k) = \frac{1}{L} \sum_{ij} e^{ik(i-j)} \hat{b}_i^\dagger \hat{b}_j$$

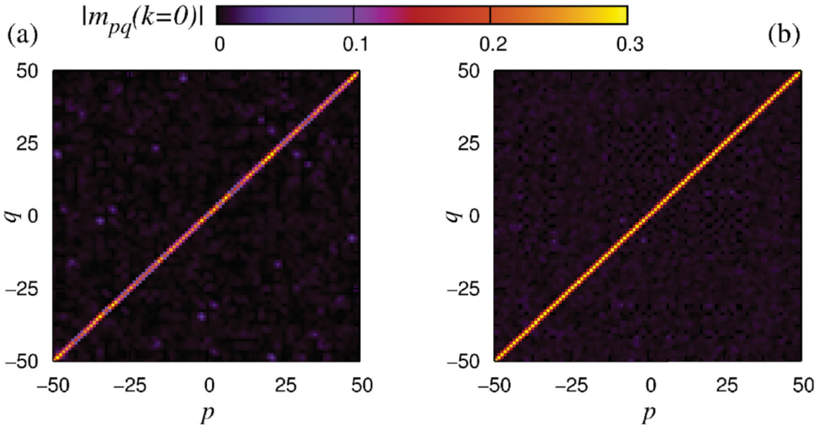
Kinetic energy per site:

$$\hat{K} = \sum_{j=1}^{L} \left[-J \left(\hat{b}_j^\dagger \hat{b}_{j+1} + \mathrm{h.c.} \right) - J' \left(\hat{b}_j^\dagger \hat{b}_{j+2} + \mathrm{h.c.} \right) \right]$$

Diagonal elements



Off-diagonal elements



Take home messages

- Chaos, ergodicity and thermalization pertain to the nature of the Hamiltonian eigenstates.
- These are *typical* in the sense that they give the same result for $\left\langle \hat{O} \right\rangle$ as the microcanonical ensemble.
- Relaxation of observables to their equilibrium values is nothing but the result of *dephasing*.
- The information about the eventual thermal state is encoded in the system from the very beginning: the time evolution simply reveals it.

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Thanks for your attention!

The RMT limit

The ETH ansatz reduces to RMT if one focuses on a narrow energy window where

$$f_0(\bar{E}, \omega) \simeq \texttt{const.}$$

For a single-particle diffusive system, this scale is set by the Thouless energy

$$\omega < E_T \equiv \frac{\hbar}{\tau_{\texttt{diff}}} = \frac{\hbar \mathcal{D}}{L^2}$$

which vanishes in the thermodynamic limit. Since the level spacing vanishes faster, there are still many energy levels within the region where RMT applies.

Semiclassical limit

Define the Wigner function as the Wigner-Weyl transform of the density matrix:

$$W(\boldsymbol{x},\boldsymbol{p}) = \frac{1}{(2\pi\hbar)^{3N}} \int \mathrm{d}^{3N}y \left\langle \boldsymbol{x} - \frac{\boldsymbol{y}}{2} \middle| \hat{\rho} \middle| \boldsymbol{x} + \frac{\boldsymbol{y}}{2} \right\rangle e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{y}}$$

$$\xrightarrow[\hat{\rho}=|\psi\rangle\langle\psi|]{} \frac{1}{(2\pi\hbar)^{3N}} \int \mathrm{d}^{3N}y\, \psi^* \left(\boldsymbol{x} - \frac{\boldsymbol{y}}{2}\right) \psi \left(\boldsymbol{x} + \frac{\boldsymbol{y}}{2}\right) e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{y}} .$$

Given a wavefunction, this defines a corresponding quasi-probability distribution in phase space:

$$\left\langle \hat{O} \right\rangle = \int \mathrm{d}^{3N}x\, \mathrm{d}^{3N}p\, O_w(\boldsymbol{x},\boldsymbol{p}) W(\boldsymbol{x},\boldsymbol{p})$$

$$O_w(\boldsymbol{x},\boldsymbol{p}) \equiv \frac{1}{(2\pi\hbar)^{3N}} \int \mathrm{d}^{3N}y \left\langle \boldsymbol{x} - \frac{\boldsymbol{y}}{2} \middle| \hat{O} \middle| \boldsymbol{x} + \frac{\boldsymbol{y}}{2} \right\rangle e^{-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{y}}$$

Semiclassical limit

Berry's conjecture: in the semiclassical limit of a quantum system with a chaotic counterpart,

$$\boxed{\overline{W(\boldsymbol{X},\boldsymbol{P})} = \int_{\boldsymbol{\Delta\Omega}} \frac{\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^{3N}} W(\boldsymbol{x},\boldsymbol{p}) \xrightarrow[\hbar\to 0]{} \frac{1}{\mathcal{N}}\delta\left[E-\mathcal{H}(\boldsymbol{X},\boldsymbol{P})\right]}$$

where $\boldsymbol{\Delta\Omega}$ is a small phasespace volume centered around $(\boldsymbol{X},\boldsymbol{P})$, with

$$\frac{\hbar}{\boldsymbol{\Delta\Omega}} \to 0 \quad \text{as } \hbar, \boldsymbol{\Delta\Omega} \to 0 \,.$$

$\rightarrow$ The energy eigenstate expectation value of any observable in the semiclassical limit of a quantum system whose classical counterpart is chaotic is the same as a MC average.