Resummation techniques in scalar Field Theory

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Outline

Instantons in Quantum Mechanics

- Semiclassical *vs* weak-coupling expansion
- Quantum Mechanics as a 1d QFT

Borel summability

- Asymptotic series and Borel transform
- Large order behavior
- Borel summability in d<4 scalar FT

ϕ^4 theory: the symmetry broken phase

- Chang duality
- Instantons and large order behavior
- Weak coupling: conformal mapping method
- Strong coupling: Exact Perturbation Theory

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Semiclassical approximation

Consider a scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4$$
$$\xrightarrow[\phi'=\sqrt{\lambda}\phi]{} \frac{1}{\lambda}\left\{\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2\phi'^2 - \phi'^4\right\} .$$

For classical Physics, λ is an irrelevant parameter: it does not enter the EOM. λ only becomes important in quantum Physics, where the scale $\hbar$ appears:

$$\frac{\mathcal{L}}{\hbar} = \frac{1}{\lambda\hbar}\left\{\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \dots\right\} .$$

Small $\hbar$ (semiclassical) approximations are tantamount to small λ (weak coupling) approximations.

Why do we need instantons?

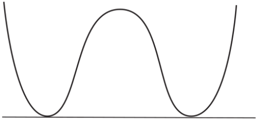
Some physical phenomena are not captured by perturbative series in λ (or $\hbar$). For example:

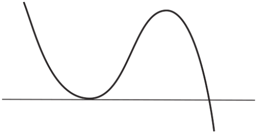
- The tunneling amplitude through a potential barrier reads

$$|T(E)| = e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} \mathrm{d}x \sqrt{2(V(x)-E)}}\left(1+\mathcal{O}(\hbar)\right) \sim e^{-\frac{A}{\hbar}}$$

  and this is relevant for *false vacua*.
- The ground state degeneracy in a *double well* is also nonperturbative in the quartic coupling λ:

$$E_1(\lambda) - E_0(\lambda) \sim e^{-\frac{A}{\lambda}} \; .$$





Quantum Mechanics as a 1d QFT

Given a quantum mechanical Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + W(q)$$

we can construct the propagator

$$\langle q_f | e^{-\frac{\mathcal{H}T}{\hbar}} | q_i \rangle = \mathcal{N} \int \mathcal{D}q(t) e^{-\frac{S(q)}{\hbar}}$$

where the path integral is performed over trajectories obeying $q(-\frac{T}{2}) = q_i\,,\ q(\frac{T}{2}) = q_f\,,$ and the action

$$S(q) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{d}t \left\{ \frac{1}{2}\dot{q}(t)^2 + W(q) \right\}$$

resembles a Lagrangian for the inverted potential

$$V(q) \equiv -W(q)\,.$$

Path integral around an instanton

Nontrivial saddle points are *finite action* solutions of the Euclidean EOM in the *inverted* potential

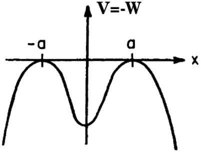
$$\ddot{q}_c(t) + V'(q_c) = 0\ .$$

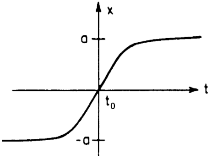
Expanding $q(t) = q_c(t) + \eta(t)$, $\mathcal{D}\eta = \prod_{n\geq 0} \mathrm{d}c_n$,

$$\langle q_f | e^{-\frac{\mathcal{H}T}{\hbar}} | q_i \rangle \simeq \mathcal{N} e^{-\frac{S(q_c)}{\hbar}} \int \mathcal{D}\eta(t) e^{-\frac{1}{\hbar}\int \eta \left.\frac{\delta^2 S}{\delta q^2}\right|_{q_c} \eta}$$

$$= \mathcal{N} e^{-\frac{S(q_c)}{\hbar}} \left[\det\left(-\partial_t^2 + W''(q_c)\right)\right]^{-\frac{1}{2}} (1 + \mathcal{O}(\hbar))\ .$$

Example: $E_{\text{GS}} = -\lim_{\beta\to\infty} \frac{1}{\beta} \log\left[\operatorname{Tr} e^{-\beta\mathcal{H}}\right]$





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Asymptotic expansions

The power series

$$\varphi(\lambda) \to \sum_{n=0}^{\infty} a_n \lambda^n$$

is said to be asymptotic to $f(\lambda)$ as $\lambda \to 0$ iff

$$f(\lambda) = \underbrace{\sum_{n=0}^{N} a_n \lambda^n}_{\varphi_N(\lambda)} + \mathcal{O}\left(\lambda^{N+1}\right) .$$

As we vary N, in contrast to convergent series, the partial sum $\varphi_N(\lambda)$ will first approach $f(\lambda)$ and then diverge. Assuming

$$a_n \sim A^{-n} n!$$

one can minimize the error $\Delta_N = |a_N \lambda^N|$ and find the optimal truncation $N^* = |\frac{A}{\lambda}|$. The next term differs by $\sim e^{-|\frac{A}{\lambda}|}$, which is the *maximal resolution*.

Borel transform

Problem: all the information in the terms $n > N^*$ we discarded is lost!
Solution: define the *Borel transform*

$$B(t) \to \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \ .$$

- If $a_n \sim A^{-n} n!$, its radius of convergence is $\rho = |A|$.
- Suppose its analytic continuation $B(t)$ has no singularities on the real semiaxis $t > 0$. Then

$$\varphi_B(\lambda) = \int_0^{\infty} \mathrm{d}t\, e^{-t} B(\lambda t) = \frac{1}{\lambda} \int_0^{\infty} \mathrm{d}t\, e^{-\frac{t}{\lambda}} B(t)$$

$$\varphi_B^{(N)}(\lambda) = \frac{1}{\lambda} \sum_{n=0}^{N} \frac{a_n}{n!} \int_0^{\infty} e^{-\frac{t}{\lambda}} t^n \, \mathrm{d}t = \sum_{n=0}^{N} a_n \lambda^n$$

and $\varphi(\lambda)$ is said to be Borel summable to $\varphi_B(\lambda)$.

Borel transform

- In general, $\varphi(\lambda) \neq \varphi_B(\lambda)$ as they may differ by singular (nonperturbative) terms.
- In some lucky cases, we are able to express

$$\varphi(\lambda) = \int_0^\infty \mathrm{d}t\, e^{-t} \underbrace{(\dots)}_{B(\lambda t)} .$$

Then we already know that $\varphi_B(\lambda) = \varphi(\lambda)$, *i.e.* $\varphi(\lambda)$ is Borel summable to the exact result.

- In practice, we don't have all the a_n terms, and if we naively Laplace transform the truncated series for $B(\lambda t)$ term-by-term, we get back the original asymptotic expansion!
- We will need to do *something* first.

Large order behavior

The analytic structure of the Borel transform is connected to the large order behavior of $\varphi(\lambda)$.

(i) The behavior of the Borel transform $B(t)$ near a singularity $t = A$ determines the nonperturbative term of order $e^{-\frac{A}{\lambda}}$.

(ii) The large n behavior of the coefficients a_n in $\varphi(\lambda)$ is controlled by the leading nonperturbative contribution.

Example: $B(t)$ has a branch cut starting at $t = A > 0$,

$$B(A+t) = (-t)^{-b} \sum_{n \geq 0} c_n t^n + \text{ (regular) }.$$

Then, calling $\hat{c}_k = 2\sin(\pi b)\Gamma(k+1-b)c_k$,

$$a_n \sim \frac{1}{2\pi} \sum_{k \geq 0} A^{k-b-n} \hat{c}_k \Gamma(n+b-k) \,, \quad disc(\varphi)(\lambda) = ie^{-\frac{A}{\lambda}} \lambda^{-b} \sum_{n=0}^{\infty} \hat{c}_n \lambda^n \,.$$

Borel summability in d<4 scalar FT

$$\mathcal{I} = \int \mathcal{D}\phi G[\phi] e^{-\frac{1}{\hbar}S[\phi]}\,, \quad S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\}$$

- Focus on *super-renormalizable* theories, $d = 2, 3$ (counterterms can be reabsorbed into $G[\phi]$).
- Let $\tilde{\phi} \equiv 0$ be the unique, absolute minimum of $V(\phi)$ *s.t.* $S[\tilde{\phi} = 0] = 0$. Construct

$$\mathcal{I} = \int_0^\infty \mathrm{d}t\, e^{-t} \underbrace{\int \mathcal{D}\phi G[\phi] \delta\left(t - \frac{S[\phi]}{\hbar} \right)}_{B(\hbar t)}$$

- Is this legit? Yes if the change of variables $t = \frac{S[\phi]}{\hbar}$ is nonsingular ($S'[\phi] \neq 0$), *i.e.* if there are no finite action critical points for real field configurations other than $\tilde{\phi} = 0$.

Generalized Derrick Theorem *[Brézin et al.]*

$S[\phi]$ *has no nontrivial real saddles with finite action.*

$$S[\phi] = \underbrace{\int \mathrm{d}^d x \, \frac{1}{2} \partial_\mu \phi_a \mathcal{M}^{ab} \partial^\mu \phi_b}_{A} + \underbrace{\int \mathrm{d}^d x \, \frac{1}{\lambda} V(\sqrt{\lambda}\phi)}_{B}$$

Proof: assume $\tilde{\phi}$ is a real, classical solution. Then

$$S(\alpha) \equiv S[\tilde{\phi}(\alpha x)] = \alpha^{2-d} A + \alpha^{-d} B \, .$$

Since $S[\phi]$ is stationary under variations at $\alpha = 1$,

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=1} = -\left[(d-2) \cdot A + d \cdot B\right] \equiv 0 \quad \rightarrow \quad \boxed{B = \frac{2-d}{d} A}$$

$$\left.\frac{\mathrm{d}^2 S}{\mathrm{d}\alpha^2}\right|_{\alpha=1} = 2d \cdot B \geq 0 \qquad \text{(stability condition)}$$

- Since $A \geq 0$, there are no real solutions for $d > 2$.
- $d = 2 \rightarrow B = 0 \rightarrow V(\tilde{\phi}) \equiv 0 \rightarrow V(\phi)$ has a set of degenerate vacua continuously connected.

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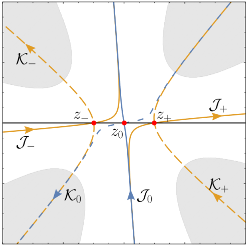
ϕ^4 theory

If the minimum is not the global one, we generally have more real finite action critical points. The domain of integration is said to be on a *Stokes line* and the perturbative series will not be Borel summable.



- This is what happens in the presence of SSB. In order to restore summability, we have to introduce a small SB term as

$$S_{V,\epsilon}[\phi] = S_V[\phi] + \epsilon \int_V \mathrm{d}^d x\, \phi(x) \quad \rightarrow \quad \mathcal{I}_{\mathrm{SSB}} = \lim_{\epsilon \to 0} \lim_{V \to \infty} \mathcal{I}(V, \epsilon)\,.$$

- This the case in $d = 2$ for

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \lambda \phi^4\,, \qquad m^2 > 0\,.$$

For $\lambda \geq \lambda_c$ we get SSB because of a 2^{nd} order P.T.

Normal ordering

- In any scalar theory in 2d with nonderivative interactions, divergencies which occur at any order in perturbation theory can be removed by simply normal ordering the Hamiltonian.
- To do so, one has to specify the particle mass of the free Hamiltonian through which creation and annihilation operators are defined.
- Coleman (1975) proved the relations

$$\mathcal{N}_m\left(e^{i\beta\phi}\right) = \left(\frac{\mu^2}{m^2}\right)^{\frac{\beta^2}{8\pi}} \mathcal{N}_\mu\left(e^{i\beta\phi}\right)$$

$$\begin{aligned}\mathcal{N}_m(\mathcal{H}_0) &= \mathcal{N}_\mu(\mathcal{H}_0) + E_0(\mu) - E_0(m) \\ &= \mathcal{N}_\mu(\mathcal{H}_0) + \frac{1}{8\pi}(\mu - m)\end{aligned}$$

where

$$\mathcal{H}_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 , \quad E_0(m) = \int \frac{\mathrm{d}k}{8\pi} \frac{2k^2 + m^2}{\sqrt{k^2 + m^2}} .$$

Chang duality *(1976)*

The 2d theory described by the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \lambda\mathcal{N}_m(\phi^4)$$

with $m^2, \lambda > 0$ *admits a dual representation as*

$$\tilde{\mathcal{L}} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\mu^2\phi^2 + \lambda\mathcal{N}_\mu(\phi^4)\ .$$

Proof: using Coleman's relations, one can check that $\mathcal{L}$ maps on $\tilde{\mathcal{L}}$ provided

$$\frac{1}{2}m^2 + \frac{3\lambda}{2\pi}\log\frac{m^2}{\mu^2} = -\frac{1}{4}\mu^2\ .$$

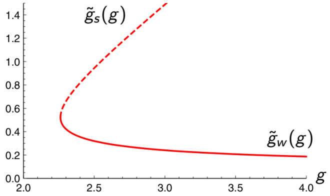
The dual mass μ is found by solving ($g \equiv \frac{\lambda}{m^2}$, $\tilde{g} \equiv \frac{\lambda}{\mu^2}$)

$$f_1(g) = f_2(\tilde{g})\ , \qquad \begin{cases} f_1(g) = \log(g) - \frac{\pi}{3g} \\ f_2(\tilde{g}) = \log(\tilde{g}) + \frac{\pi}{6\tilde{g}}\ . \end{cases}$$

Chang duality: dual mass

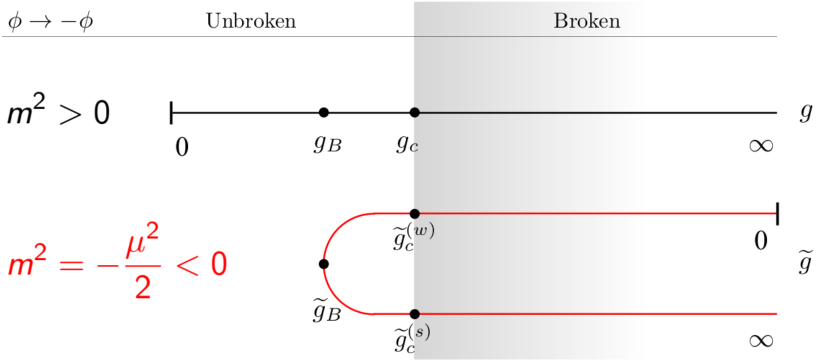
For $g \geq g_B \simeq 2.26$, there are two solution branches with different asymptotic behavior as $g \to \infty$:

$$\begin{cases} \tilde{g}_W(g) \sim \frac{\pi}{6\log(g)} \\ \tilde{g}_s(g) \sim g \end{cases}$$

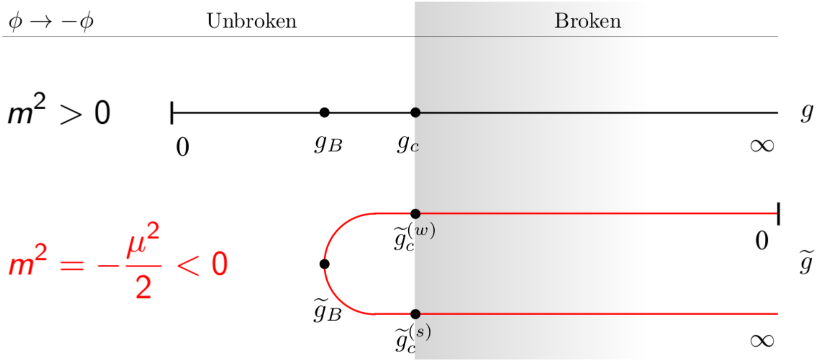


- For small g, the theory is in the symmetric phase with $\langle \phi \rangle = 0$.
- For large g, we use the dual description and get a weakly coupled double well: the symmetry is spontaneously broken and $\langle \phi \rangle = \pm \frac{\mu}{\sqrt{8\lambda}}$.
- By continuity, there must be a phase transition point in between!
- Simon-Griffiths showed that it can't be 1^{st} order.

Chang duality: summary



Chang duality: summary



Aim: use this to compute perturbative expansions in the symmetry broken phase!

Borel summability in the broken phase

- Observables: n-point functions. They admit a divergent series expansion

$$\mathcal{I}(\tilde{g}) \to \sum_{n=0}^{\infty} I_n \tilde{g}^n$$

- Borel summability means they can be recovered as

$$\mathcal{I}(\tilde{g}) \equiv \mathcal{I}_B(\tilde{g}) = \frac{1}{\tilde{g}} \int_0^{\infty} \mathrm{d}t \, e^{-\frac{t}{\tilde{g}}} B(t) \,, \qquad B(t) \to \sum_{n=0}^{\infty} \frac{I_n}{n!} t^n$$

- This is possible if $B(t)$ is regular over the positive t real axis, but in general singularities are present in the complex t plane.
- *Their position corresponds to the value of the action on (complex) instantons.*

Large order behavior

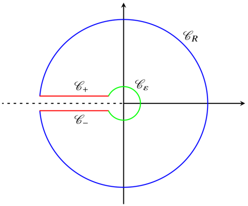
$$Z(g) = \int \mathcal{D}\phi e^{-S(\phi)} \equiv \sum_{k=0}^{\infty} Z_k g^k$$

By Cauchy integral formula,

$$Z_k = \frac{1}{2\pi i}\int_{\mathcal{C}} \mathrm{d}g \,\frac{Z(g)}{g^{k+1}} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\mathrm{d}g}{g}\int \mathcal{D}\phi e^{-\overbrace{\{k\log g + S(\phi)\}}^{f(\phi,g)}} .$$

Solving the saddle point conditions

$$\begin{pmatrix} \frac{\delta}{\delta\phi(x)} \\ \frac{\delta}{\delta g} \end{pmatrix} f(\phi, g) \equiv 0$$



one finds the leading contribution for large k

$$\boxed{Z_k \sim k!\,\frac{1}{(-S_c)^k}}$$

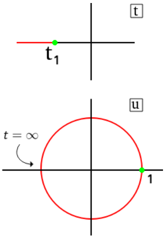
Conformal mapping method

- The saddle with the minimum value for the action, say $t_1 = S[\phi_c^{(1)}]$, determines the radius of convergence for $B(t)$.
- $B(t)$ can be approximated by a truncated series expansion (at any order) within $0 < t < |t_1|$.
- $\mathcal{I}(\tilde{g})$ can only be reconstructed up to $\mathcal{O}(e^{-\frac{|t_1|}{\tilde{g}}})$, which was the accuracy of the original expansion!
- Let's instead enlarge the radius of convergence of $B(t)$ beyond $|t_1|$ by a conformal mapping

$$t(u) \;:\; t \in [0,\infty) \mapsto u \in [0,1)$$

with all the singularities on $|u| = 1$:

$$\mathcal{I}(\tilde{g}) \equiv \frac{1}{\tilde{g}} \int_0^1 \mathrm{d}u \, |t'(u)| e^{-\frac{t(u)}{\tilde{g}}} B(t(u)) \, .$$



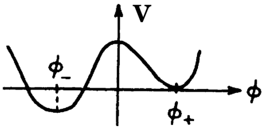
Construction of the instanton solution

$$S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\} \quad \rightarrow \quad \left(\nabla^2 + \partial_\tau^2\right)\phi = V'(\phi)$$

We look for an instanton solution *s.t.*

$$\lim_{\tau \to \pm\infty} \phi(\mathrm{x}, \tau) = \phi_+$$

$$\lim_{|\mathrm{x}| \to \infty} \phi(\mathrm{x}, \tau) = \phi_+$$



Focusing on O(d) invariant solutions, the EOM becomes

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = V'(\phi)\,.$$

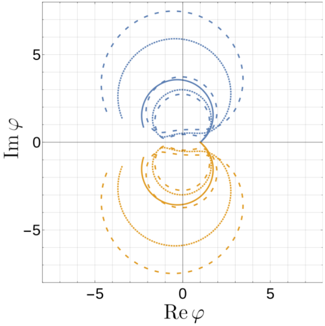
If the particle starts "still" in a well chosen $\phi_0 > \phi_-$ at "time" $r = 0$, *i.e.* $\left.\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=0} = 0$, then it will come to rest in $\phi = \phi_+$.

Instanton solution for $\lambda\phi^4$

To study the broken phase of $\lambda\phi^4$ where $\langle\phi\rangle = \pm v$, we solve the radial ODE numerically subjet to the b.c.

$$\nabla\phi(r=0) = 0\,, \quad \phi(r\to\infty) = v\,.$$

Computing the action for different initial points $\phi(r=0)$ in the complex plane gives the position of the leading singularities of the Borel transform:



[Serone et al.]

$$t_i^{\pm} = \frac{1}{\tilde{g}}\left\{S[\phi_{\text{bounce}}] - S[\phi \equiv v]\right\}$$

$$|t_1| \simeq 1.6\,, \quad |t_2| \simeq 8.9\ \ldots$$

Weak coupling: conformal mapping

We define a *Schwarz-Christoffel transformation*

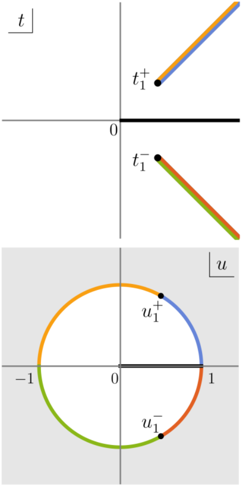
$$t(u) = 4|t_1|u \left[\frac{\alpha_1}{(1-u)^2}\right]^{\alpha_1} \left[\frac{1-\alpha_1}{(1+u)^2}\right]^{1-\alpha_1}$$

which maps away the leading saddles $t_1^{\pm} = |t_1^{\pm}|e^{\pm i\pi\alpha_1}$.

We thus enlarged the radius of convergence to $|u(t_2^{\pm})| < 1$, where $t_2^{\pm}$ are the next saddles.

This produces, at small couplings, an irreducible error

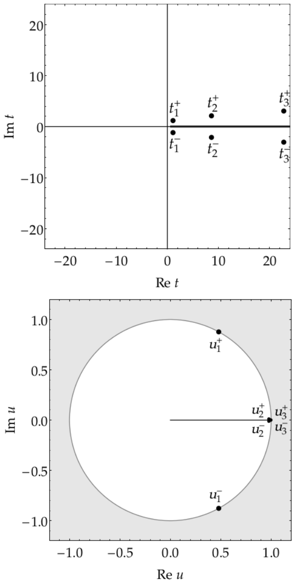
$$\mathcal{O}(e^{-\frac{|t_2|}{\tilde{g}}}) \ll \mathcal{O}(e^{-\frac{|t_1|}{\tilde{g}}})$$



[Serone et al.]

Weak coupling: conformal mapping

- If the singularities were radially aligned, they would get mapped on the unit disc and $B(t(u))$ would converge everywhere.
- This is what happens in the $\mathbb{Z}_2$ unbroken phase of the theory, where singularities lie on the negative real t axis.



Strong coupling: Exact Perturbation Theory

Expanding $\tilde{\mathcal{L}}$ around a classical minimum $\phi_{\text{cl}} = \frac{\mu}{\sqrt{8\lambda}}$,

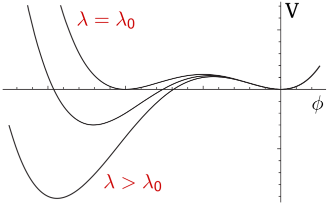
$$\tilde{\mathcal{L}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \lambda_3\phi^3 + \lambda\phi^4 \,, \qquad \lambda_3 = \sqrt{2\lambda}\mu \,.$$

Define instead a modified Lagrangian

$$\hat{\mathcal{L}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \hat{\lambda}_3\phi^3 + \lambda\phi^4 \,, \qquad \hat{\lambda}_3 = \mu\lambda\sqrt{\frac{2}{\lambda_0}}$$

$$\xrightarrow[\phi'=\sqrt{\lambda}\phi]{} \frac{1}{\lambda}\left\{\frac{1}{2}(\partial\phi')^2 + \frac{1}{2}\mu^2\phi'^2 + \phi'^4 + (\text{const})\cdot\sqrt{\lambda}\phi'^3\right\} \,.$$

At fixed λ_0, the cubic term turns "quantum" and the classical finite action configurations are those of the $\mathbb{Z}_2$ unbroken phase.



Diagrammatic expansions
(beyond my pay grade)

- Vacuum energy

$$\frac{\tilde\Lambda}{\mu^2} = -\left(\frac{\psi^{(1)}(1/3)}{4\pi^2} - \frac{1}{6}\right)\tilde g - 0.042182971(51)\tilde g^2 - 0.0138715(74)\tilde g^3 - 0.01158(19)\tilde g^4 + \mathcal{O}\left(\tilde g^5\right)$$

$$\left.\frac{\tilde\Lambda}{\mu^2}\right|_{\mathrm{EPT}} = -\left[\frac{1}{\tilde{g_0}}\left(\frac{\psi^{(1)}(1/3)}{4\pi^2} - \frac{1}{6}\right) + \frac{21\zeta(3)}{16\pi^3}\right]\tilde g^2 + \left(\frac{0.15991874}{\tilde{g_0}} + \frac{27\zeta(3)}{8\pi^4}\right)\tilde g^3 + \ldots(\cdot)\tilde g^8$$

- 1-point Tadpole

$$\frac{\langle\phi\rangle}{\phi_{\mathrm{cl}}} = 1 - 0.712462426(83)\tilde g^2 - 2.152451(65)\tilde g^3 - 6.5422(59)\tilde g^4 + \mathcal{O}\left(\tilde g^5\right)$$

but, since in 2d Ising $\langle\phi\rangle \sim \mu^{\frac{1}{8}}$, we'll plot

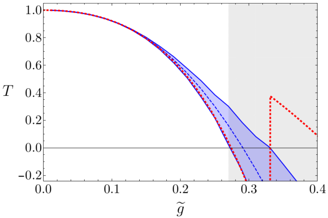
$$T \equiv \left(\frac{\langle\phi\rangle}{\phi_{\mathrm{cl}}}\right)^8 \sim |\tilde g_c - \tilde g| \; .$$

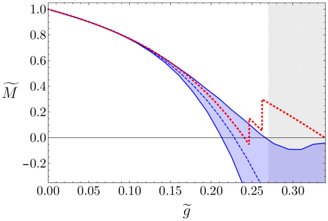
- Physical mass

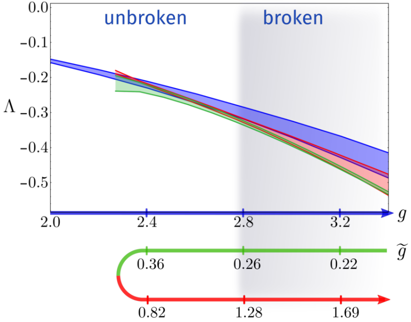
$$\frac{\tilde M^2}{\mu^2} = 1 - 2\sqrt{3}\tilde g - 4.1529(18)\tilde g^2 - 14.886(30)\tilde g^3 - 50.62(99)\tilde g^4 + \mathcal{O}\left(\tilde g^5\right)$$

Results

[Serone et al.]







Take home messages

- Observables in QFT are often computed in terms of asymptotic series, so that the strongly coupled region is inaccessible.
- The Borel transform technique is a way to circumvent this problem in a large class of scalar field theories.
- Its effectiveness depends on the singularity structure of the Borel function, which in turn is determined by the instanton configurations of the theory.
- This can be used in ϕ^4 theory in $d = 2$ to study its symmetry broken phase.

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- This can be used in ϕ^4 theory in $d = 2$ to study its symmetry broken phase.

Thanks for your attention!

$$\mathcal{I}(\lambda) = \int \mathcal{D}x(\tau) G[x(\tau)] e^{-\frac{1}{\lambda} S[x(\tau)]} \,, \quad S[x] = \int \mathrm{d}\tau \left\{ \frac{1}{2}\dot{x}^2 + V(x) \right\}$$

- Derrick's theorem doesn't cover the $d = 1$ case.
- Still, if the action $S[x(\tau)]$ has only one real saddle *s.t.* $\det(S''[x_c(\tau)]) \neq 0$, then the series expansion of $\mathcal{I}(\lambda)$ is Borel summable to the exact result.
- This requirement is met whenever $V(x)$ has a single, nondegenerate critical point (minimum).

Exact Perturbation Theory

Let's split the potential as $V = V_0 + \Delta V$ so that

(1) V_0 has a single, nondegenerate minimum.

(2) $\lim_{|x|\to\infty} \frac{\Delta V}{V_0} = 0$.

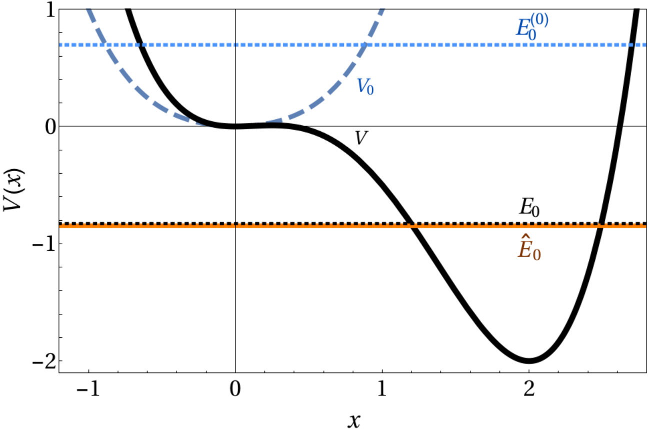
Introducing the modified potential

$$\hat{V} \equiv V_0 + \frac{\lambda}{\lambda_0}\Delta V \equiv V_0 + \lambda V_1$$

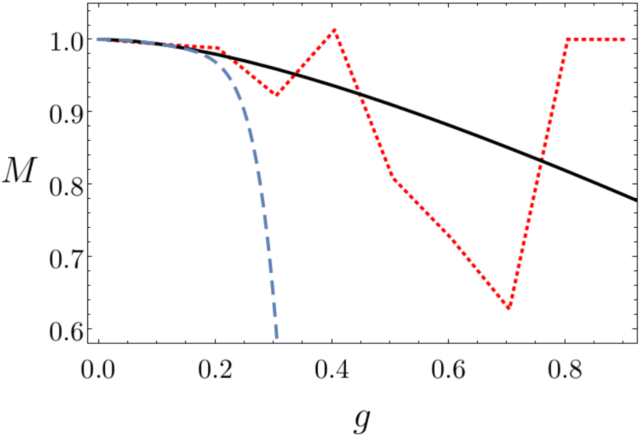
$$\hat{\mathcal{I}}(\lambda, \lambda_0) = \int \mathcal{D}x G[x] e^{-\frac{1}{\lambda_0}\int \mathrm{d}\tau \Delta V - \frac{1}{\lambda} S_0}\,, \quad S_0 \equiv \int \mathrm{d}\tau \left\{ \frac{1}{2}\dot{x}^2 + V_0 \right\}$$

we obtain that $\mathcal{I}(\lambda) = \hat{\mathcal{I}}(\lambda, \lambda)$ becomes Borel summable to the exact result.

Exact Perturbation Theory



Mass in the unbroken phase



Kink mass from the unbroken phase?