Random matrices, quantum chaos, and localization

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In this 4-hours lecture, I aim to provide a bird's-eye view on chaos in quantum many-body systems and Anderson localization, using random matrices as a tool to understand their physics.

The intent is pedagogical rather than exhaustive, and I list below a few references where these concepts are explored in greater depth. In particular, this review [1] offers a contained and colloquial introduction to quantum chaos and the Eigenstate Thermalization Hypothesis, whereas this second review [2] provides a more comprehensive discussion of these topics (at the cost of being more lengthy). These slides [7] summarize the same main ideas, following a storyline similar to the one adopted in the first part of my lecture. A great introduction to random matrices can be found in this book [4]; see also these lecture notes [6], which are based on a course given by Pierpaolo Vivo, one of the authors of [4]. Unfortunately, there are not many pedagogical introductions around to the topic of Anderson and many-body localization; in the second part of my lecture, I initially follow [5], and then the storyline summarized in these slides [8].

Throughout the lecture I propose several guided exercises, the solutions to which (mostly presented in class) can also be found in the references given below.

Syllabus

- Reminder on the concept of thermalization in classical many-body systems (ergodicity, typicality).
- Reminder on classical integrable and chaotic systems (Liouville-Arnold theorem, Lyapunov exponents).
- Problems with the naive application of these concepts to quantum mechanics.
- How does a quantum system reflect the regular/chaotic behavior of its classical counterpart? Energy spectrum and level spacing statistics.
- **Exercise** [4, 6]. Consider a set of i.i.d. random energy levels E_n. Derive the distribution $P(s)$ of the spacing $s = E_{n+1} - E_n$ between adjacent levels.
- Berry-Tabor conjecture (spectrum of quantum systems whose classical counterpart is integrable).
- More complex Hamiltonians: why introducing random matrices. Generalities on the Gaussian ensembles.
- **Exercise** [6]. Take a symmetric 2×2 matrix H whose entries are Gaussian random variables. Write the joint pdf $\mathcal{P}(H)$ of its *entries*, and deduce the compatibility between the two requirements of *independent Gaussian entries*, and *rotational invariance* of the statistical weight of the ensemble.
- Joint distribution of the eigenvalues in the Gaussian Orthogonal Ensemble (GOE), eigenvalue repulsion.
- **Exercise** [4]. Use such joint pdf to derive $P(s)$ for the GOE in the 2×2 case. Show that it follows the so-called Wigner's surmise.
- Bohigas-Giannoni-Schmit conjecture (spectrum of quantum systems whose classical counterpart is chaotic).
- **Exercise** [3, 4]. Given the distribution $\mathcal{P}_\psi(\boldsymbol{\psi})$ of a generic eigenvector in the GOE, marginalize it to compute the PDF $P_1(x = \psi_1)$ of a single component. Take the thermodynamic limit and compute its first four moments, then obtain $P_y(y = N|\psi_1|^2)$ (known as the Porter-Thomas distribution).
- Structure of the matrix elements of an observable averaged over GOE eigenvectors.

- **Exercise** [2]. Prove that the fluctuations of such observables scale in the GOE as $\sim 1/N$, where N is the system size.
- Reminder on the density matrix in quantum mechanics. Time evolution (projection on the diagonal ensemble), apparent clash with ergodicity.
- Statement of the eigenstate thermalization hypothesis (ETH), and its prediction for the expectation value of (a certain class of) observables.
- **Exercise** [2]. Use the ETH ansatz to show that fluctuations in the observables are suppressed exponentially with the system size.
- Cases in which the ETH is known to fail: integrable systems, phase transitions, quantum scars, and localized systems. (Brief explanation of what a quantum scar is.)
- Anderson localization, statement of the problem and state of the art.
- **Exercise** [5]. Consider the infinite-disorder limit. Compute the return probability of a quantum particle initially localized at the origin, and the local density of states.
- **Exercise** [5]. Consider the disorder-free limit. Compute the return probability and the local density of states (begin by considering a finite system with periodic boundary conditions, then take the thermodynamic limit).
- Many-body localization in a nutshell. Analogies with the problem of Anderson localization on a sparse random graph and in the presence of correlated random energies and hopping terms.
- Diagnostics of localization: inverse participation ratio (IPR), multifractal exponents, spectral statistics.
- **Exercise** [3]. Use the Porter-Thomas distribution to compute the IPR in the GOE.

References

[1] Joshua M Deutsch. Eigenstate thermalization hypothesis. *Reports on Progress in Physics*, 81(8):082001, July 2018. URL `http://dx.doi.org/10.1088/1361-6633/aac9f1`.

[2] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Advances in Physics*, 65(3):239–362, May 2016. URL `http://dx.doi.org/10.1080/00018732.2016.1198134`.

[3] Fritz Haake, Sven Gnutzmann, and Marek Kuś. *Quantum Signatures of Chaos*. Springer International Publishing, 2018. doi: 10.1007/978-3-319-97580-1. URL `http://dx.doi.org/10.1007/978-3-319-97580-1`.

[4] Giacomo Livan, Marcel Novaes, and Pierpaolo Vivo. *Introduction to Random Matrices*. Springer International Publishing, 2018. URL `http://dx.doi.org/10.1007/978-3-319-70885-0`.

[5] Antonello Scardicchio and Thimothée Thiery. Perturbation theory approaches to Anderson and many-body localization: some lecture notes, 2017. URL `https://arxiv.org/abs/1710.01234`.

[6] Davide Venturelli. Lecture notes on the course "Random matrix theory" given in SISSA by Pierpaolo Vivo, 2020. URL `https://drive.google.com/file/d/11x4z8TKNoB_cZEeYvolxxHh7lGETtdBT/view?usp=drive_link`.

[7] Davide Venturelli. Quantum chaos and ETH, 2020. URL `https://drive.google.com/file/d/1c5V8lz7ybiN76cd9_w56NkxDZ_XiByKq/view?usp=drive_link`. Presentation @ SISSA.

[8] Davide Venturelli. Multifractality in disordered graphs: insights from two random matrix models, 2024. URL `https://sonarventu.github.io/assets/slides/2024-06-24-SIFS-Parma.pdf`. IV Conference of the Italian Society of Statistical Physics, Università di Parma.