Multifractality in disordered graphs

Insights from two random matrix models

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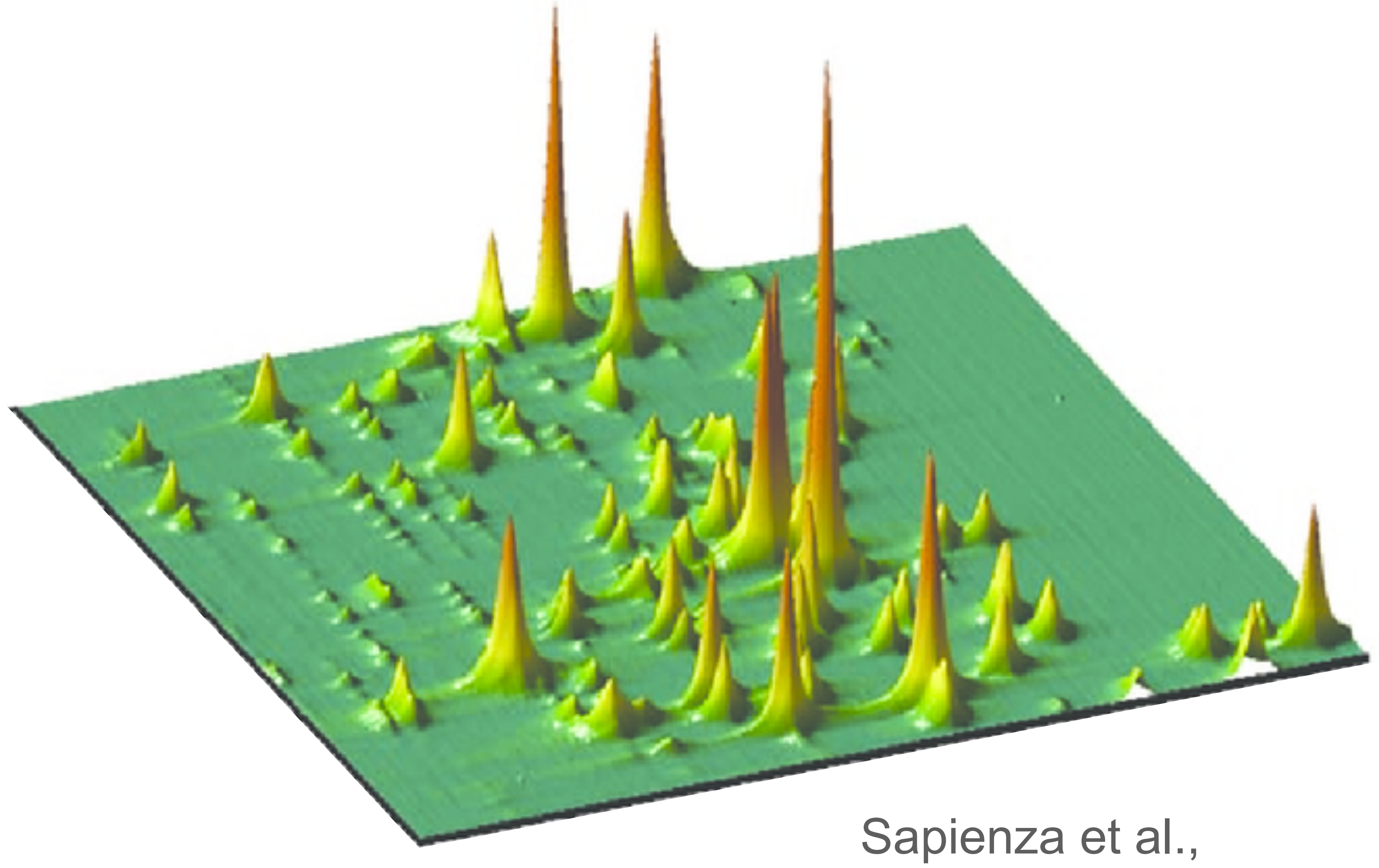
Anderson Localization

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_i \epsilon_i c_i^\dagger c_i$$

ϵ_i i.i.d. in $[-W/2, W/2]$

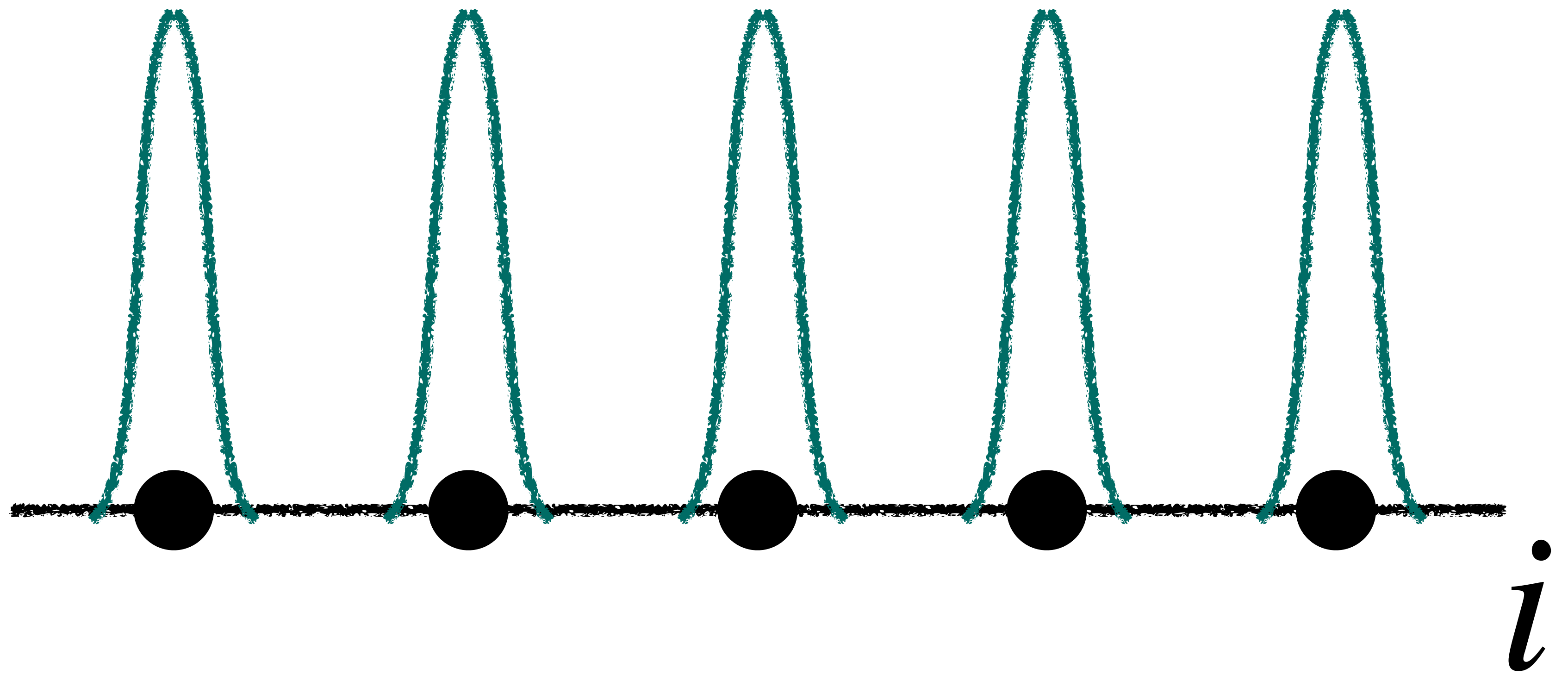


Sapienza et al., Science 327, 1352 (2010)

- **AL**: disorder-induced transition,

$$\psi(x) \sim e^{-|x-x_0|/\xi}$$

- In $d \leq 2$, infinitesimal disorder sufficient
- In $d > 2$, critical W_c between metallic/insulating phases



Anderson, Phys. Rev. 109, 1492 (1958)

Many-Body Localization

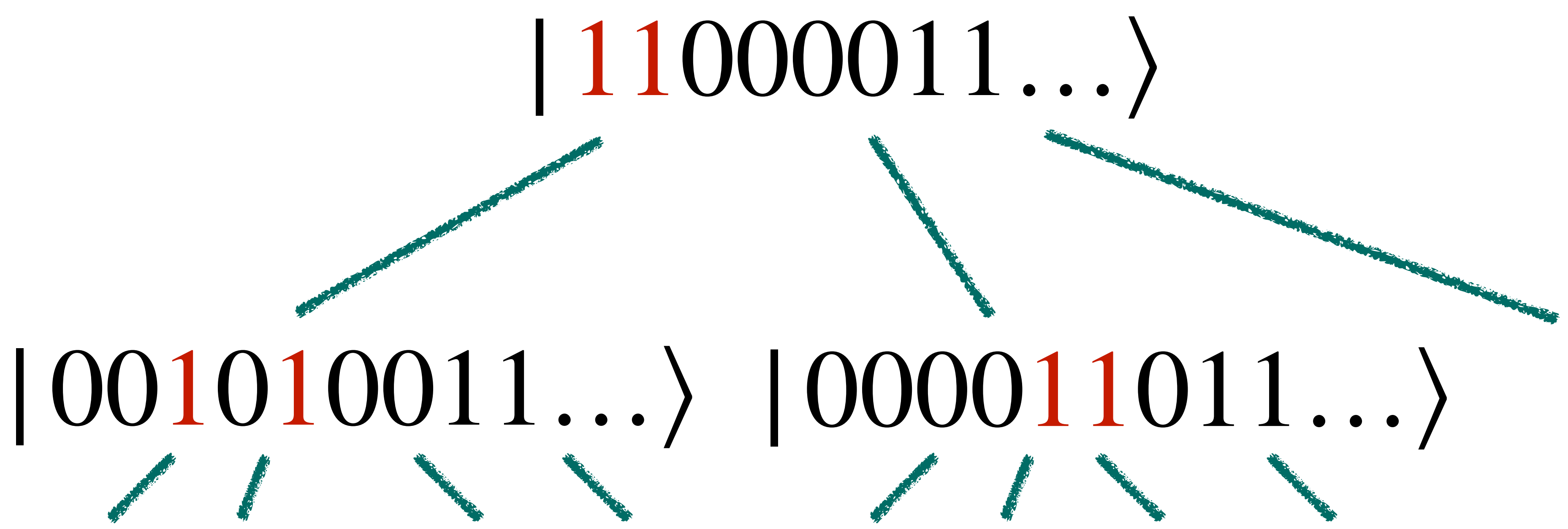
$$H = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_i \epsilon_i c_i^\dagger c_i + V \sum_{\langle i,j \rangle} c_i^\dagger c_i \, c_j^\dagger c_j$$

$$H' = \sum_{\alpha \neq \beta} T_{\alpha\beta} |\alpha\rangle\langle\beta| + \sum_\alpha E_\alpha |\alpha\rangle\langle\alpha|$$

High dimensional Fock space

$$|\alpha\rangle = |n_1, n_2, n_3, \ldots\rangle$$

Altshuler, Gefen, Kamenev, Levitov, PRL 78, 2803 (1997)



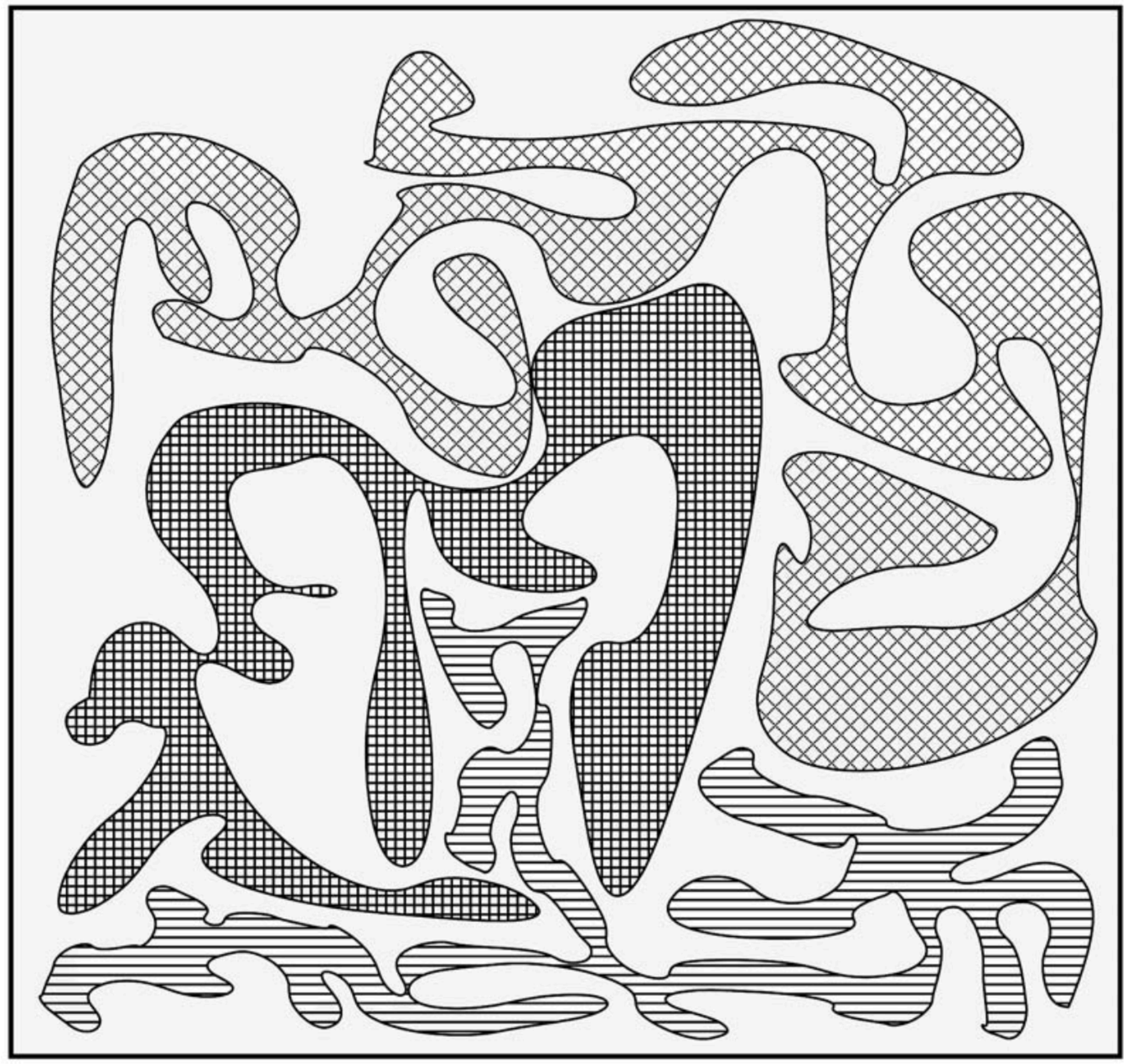
- $T_{\alpha\beta}$ hierarchical: Fock space is a **network**
- At strong disorder, $|E_\alpha - E_\beta| \gg T_{\alpha\beta}$
- Localized in Fock space $\longrightarrow$ non-ergodic, no thermalization

Basko, Aleiner, Altshuler, Ann. Phys. 321,1126 (2006)

Multifractality

- Insulating vs metallic (→ **ergodic**)
- "Bad metal" phase?
- Inverse participation ratios (**IPR**):

$$I_q = \left\langle \sum_i |\psi(i)|^{2q} \right\rangle \propto N^{-(q-1)\,D_q}$$



Cuevas, Kravtsov, PRB 76, 235119 2007

- For $q > 1$, fractal dimensions D_q $\longrightarrow$ (***multi)fractal*** if $0 < D_q < 1$

fully delocalized

- Support set: $D_1 = -\dfrac{\partial}{\partial \ln N} \left\langle \sum_i |\psi(i)|^2 \ln |\psi(i)|^2 \right\rangle$

Basko, Aleiner, Altshuler, Ann. Phys. 321,1126 (2006)

The generalized Rosenzweig-Porter model

$$H = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{NN} \end{pmatrix} + \frac{\nu}{N^{\gamma/2}} \begin{pmatrix} & & b_{ij} \\ & \text{GOE} & \\ & & \end{pmatrix}$$

$p_a(a_{ii})$

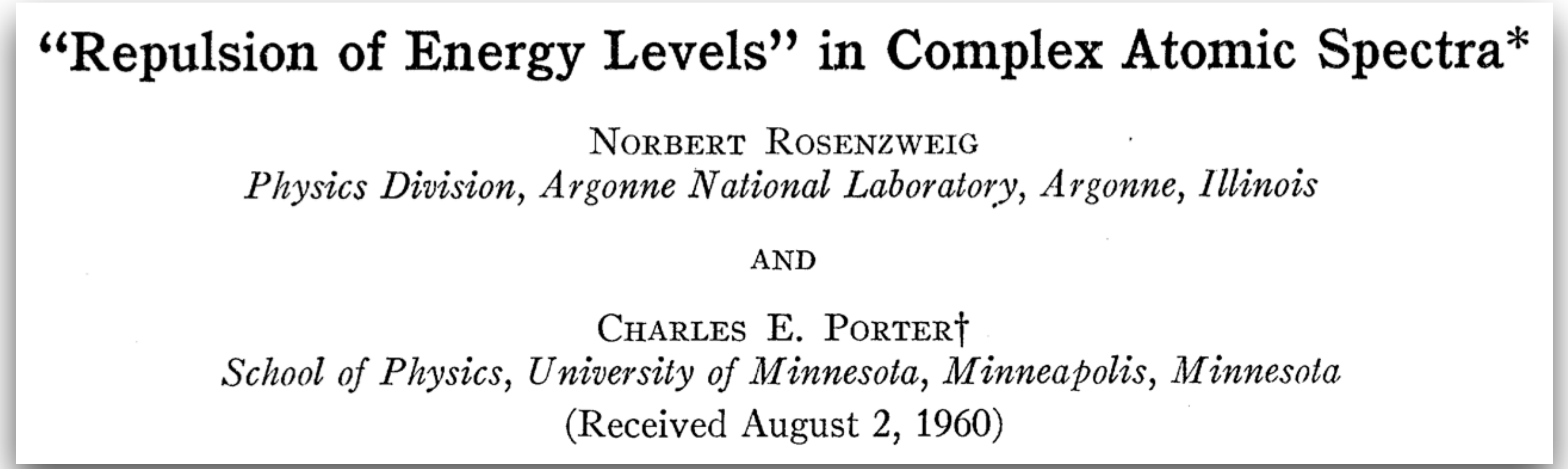
Gaussian i.i.d.

N

- A toy model for non-ergodic states
- Idea: **2-step disorder**

1. **Structural** disorder
   (Random Regular Graph + deterministic energies $\longrightarrow$ GOE)

2. Random **on-site** energies



"Repulsion of Energy Levels" in Complex Atomic Spectra*

NORBERT ROSENZWEIG
Physics Division, Argonne National Laboratory, Argonne, Illinois

AND

CHARLES E. PORTER†
School of Physics, University of Minnesota, Minneapolis, Minnesota
(Received August 2, 1960)

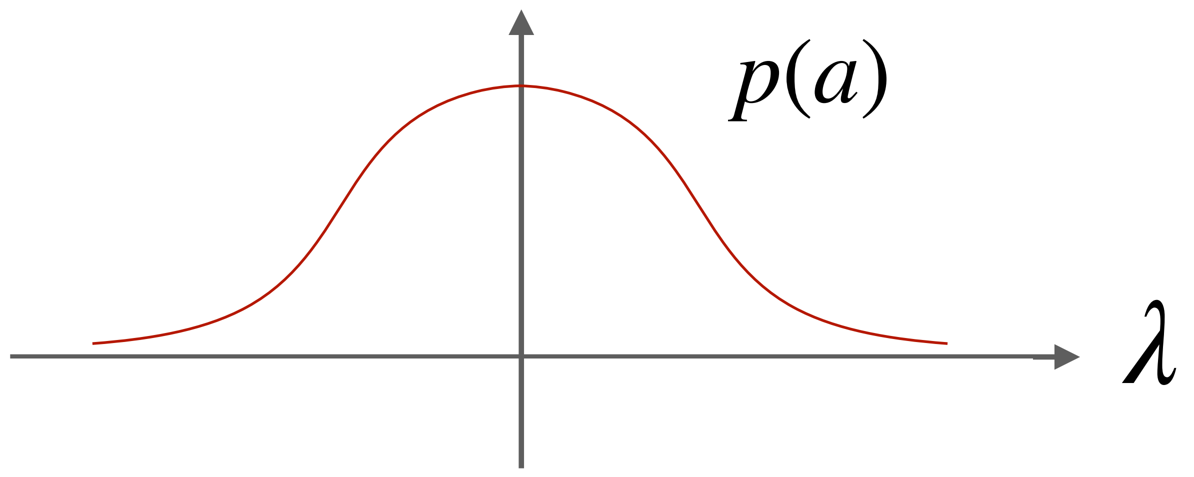
Oren, Smilansky, J. Phys. A 43 (2010) 225205

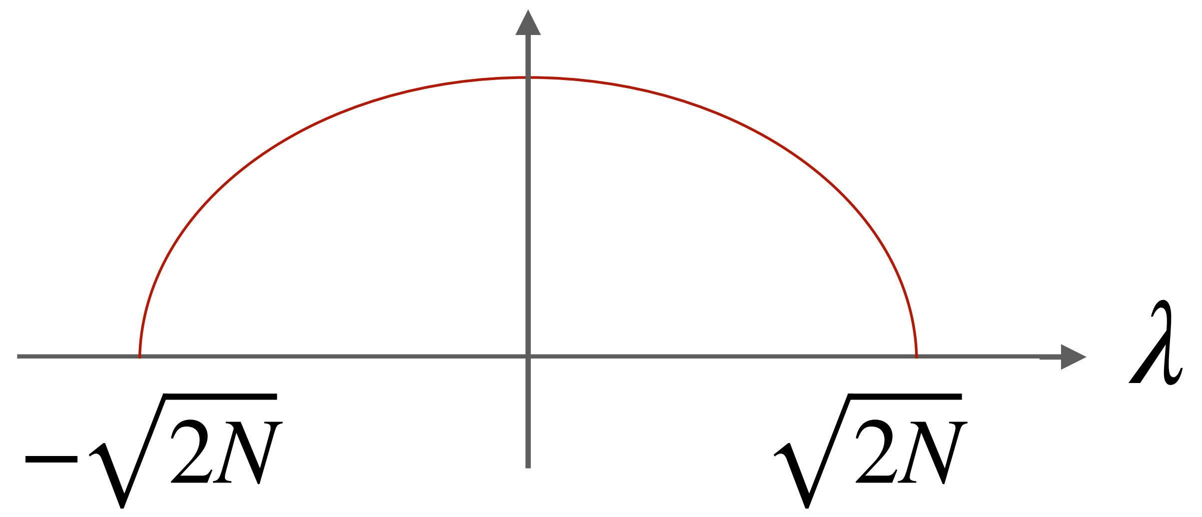
Kravtsov et al., 2015 New J. Phys. 17 122002

The generalized Rosenzweig-Porter model

$$H = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{NN} \end{pmatrix} + \frac{\nu}{N^{\gamma/2}} \underbrace{\begin{pmatrix} & \textbf{GOE} & \end{pmatrix}}_{N}$$

$p_a(a_{ii})$





$$\rho_N(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$$

Localized

Fully delocalized ($D_q = 1$)

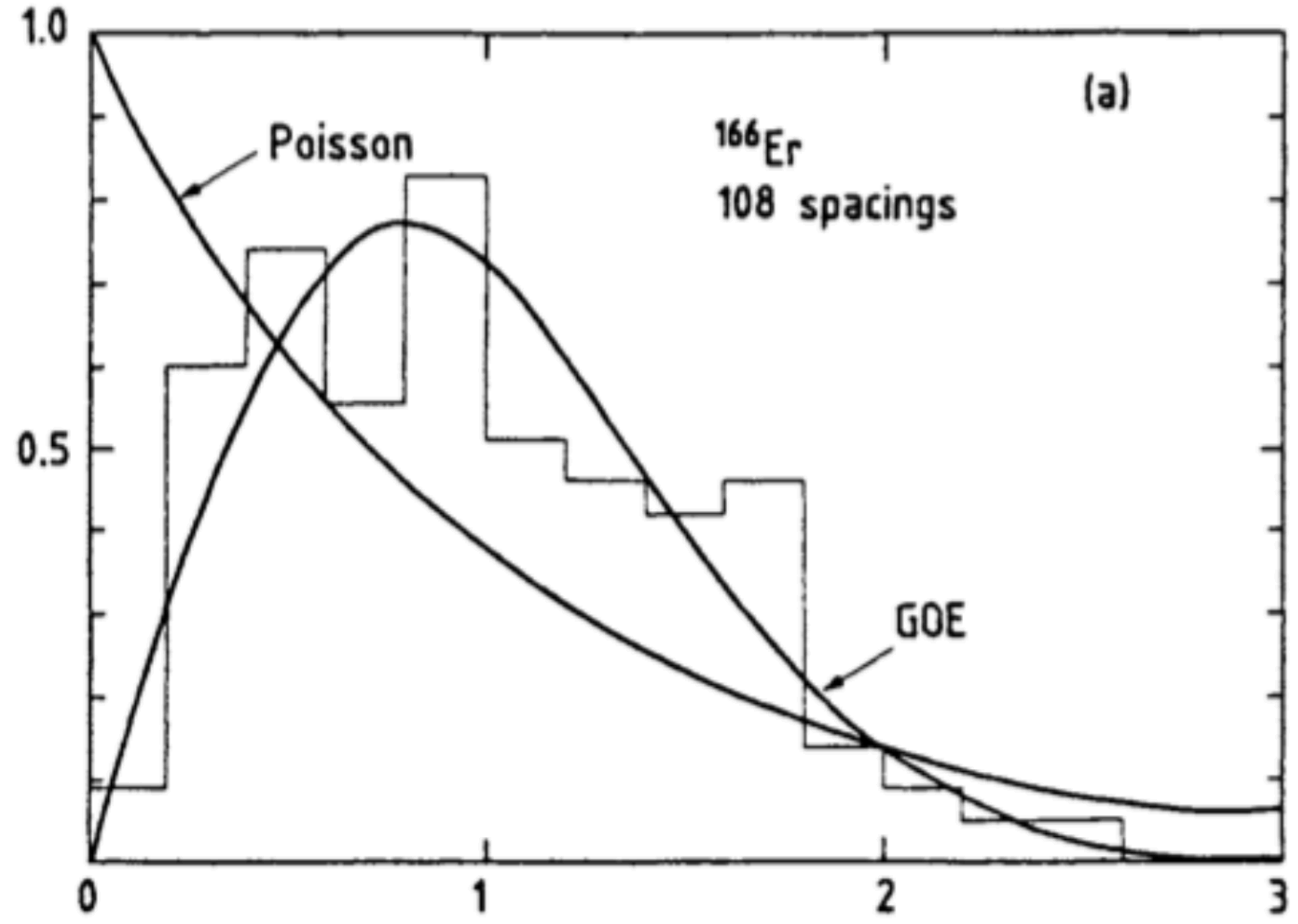
Eigenvectors

The generalized Rosenzweig-Porter model

$$H = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{NN} \end{pmatrix} + \frac{\nu}{N^{\gamma/2}} \begin{pmatrix} \\ \textbf{GOE} \\ \\ \end{pmatrix}$$

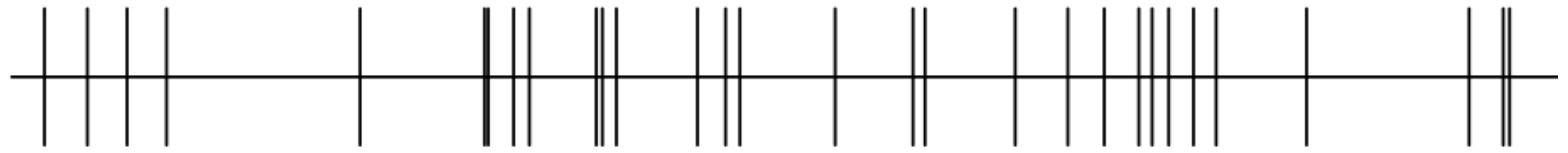
N

$p_a(a_{ii})$



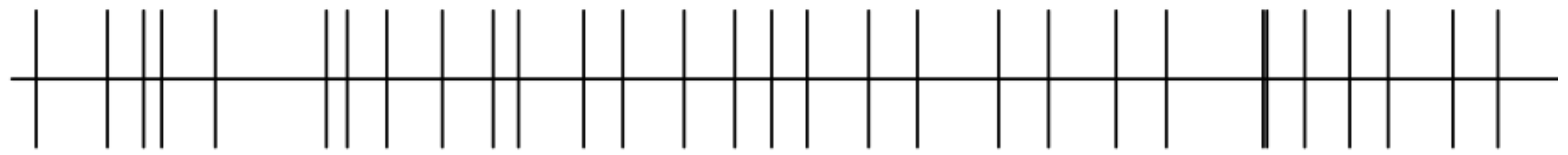
$$P(s) = e^{-s}$$

Poisson



$$P(s) = \frac{\pi}{2} s \, e^{-\pi s^2/4}$$

Wigner-Dyson



Level spacing

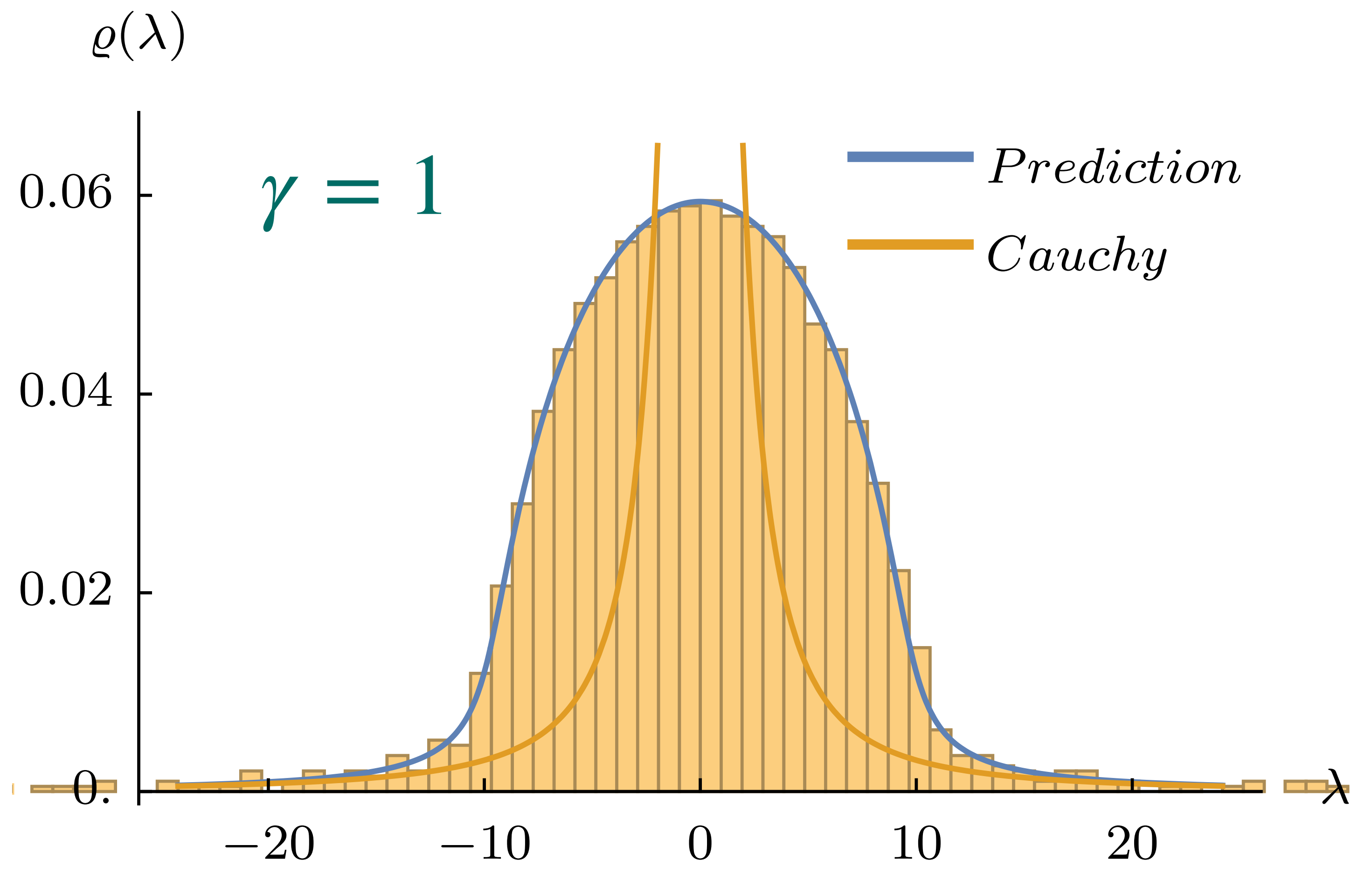
$$s_i = \lambda_{i+1} - \lambda_i$$

The generalized Rosenzweig-Porter model

$$H = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{NN} \end{pmatrix} + \frac{\nu}{N^{\gamma/2}} \begin{pmatrix} \\ & \textbf{GOE} & \\ \\ \end{pmatrix}$$

$p_a(a_{ii})$

N



	$\gamma < 1$	$1 < \gamma < 2$	$\gamma > 2$
	Delocalized eigenvectors		**Localized eigenvectors**
Spectral density	Semi-circle	$p_a(\lambda)$	
Level statistics	Wigner-Dyson	Wigner-Dyson	Poisson

$1 < \gamma < 2$

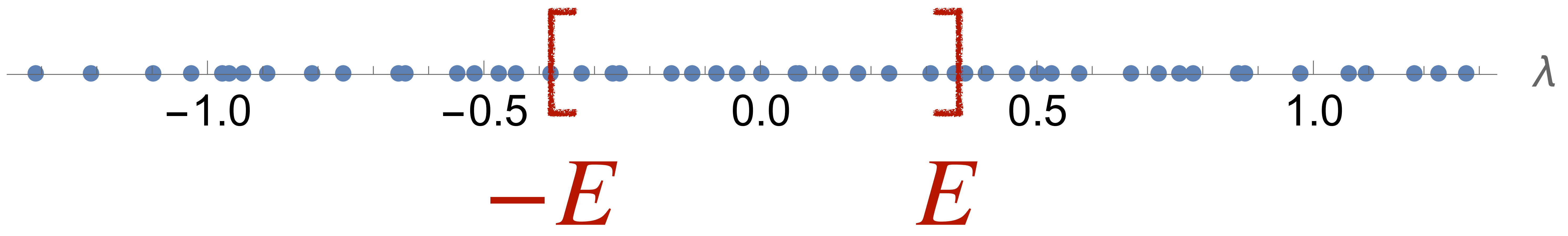
Fractal phase: partially delocalized eigenvectors

$D_q = 2 - \gamma < 1$

(for $q > 1/2$)

Kravtsov et al., 2015 New J. Phys. 17 122002

Level statistics: # of eigenvalues in an interval



$$I_N(E) = N \int_{-E}^{E} \mathrm{d}\lambda \, \rho_N(\lambda) = \sum_{i=1}^{N} \left[\theta(E - \lambda_i) - \theta(-E - \lambda_i)\right]$$

Cumulant generating function:

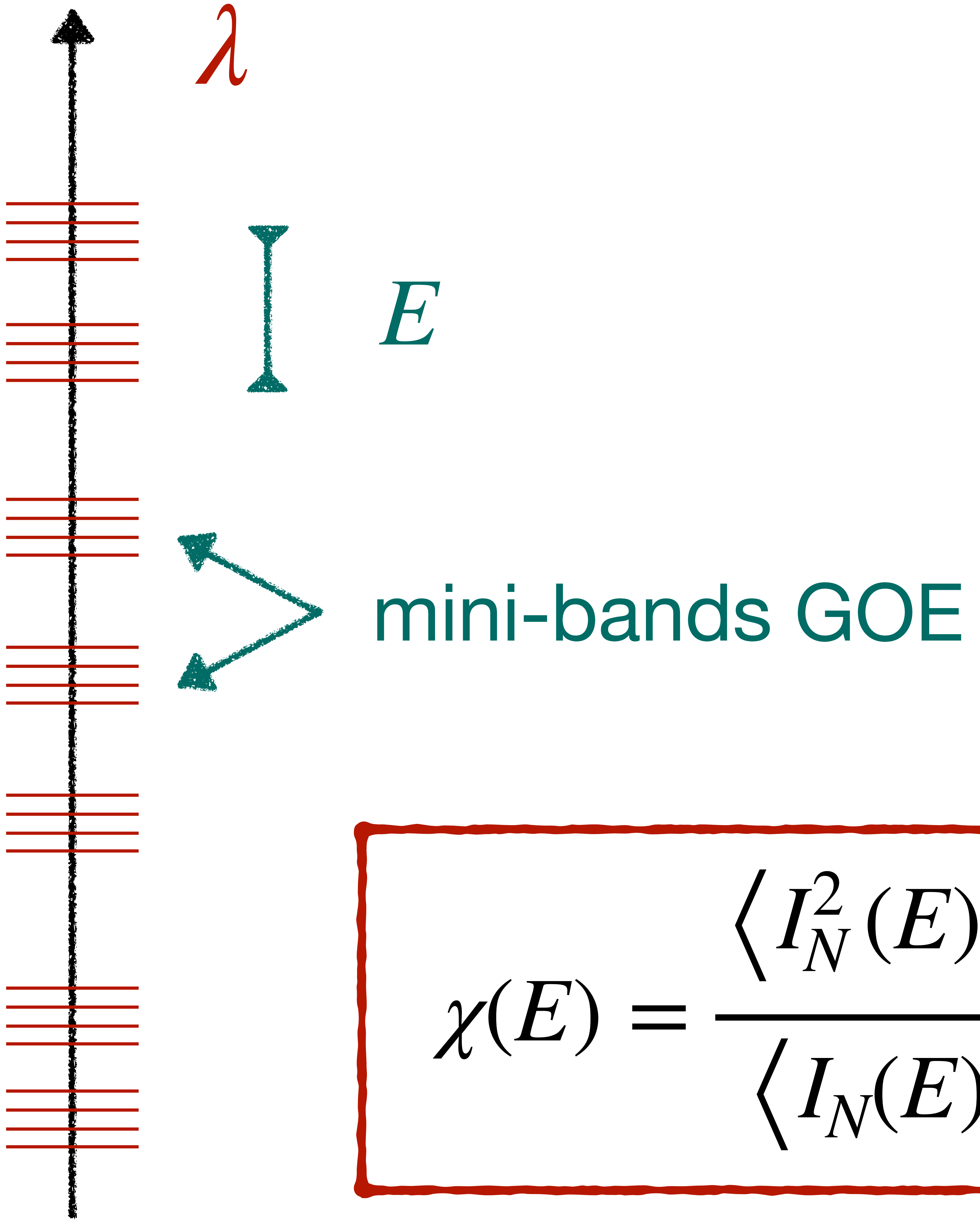
$$F_{N,E}(s) = \frac{1}{N} \ln \left\langle e^{-s\, I_N(E)} \right\rangle$$

leading order for large N using **replicas**

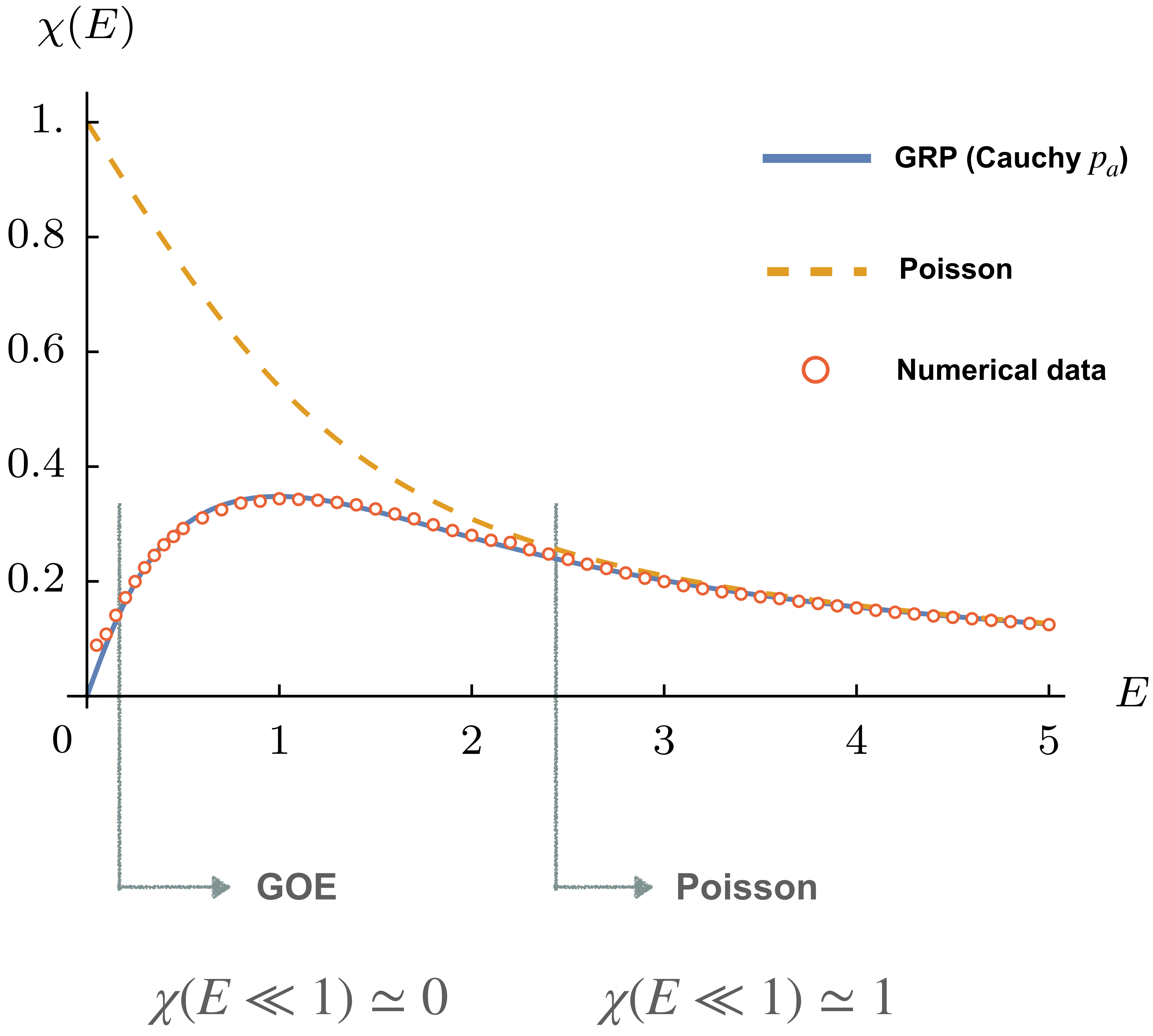
nonlinear in s already at 0-loops,
can access fluctuations!

Cavagna, Garrahan, Giardina, Phys. Rev. B 61, 3960 (2000)
Metz 2017 J. Phys. A: Math. Theor. 50 495002

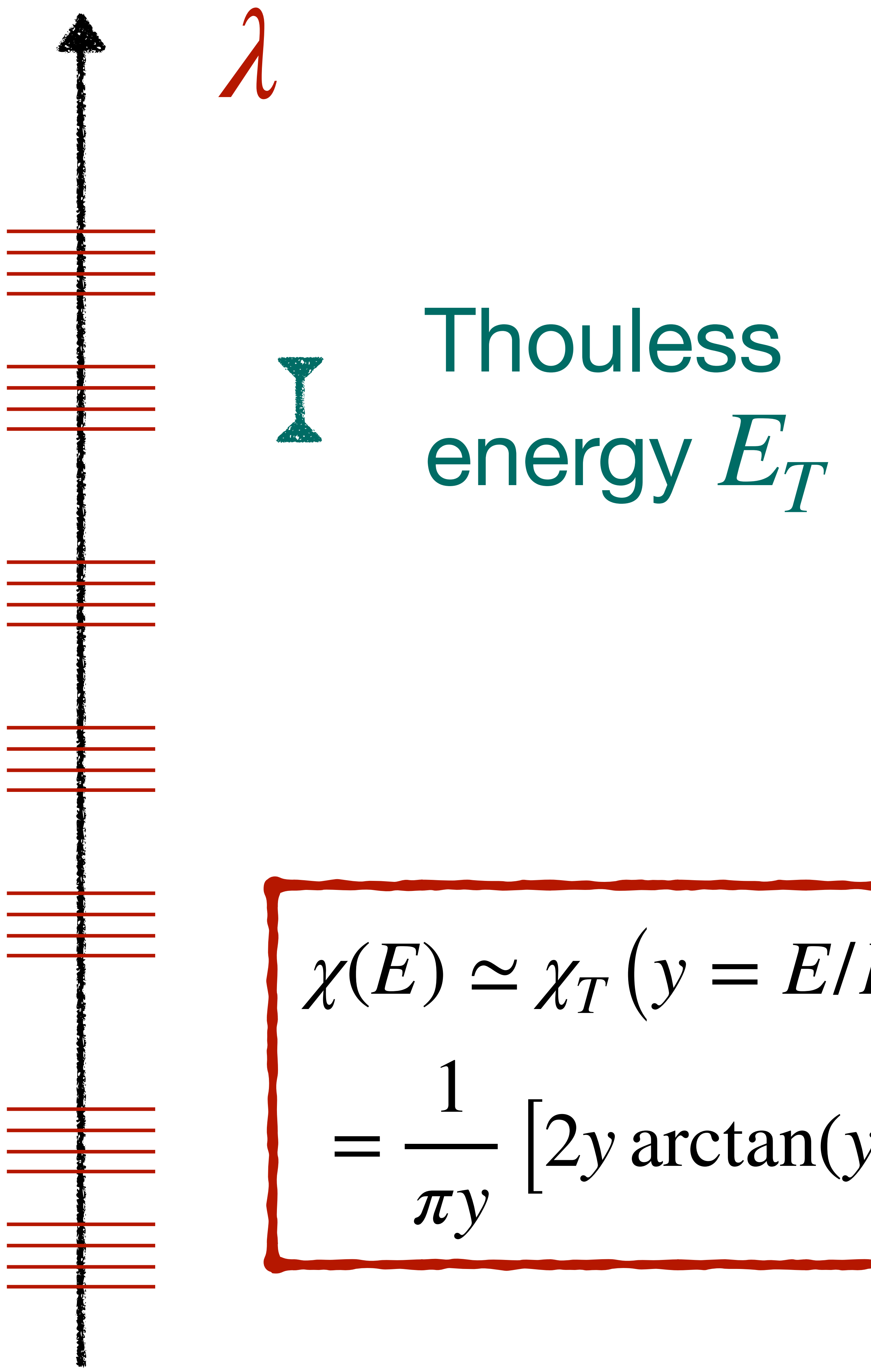
Level compressibility



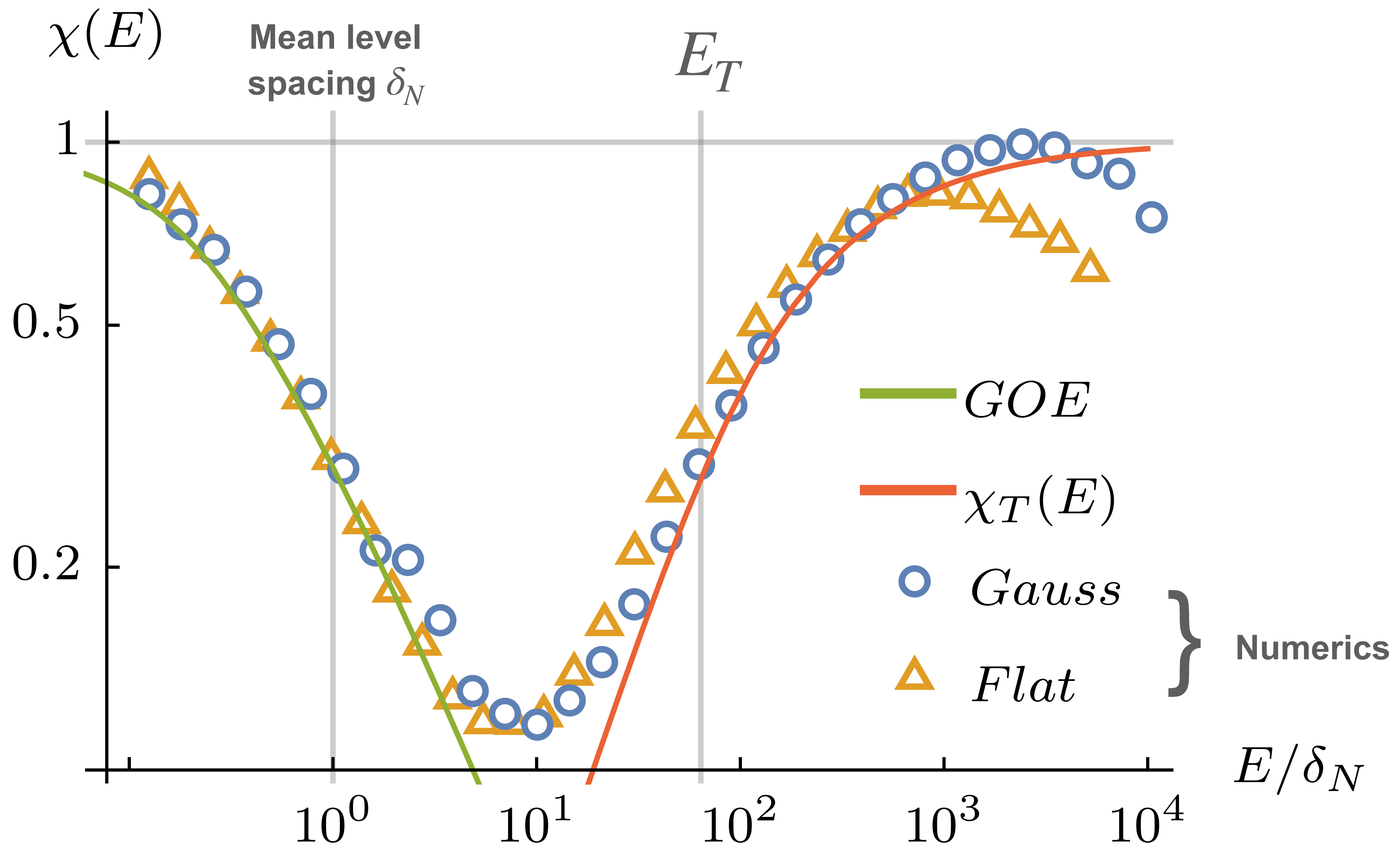
$$\chi(E) = \frac{\langle I_N^2(E) \rangle_c}{\langle I_N(E) \rangle}$$



Level compressibility



$$\chi(E) \simeq \chi_T\left(y = E/E_T\right) = \frac{1}{\pi y}\left[2y\arctan(y) - \ln(1+y^2)\right]$$



universal for $E \sim E_T$, independent of $p_a(a_{ii})$

Venturelli, Cugliandolo, Schehr, Tarzia, SciPost Phys. 14, 110 (2023)

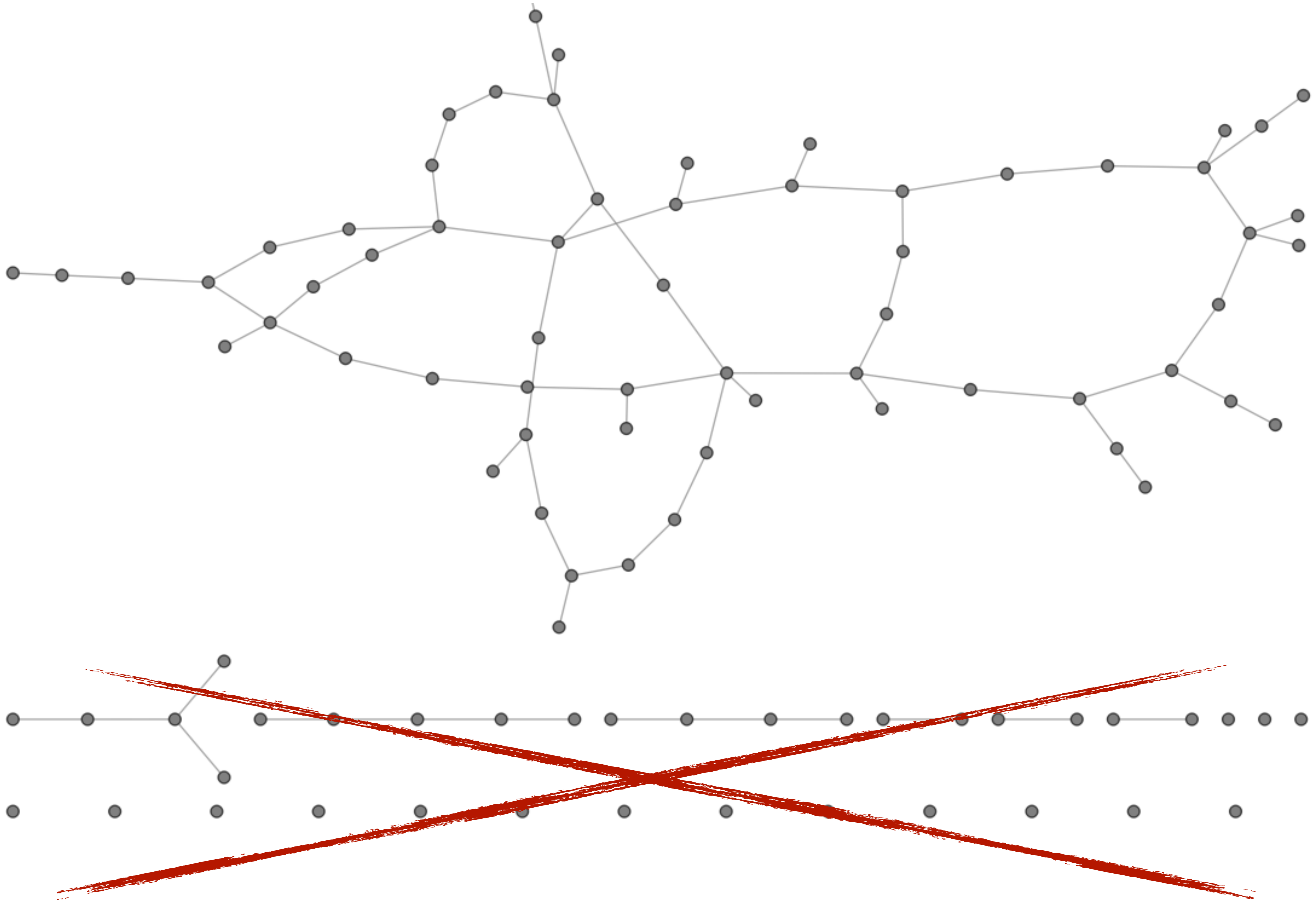
Weighted Erdös-Rényi graphs

$$H_{ij} = \frac{1}{\sqrt{p}} \sigma_{ij} h_{ij} \,,$$

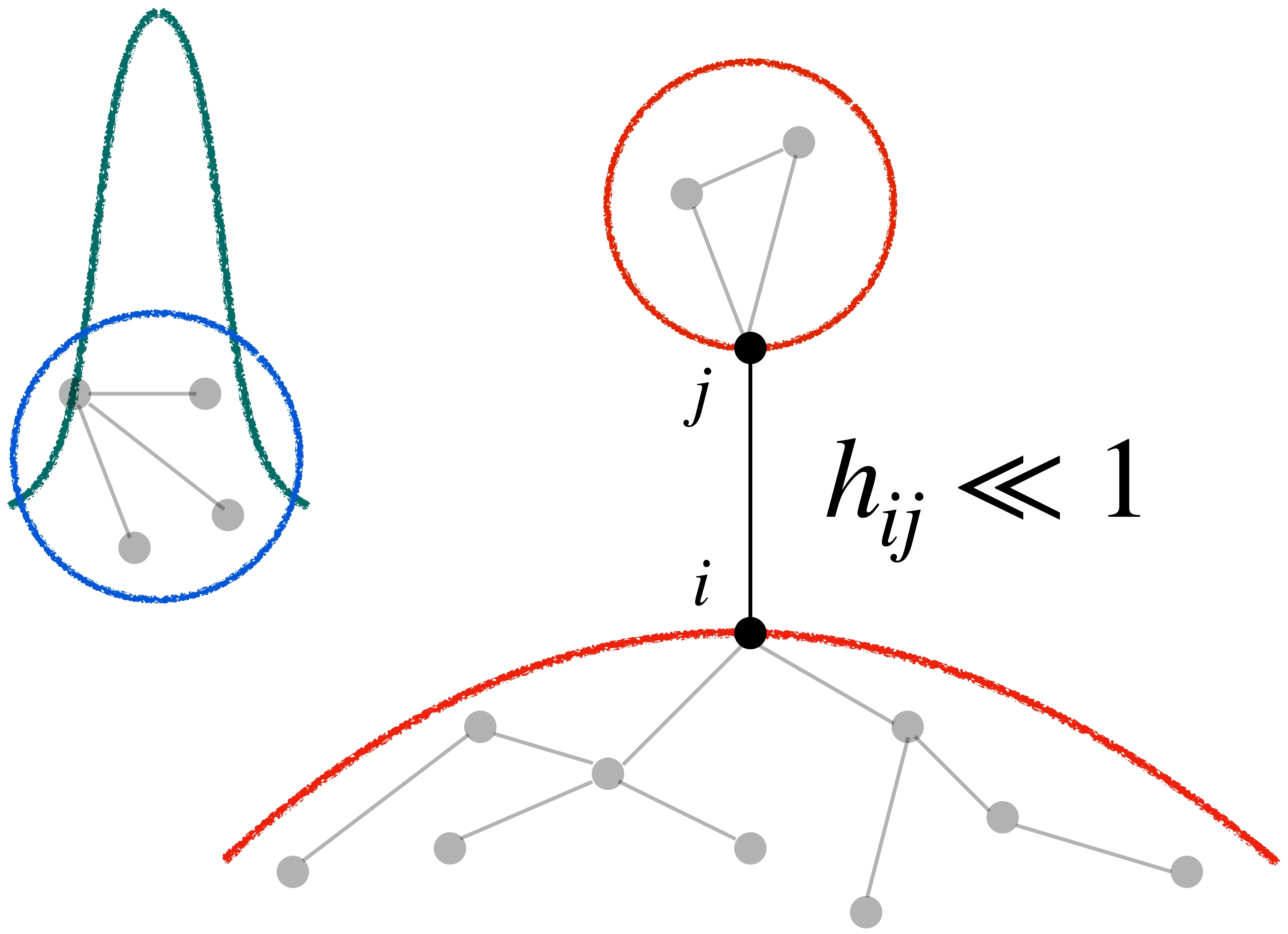
$\longleftarrow$ $p(h_{ij})$ (GOE)

$$\sigma_{ij} = \begin{cases} 1, & \text{prob} = p/N \\ 0, & \text{prob} = 1 - p/N \end{cases}$$

- Connectivity $p > 1$
- Isolate giant cluster
- Hierarchical lattice + random hoppings



Multifractality?



- Expected because of **fragmentation**
- **Aim**: measure D_q via the **IPR**

$$I_q(E) = \frac{\left\langle \sum_{i,\alpha} |\psi_\alpha(i)|^{2q} \delta(E-\lambda_\alpha) \right\rangle}{\left\langle \sum_\alpha \delta(E-\lambda_\alpha) \right\rangle} \propto N^{(1-q)D_q}$$

- Introduce **local density of states**

$$\rho_i(E) = \sum_\alpha |\psi_\alpha(i)|^2 \delta_\eta(E-\lambda_\alpha)$$

$$\delta_\eta(x) = \frac{\eta}{\pi(x^2+\eta^2)}, \quad \eta \sim \frac{1}{N}$$

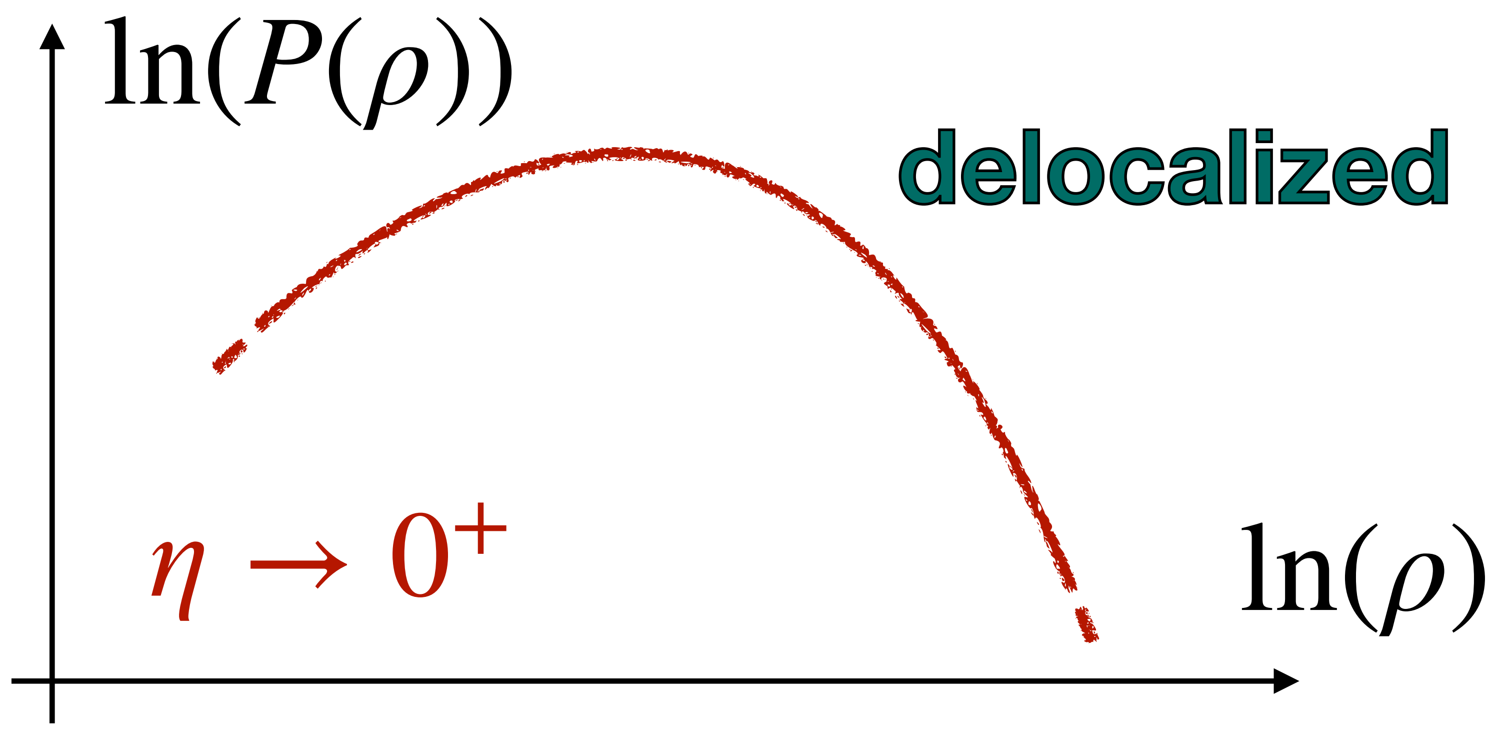
$$\longrightarrow \quad I_q(E) \propto \lim_{\eta\to 0^+} \eta^{q-1} \int \mathrm{d}\rho\, P(\rho)\, \rho^q$$

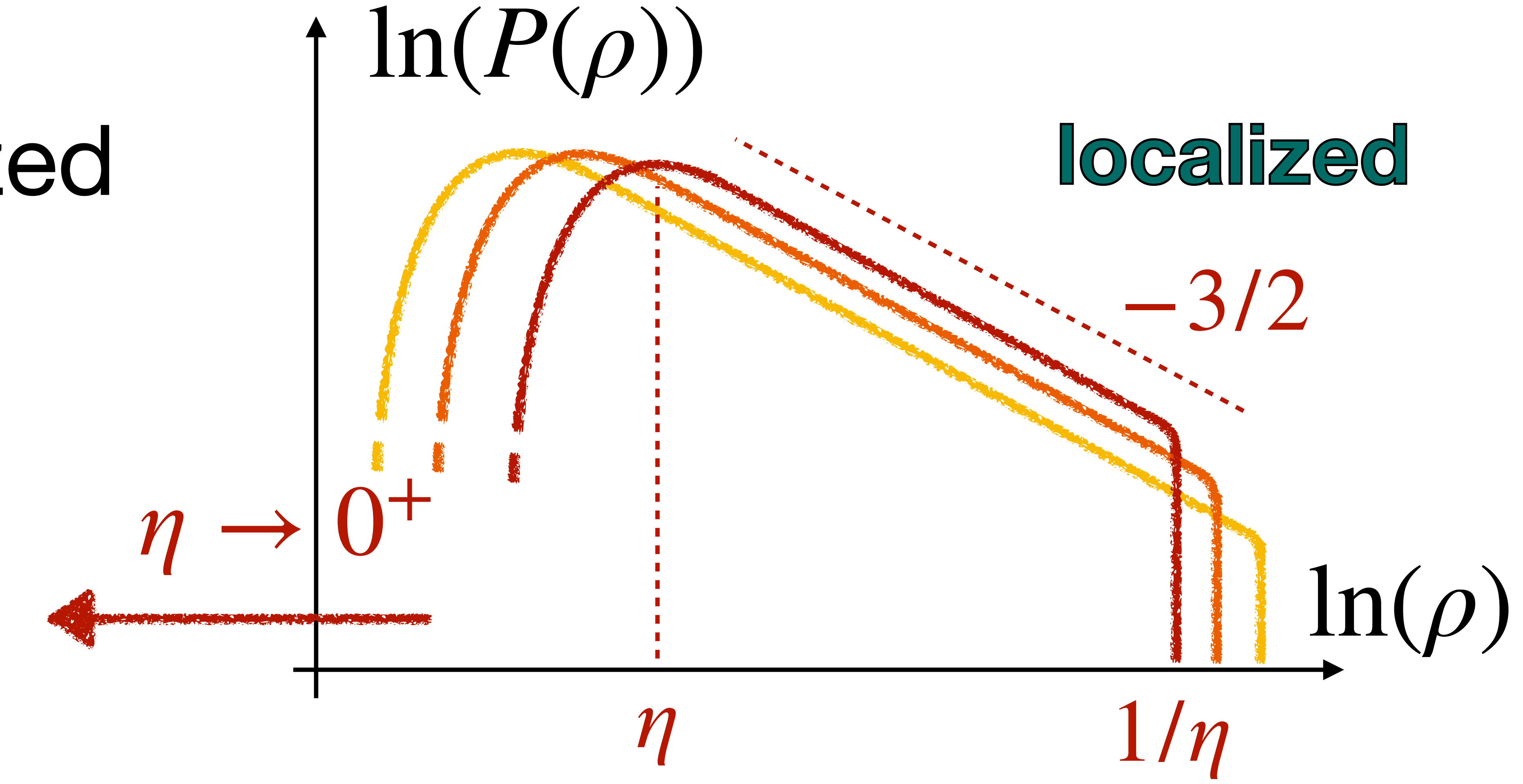
moments of $P(\rho)$

Cavity calculation

$$N^{(1-q)\,D_q} \sim I_q(E) \propto \lim_{\eta\to 0^+} \eta^{q-1} \int^{1/\eta} d\rho\, P(\rho)\, \rho^q$$

- Using **cavity method**, compute $\rho_i(E)$ (numerical recursion, no population dynamics!)
- $P(\rho)$ depends on regulator η: different behavior for $\eta \to 0^+$ if localized/delocalized
- Cutoff $1/\eta$ in localized phase



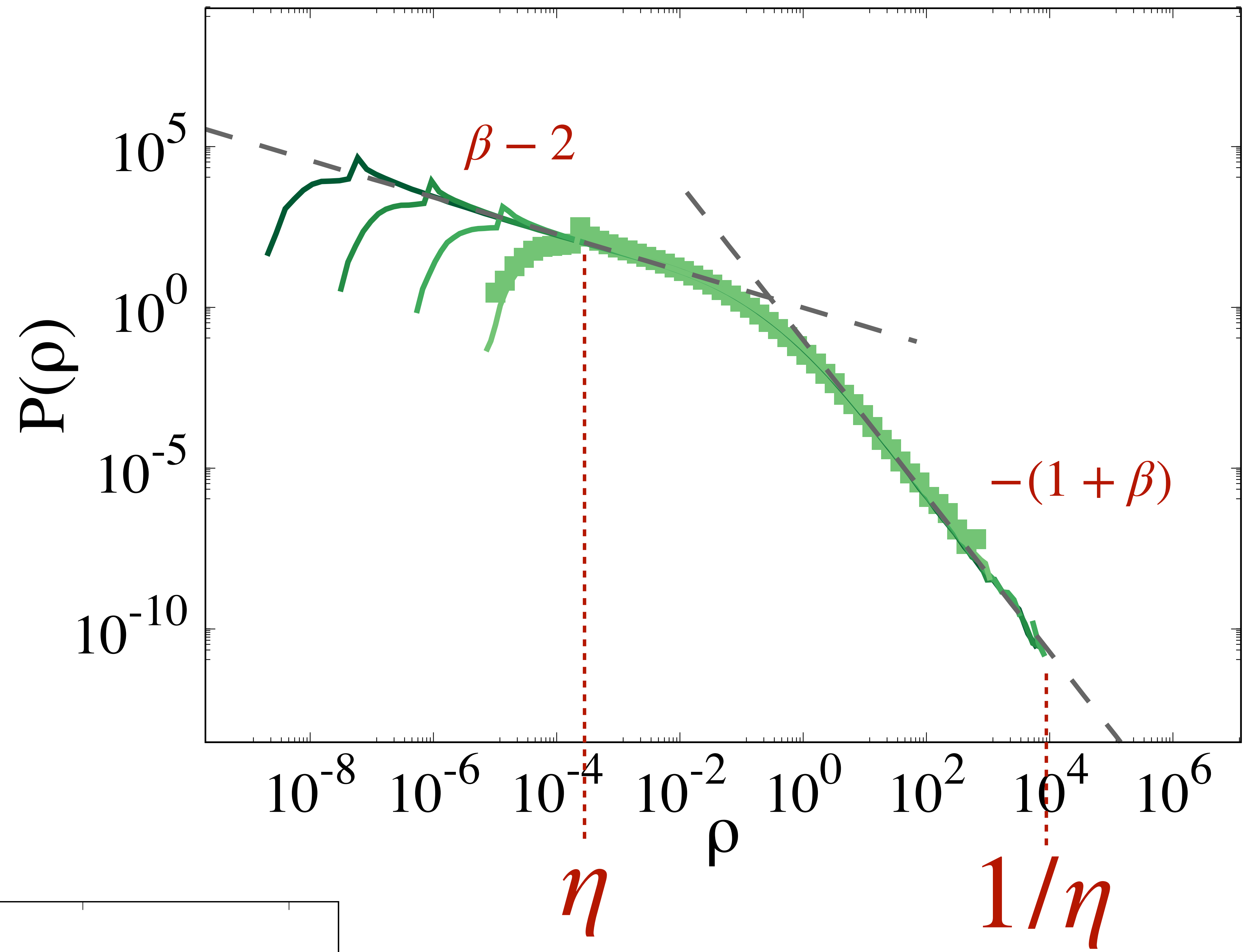


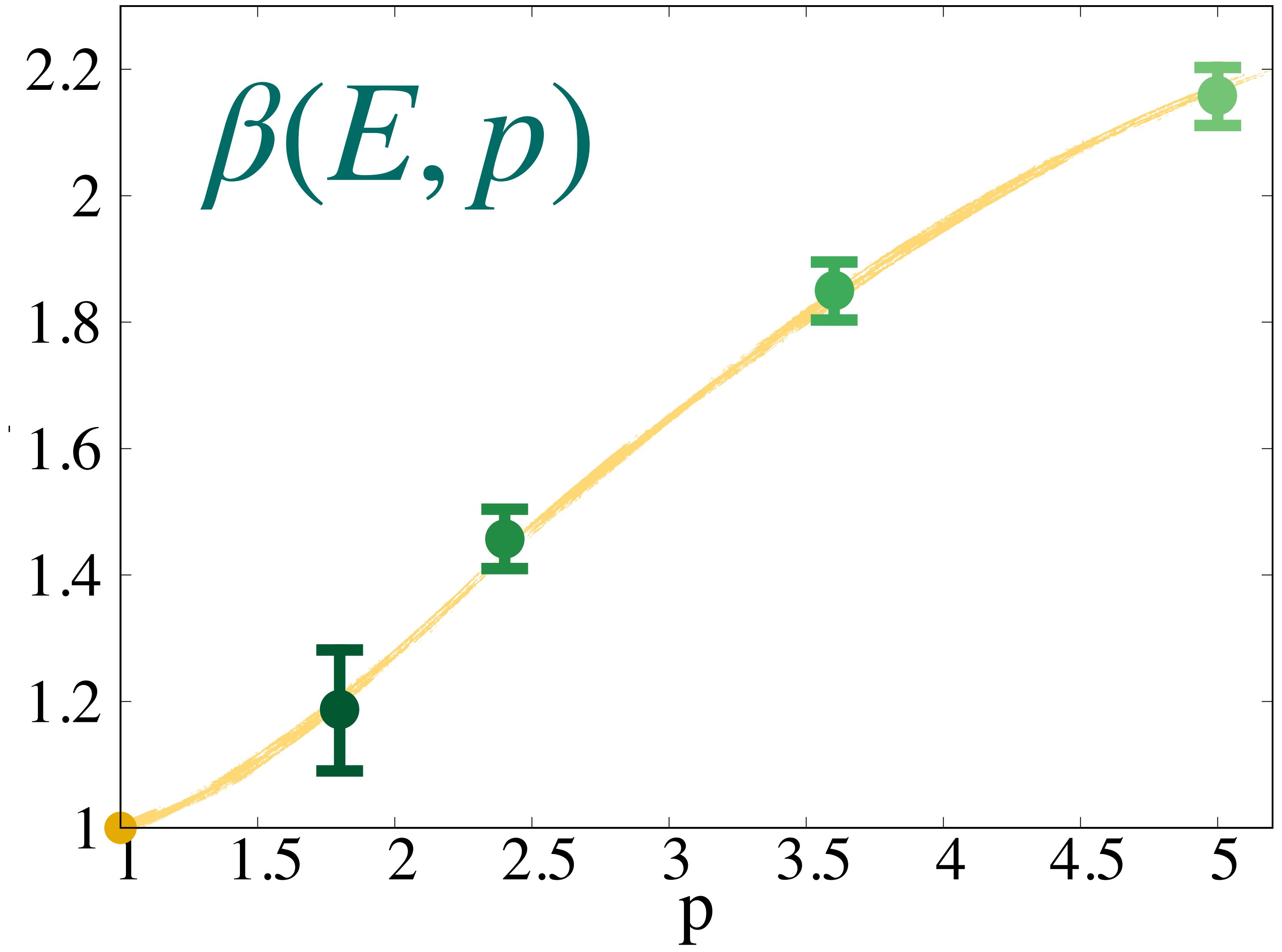
Abou-Chacra, Thouless, Anderson, J. Phys. C 6, 1734 (1973)

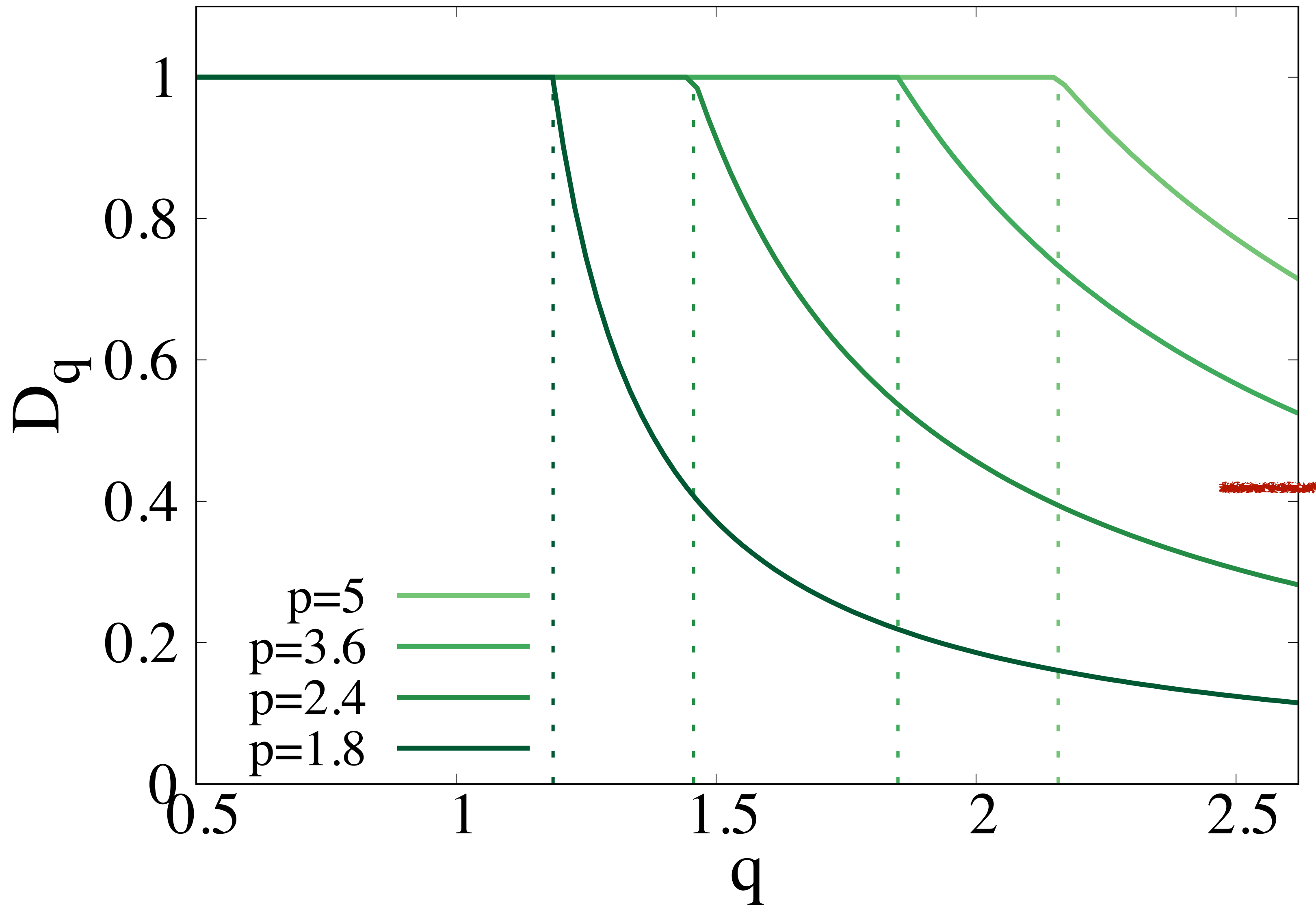
Measuring D_q

$$N^{(1-q)\,D_q} \sim I_q(E) \propto \lim_{\eta\to 0^+} \eta^{q-1} \int^{1/\eta} d\rho\, P(\rho)\, \rho^q$$

$$\eta \sim \frac{1}{N} \quad \longrightarrow \quad D_q = \begin{cases} \dfrac{\beta-1}{q-1}, & q \geq \beta, \\ 1, & q < \beta \end{cases}$$





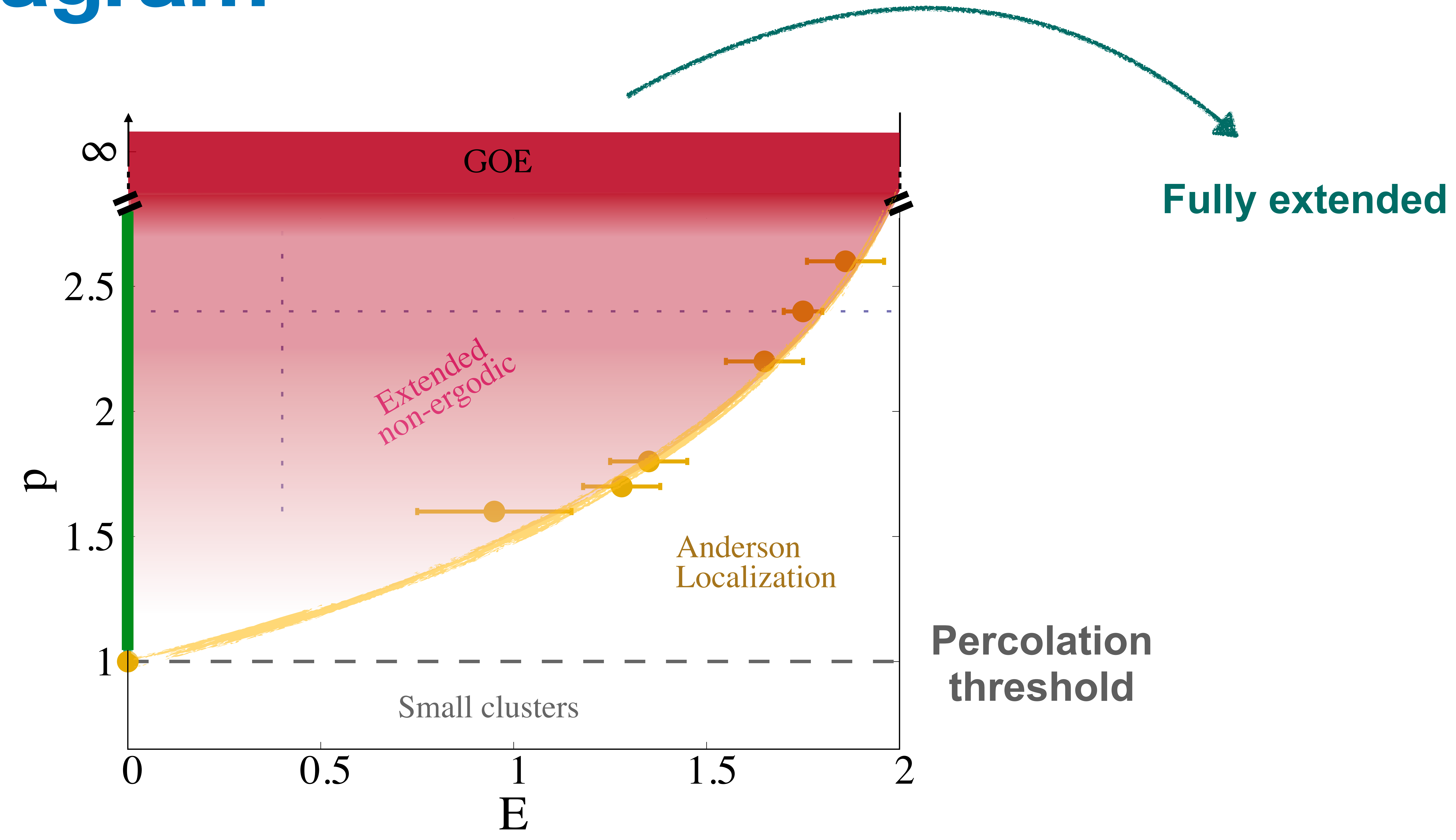


$p \gg 1$ ⟶

ER ≃ GOE

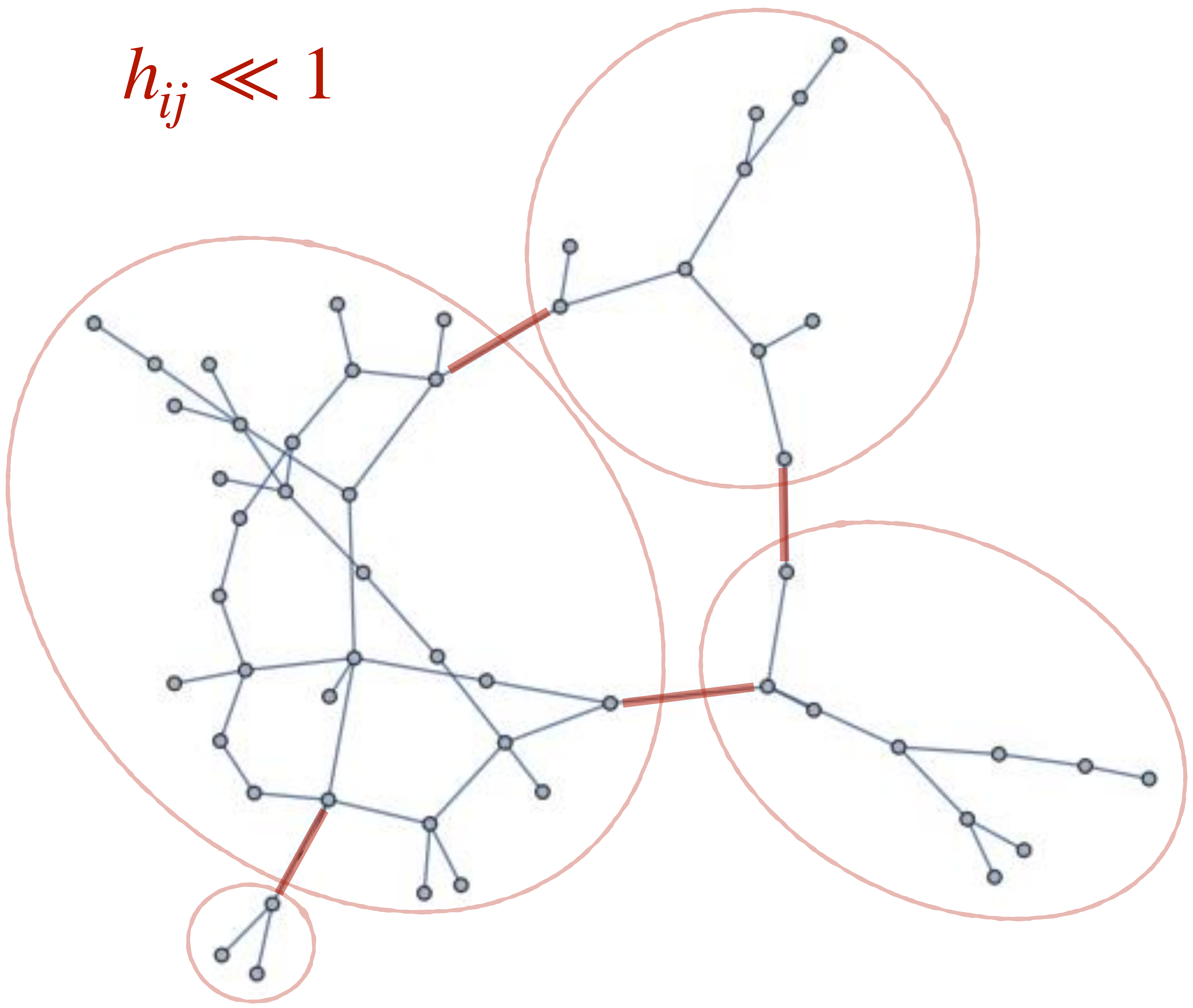
$D_q = 1$

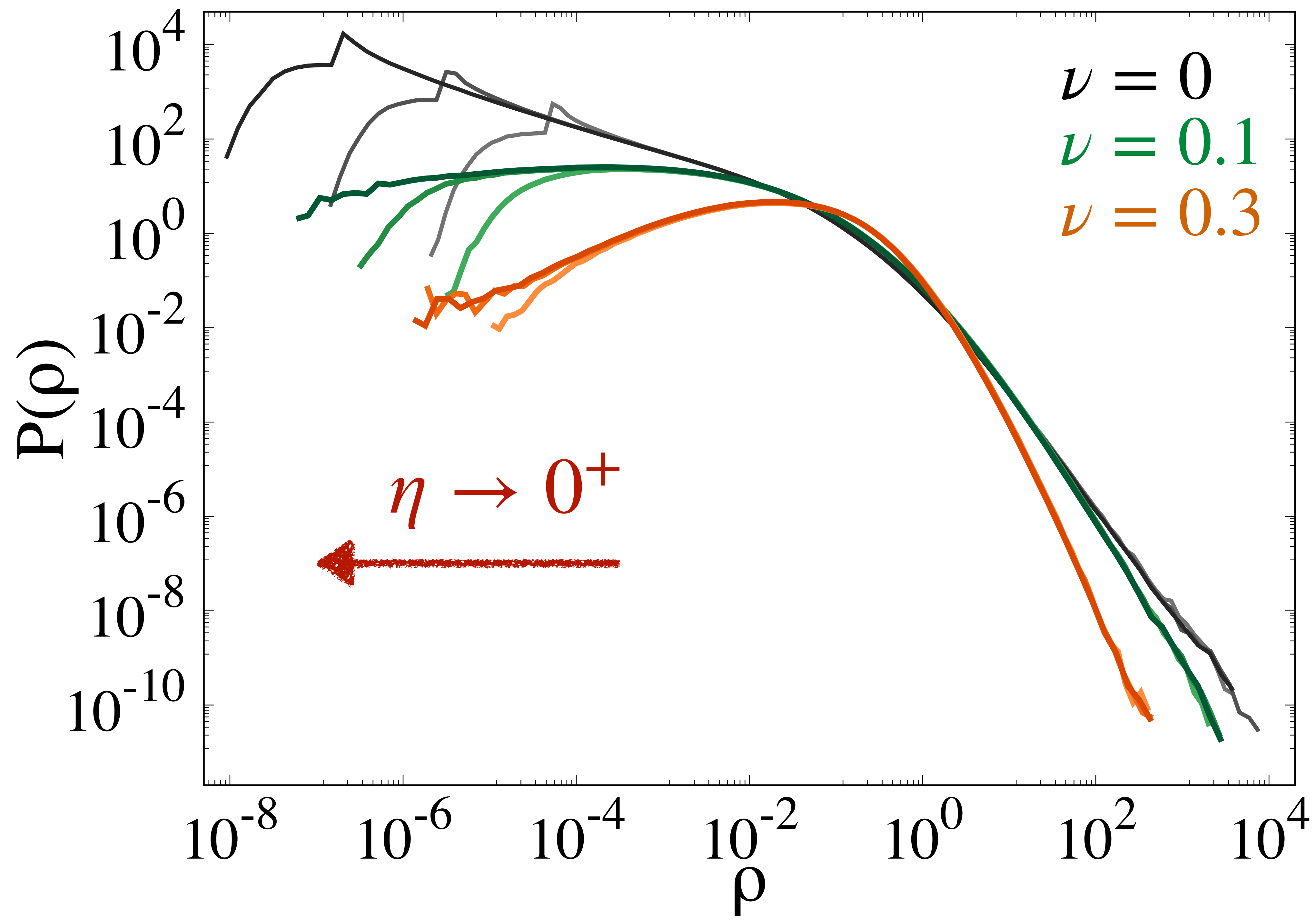
Phase diagram

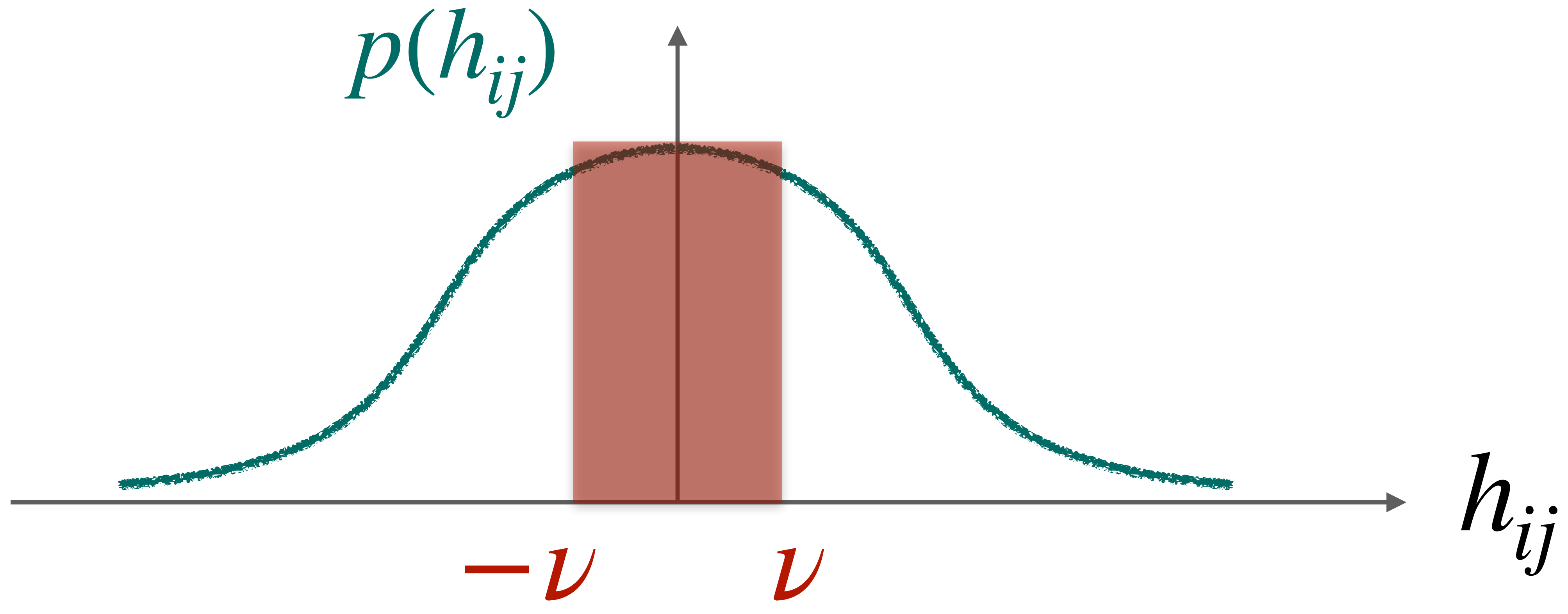


Cugliandolo, Schehr, Tarzia, Venturelli, arXiv: 2404.06931

Physical mechanism







Summing up

- Generalized Rosenzweig-Porter model
  - ☑ characterized **level correlations** using **replicas**
  - ☑ **universal** scaling form for level compressibility $\chi(E/E_T)$
  - ☐ other ensembles, e.g. Wishart-RP?
- Weigthed Erdös-Rényi graphs
  - ☑ measured **multifractal exponents** D_q via **cavity** approach
  - ☑ new multifractal phase + physical mechanism
  - ☐ other ensembles? Does the cutoff ν scale with N ?

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